A finiteness result for Heegaard splittings

Martin Lustig\textsuperscript{a,1}, Yoav Moriah\textsuperscript{b,*,2}

\textsuperscript{a}Mathématiques (L.A.T.P.), Université d’Aix-Marseille III, Ave. E. Normandie-Niemen, 13397 Marseille 20, France
\textsuperscript{b}Department of Mathematics, Technion, Haifa 32000, Israel

Received 9 January 2003; accepted 13 January 2004

Abstract

In this paper we show that for a given 3-manifold and a given Heegaard splitting there are finitely many preferred decomposing systems of $3g - 3$ disjoint essential disks. These are characterized by a combinatorial criterion which is a slight strengthening of Casson–Gordon’s rectangle condition. This is in contrast to fact that in general there can exist infinitely many such systems of disks which satisfy just the Casson–Gordon rectangle condition.

\textsuperscript{*}Corresponding author. Tel.: +972-48294088; fax: +972-48233388.
\textsuperscript{E-mail addresses:} martin.lustig@univ.u-3mrs.fr (M. Lustig), ymoriah@tx.technion.ac.il (Y. Moriah).
\textsuperscript{1}Supported by the Subvention 01/00495 for international scientific cooperation of the Région Provence (France).
\textsuperscript{2}Supported by The Fund for Promoting Research at the Technion, grant 100-127 and the Technion VRP fund, grant 100-127.

MSC: 57N25

Keywords: Heegaard splittings; Rectangle condition; Double rectangle condition; Pair of pants decomposition

1. Introduction

Every closed orientable three-dimensional manifold $M$ admits a \textit{Heegaard splitting}, i.e., a decomposition into two handlebodies $H_1$ and $H_2$ which meet along their boundary. This common boundary is called a \textit{Heegaard surface} in $M$ and is usually considered only up to isotopy in $M$.

Heegaard splittings are a convenient way to define a 3-manifold, but a priori it is difficult to get structural information about the manifold from them. In the last fifteen years a lot of progress was made in understanding the structural aspects of Heegaard splittings. A breakthrough was achieved in the work of Casson and Gordon [1] which ties Heegaard splittings to the existence of incompressible
surfaces. In particular, for non-Haken 3-manifolds strongly irreducible Heegaard surfaces are now considered as suitable analogues of essential surfaces in the Haken case, thus establishing them as an important tool in the study of these manifolds.

The main difficulty with Heegaard splittings is that a Heegaard splitting corresponds to a double coset \( \mathcal{H} \phi \mathcal{H} \) of an element \( \phi \) in the mapping class group \( \mathcal{MCG}(\Sigma_g) \) of a closed surface \( \Sigma_g \) of genus \( g \geq 2 \), where \( \mathcal{H} \) is the subgroup of surface homeomorphisms which extend to a handlebody \( H \) via a properly chosen identification \( \Sigma_g = \partial H \). This subgroup is not normal in \( \mathcal{MCG}(\Sigma_g) \), and it is not well understood at all. The geometric analogue of this problem is the absence of a canonical "coordinate system", that is a preferred choice of disks which define the handle structure in each of the two handlebodies of the splitting.

It is this problem that we wish to address. We choose a complete decomposing system \( \mathcal{D} \), of \( 3g - 3 \) disjoint non-parallel essential disks for each of the two handlebodies. These systems \( \mathcal{D}_1 \subset H_1 \) and \( \mathcal{D}_2 \subset H_2 \) decompose each of the handlebodies into \( 2g - 2 \) solid pairs of pants. Thus we obtain a Heegaard diagram for \( M \), i.e., a finite set of combinatorial data which determine \( M \). There are infinitely many such distinct complete decomposing systems in each handlebody, so that the idea to recover characteristic data for \( M \) from a Heegaard diagram might seem hopeless. It is in this light that the following main result of this paper should be seen:

**Theorem 2.6.** For any closed orientable 3-manifold \( M \) and any Heegaard splitting \( M = H_1 \cup_{\partial H_1 = \partial H_2} H_2 \) there are only finitely many pairs of complete decomposing systems \( \mathcal{D}_1 \subset H_1 \) and \( \mathcal{D}_2 \subset H_2 \) which satisfy the double rectangle condition.

The double rectangle condition, defined precisely in Section 2 below, is a slight strengthening of the rectangle condition introduced by Casson and Gordon in [1]. The statement that Casson–Gordon’s rectangle condition is generic, can be given a precise meaning using Thurston’s measure on the boundary of Teichmüller space. The question, whether the existence of complete decomposing systems which satisfy the double rectangle condition is a generic property for Heegaard splittings, is at present open (see Remark 5.4).

As a corollary we obtain:

**Corollary 1.1.** Let \( M \) be an atoroidal closed 3-manifold which admits a Heegaard splitting with two complete decomposing systems that satisfy the double rectangle condition. Then the mapping class group of \( M \) is finite.

**Proof.** It follows from a result of Jaco and Rubinstein [2] that an atoroidal 3-manifold has only finitely many Heegaard splittings of any given genus. Any self-homeomorphism of \( M \) must take two complete decomposing systems \( \mathcal{D}_1, \mathcal{D}_2 \) that satisfy the double rectangle condition to two other such systems and, by Theorem 2.6, there are only finitely many of those. But every mapping class which fixes \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) is easily seen to be trivial. \( \Box \)

### 1.1. Organization of the paper

In Section 2 we define the basic terminology and state our main result. We give a counterexample to the conclusion of Theorem 2.6 if the “double rectangle condition” is replaced by the weaker
“Casson–Gordon rectangle condition”. This shows that the rectangle condition is not sufficient to characterize a finite collection of “preferred” decomposing disk systems.

In section 3 we investigate how the disks of a second complete decomposing system $D_1'$ in $H_1$ intersect the complementary components $B_k$ of the fixed decomposing system $D_1$ in $H_1$ (these are solid pairs of pants). Any connected component $A' \subset D_1'$ of this intersection is a disk which has as boundary an alternating sequence of arcs from $D_1 \cap D_1'$ and from $D_1' \cap \partial H_1$. The number of such arcs can be used as a measure of complexity for $A'$. A priori there is no bound on this complexity, which is one of the main reasons why homeomorphisms of three-dimensional handlebodies remain a mysterious and little understood topic. In our context, however, one can exploit the rectangle condition to get an upper bound on this complexity which depends on $D_1$ and $D_2$ only. Even better, we show in section 3 that, up to proper isotopy, the disk $A'$ must come from a finite collection which depends again only on $D_1$ and $D_2$.

In section 4 we investigate the complementary components of $D_1'$ in each solid pair $B_k$. They are called parts, and we distinguish thin and thick parts. In the presence of the rectangle condition the possible nature and number of thick parts are both determined by $D_1$ and $D_2$, while the number of thin parts depends in an essential way also on $D_1'$.

Bounding the number of the thin parts is the main problem in the proof of theorem 2.6 and is the only place where the double rectangle condition is used. This is accomplished in section 5.

Remark 1.2. The intersection pattern induced by the disks from $D_1'$ on every solid pair of pants $B_k$ is strongly reminiscent of the intersection pattern on a 3-simplex given by a surface $S$ in normal position, which is cut by $S$ into a bounded number of thick blocks and an arbitrary number of thin pieces that occur in “parallel stacks” (compare e.g. [3]). One important difference is that normal surface theory is done for closed surfaces, while we work with disks in handlebodies.

2. The double rectangle condition

Let $M$ be a closed three-dimensional manifold, and $\Sigma \subset M$ be a closed orientable Heegaard surface of genus $g \geqslant 2$ cutting $M$ into two handlebodies $H_1$ and $H_2$.

Let $D_1 \subset H_1$ and $D_2 \subset H_2$ be two complete decomposing disk systems, i.e., each handlebody is decomposed by the disk system into a union of solid pairs of pants. We will always assume that $\partial D_1$ and $\partial D_2$ have only essential intersections, that is, they intersect in transverse intersection points, and one can not decrease their number by a proper isotopy of $D_1$ in $H_1$ or of $D_2$ in $H_2$.

A wave $\omega \subset \Sigma$ with respect to $D_1$ is an arc that meets $D_1$ only in its boundary points $\partial \omega$, which lie on the same component $\partial D_j \subset \partial D_1$, such that in $H_1$ the arc $\omega$ is isotopic relative endpoints to a subarc of $\partial D_j$, but not in $\Sigma$. Similarly we define waves for $D_2$.

We say that the closure of a connected component of $\Sigma - (\partial D_1 \cup \partial D_2)$ is a rectangle $R$ if it is homeomorphic to a disk, whose boundary $\partial R$ is a concatenation of precisely four arcs, two of which are subarcs on curves in $\partial D_1$ and the other two are subarcs of curves in $\partial D_2$. It is possible that two of the curves from one system belong to the same component, and even that two opposite “boundary vertices” of the rectangle are identified.

An adjacent pair of curves in $\partial D_1$ (similarly in $D_2$) consists of two curves which can be joined by an essential arc in $\Sigma - \partial D_1$ which does not meet other curves from $\partial D_1$, and which is not a wave.
Such an arc lies in one of the pair of pants of the decomposition defined by \( \partial \mathcal{D}_1 \), and is unique up to isotopy in this pair of pants, so that we usually suppress its mentioning and only note the two curves in \( \partial \mathcal{D}_1 \). Similarly, an *adjacent triple of curves* in \( \partial \mathcal{D}_1 \) consist of three curves which can be connected by an arc that intersects the middle curve transversely, and the resulting two subarcs define two adjacent pairs of curves. Note that the above two definitions include the situation where the inclusion of the pair of pants into the surface \( \Sigma \) identifies two of its boundary curves. The same definitions hold for \( \mathcal{D}_2 \subset H_2 \).

Casson and Gordon have introduced the following [1]:

**Definition 2.1.** The complete decomposing systems \( \mathcal{D}_1 \subset H_1 \) and \( \mathcal{D}_2 \subset H_2 \) satisfy the *rectangle condition* if every pair of adjacent curves in \( \partial \mathcal{D}_1 \) and any pair of adjacent curves in \( \partial \mathcal{D}_2 \) form at least one rectangle which is contained in the intersection of the respective pairs of pants.

The importance of this notion comes from Casson–Gordon’s observation that a Heegaard splitting \( M = H_1 \cup \Sigma H_2 \) which satisfies the rectangle condition is strongly irreducible: Indeed, any essential disk \( D \subset H_1 \) must either be parallel to a curve of \( \mathcal{D}_1 \) or contain a wave with respect to \( \mathcal{D}_1 \). In both cases there exist two adjacent curves of \( \mathcal{D}_1 \) such that \( D \) intersects all rectangles formed by these two curves with any adjacent pair of curves from \( \partial \mathcal{D}_2 \). As the analogue is true for any essential disk \( E \subset H_2 \), it follows from the rectangle condition that \( D \) and \( E \) must intersect in one of the rectangles, so that the Heegaard splitting is strongly irreducible. In particular all waves with respect to \( \mathcal{D}_1 \) must intersect all waves with respect to \( \mathcal{D}_2 \).

The same idea is used in the proof of the next lemma.

**Lemma 2.2.** (a) If \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) satisfy the rectangle condition, then for any disk \( D \subset H_1 \) the boundary curve \( \partial D \subset \Sigma \) does not contain a wave with respect to \( \mathcal{D}_2 \subset H_2 \).

(b) Every wave on \( \Sigma \) with respect to \( \mathcal{D}_1 \) intersects every curve which bounds a disk in \( H_2 \) at least once.

**Proof.** (a) As \( \mathcal{D}_1 \) is a complete decomposing system of \( H_1 \), the curve \( \partial D \) must either be parallel to one of the \( \partial D_i \), or it contains a wave with respect to \( \partial \mathcal{D}_1 \). In both cases there exist two adjacent curves of \( \mathcal{D}_1 \) such that \( \partial D \) intersects all rectangles formed by these two curves with any adjacent pair of curves from \( \partial \mathcal{D}_2 \). If \( \partial D \) also contains a wave with respect to \( \partial \mathcal{D}_2 \), then there exist two adjacent curves of \( \mathcal{D}_2 \) with the same property. Hence we could deduce from the rectangle condition at least one self-intersection of \( \partial D \) in one of the rectangles.

(b) The claim follows exactly from the same arguments.

**Remark 2.3.** It is possible that a given Heegaard splitting possesses infinitely many non-isotopic decomposing disk systems \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) all satisfying the rectangle condition. An example will be given at the end of this section.

To get the desired finiteness result Theorem 2.6, we have to strengthen the rectangle condition slightly: We call the union of two rectangles which have a side in common, a *double rectangle*. Thus the boundary of a double rectangle formed by \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) consists of two subarcs from an adjacent pair of curves of, say, \( \partial \mathcal{D}_1 \), and of two subarcs from the two outer curves of an adjacent triple of curves of \( \partial \mathcal{D}_2 \).
Definition 2.4. The decomposing disk systems $D_1$ and $D_2$ satisfy the double rectangle condition if every pair of adjacent curves from $\partial D_1$ forms, with every adjacent triple from $\partial D_2$, a double rectangle, and vice versa.

Note that, of course, the double rectangle condition implies the rectangle condition.

Lemma 2.5. If $D_1$ and $D_2$ satisfy the double rectangle condition, then every essential disk $D \subset H_1$ intersects every triple in $D_2$, and vice versa.

Proof. If $D$ belongs (perhaps after a proper isotopy in $H_1$) to $D_1$, then the claim is obviously true. Otherwise, the curve $\partial D$ has a wave with respect to $D_1$. This implies that there is at least one adjacent pair of curves in some pair of pants in $D_1$ which is separated by this wave. Since $D_1$ and $D_2$ satisfies the double rectangle condition, the adjacent pair and hence the curve $\partial D$ must intersect any adjacent triple of curves from $D_2$. □

It follows that on an adjacent pair of pants we have the following intersection pattern as in Fig. 1.

We can now state the main result of this paper:

Theorem 2.6. For any closed orientable 3-manifold $M$ and any Heegaard splitting $M = H_1 \cup_\Sigma H_2$ there are only finitely many pairs of complete decomposing systems $D_1 \subset H_1$ and $D_2 \subset H_2$ which satisfy the double rectangle condition.
We finish this section with a counterexample to the analogue of this result, if one replaces the double rectangle condition by the simple rectangle condition:

**Example 2.7.** Consider the genus two Heegaard diagram obtained from Fig. 2 by making the following identifications: $D_1 \equiv D'_1$, $x \equiv x'$, $y \equiv y'$ and $D_2 \equiv D'_2$, $w \equiv w'$, $z \equiv z'$.

Let $H_1$ be the genus two handlebody obtained by these identifications from Fig. 2, and let $H_2$ be an identical copy of $H_1$. Let $M = H_1 \cup_{\partial} H_2$, where $t$ is some sufficiently large integer, and $\delta'$ is the $t$-fold Dehn twist along the curve $\delta \subset \partial H_1$. Let $\mathcal{D}_1$ be the complete decomposing system given by the disks $\{D_1, D_2, D_3\}$ and $\mathcal{D}_2$ be the identical system in $H_2$. Note that our choice of the Dehn twist exponent ensures that the two systems $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy the rectangle condition.

Now consider the annulus $A \subset H_1$ as in Fig. 2 and change the system $\mathcal{D}_1$ to a system $\mathcal{D}_1^n$ by twisting $n$ times along $A$. It is immediate to see that all systems $\mathcal{D}_1^n$ together with the system $\mathcal{D}_2$ satisfy the rectangle condition for all $n \in \mathbb{Z}$.

### 3. Finiteness of disk types

We now concentrate on the handlebody $H_1$ which contains two complete decomposing disk systems $\mathcal{D}_1$ and $\mathcal{D}_1'$. We think of $\mathcal{D}_1$ as being the fixed reference system, and of $\mathcal{D}_1'$ as an alternative candidate: The goal of the paper is to show that, under the right conditions, there are only finitely many such $\mathcal{D}_1'$.

In order to simplify the terminology we define:

**Definition 3.1.** We say that a constant defined by means of $\mathcal{D}_1'$ is *uniformly bounded* if it depends only on the fixed pair of decomposing systems of disks $\mathcal{D}_1 \subset H_1$ and $\mathcal{D}_2 \subset H_2$.

As handlebodies are irreducible, we can assume that (after a suitable isotopy) $\mathcal{D}_1$ and $\mathcal{D}_1'$ are *tight*: They intersect only in arcs which terminate in essential intersection points of their boundary.
curves. Thus each disk of $\mathcal{D}_1'$ is cut by $\mathcal{D}_1$ into disk pieces which have as boundary an alternating sequence of intersection arcs from $\mathcal{D}_1 \cap \mathcal{D}_1'$ and connecting arcs from $\partial \mathcal{D}_1' \subset \partial H_1$.

Every connecting arc is contained in a single pair of pants from the decomposition of $\partial H_1$ with respect to $\mathcal{D}_1$, and it cannot be boundary parallel on this pair of pants: This follows from our assumption that $\mathcal{D}_1$ and $\mathcal{D}_1'$ are tight. For intersection arcs we prove the weaker fact that they can not be boundary parallel on $\mathcal{D}_1 - \partial \mathcal{D}_2$:

**Lemma 3.2.** Let $\mathcal{D}_1, \mathcal{D}_1' \subset H_1$ and $\mathcal{D}_2, \mathcal{D}_2' \subset H_2$ be complete decomposing systems, and assume that the pair $\mathcal{D}_1', \mathcal{D}_2$ satisfies the rectangle condition. Then for any disk $D_k \subset \mathcal{D}_1$ every intersection arc $\alpha \subset D_k$ has its endpoints in two distinct connected components of $\partial D_k - \partial D_2$.

**Proof.** It suffices to consider an intersection arc $\alpha$ which is contained in the boundary of an outermost subdisk $\Delta$ of $D_k \in \mathcal{D}_1$. Every such $\Delta$ contains in its boundary an arc $\omega = \partial \Delta - \tilde{\alpha} \subset \partial D_k$. As $\Delta$ is outermost, $\omega$ meets $\mathcal{D}_1'$ only in its boundary points, and hence is a wave on $D_k$ with respect to $\mathcal{D}_1'$.

We can apply Lemma 2.2(b) to $\mathcal{D}_1'$ and $\mathcal{D}_2'$ to conclude that $\omega$ must meet every curve of $\partial \mathcal{D}_2$ (see Fig. 3). \qed

We now use the disks from $\mathcal{D}_2$ to group the intersection and the connecting arcs, defined above, into equivalence classes: Given a disk $D_i \subset \mathcal{D}_1$, two intersection arcs $\alpha, \alpha' \subset D_i \cap \mathcal{D}_1'$ will be called parallel if the pair $(\alpha, \partial \alpha)$ is isotopic to the pair $(\alpha', \partial \alpha')$ in $(D_i, \partial D_i - \partial \mathcal{D}_2)$. Similarly, two connecting arcs $\beta, \beta'$ will be call parallel if the pair $(\beta, \partial \beta)$ is isotopic to the pair $(\beta', \partial \beta')$ in $(\partial H_1, \partial \mathcal{D}_1 - \partial \mathcal{D}_2)$. Such an isotopy class of parallel arcs will be called the arc type of an intersection arc or of a connecting arc.

It follows from Lemma 3.2 and from the stronger fact for connecting arcs, stated in the paragraph just before Lemma 3.2, that two arcs $\alpha$ and $\alpha'$ which belong to the same arc type are indeed parallel:
They span a band (in $\partial H_1$ or in $\mathcal{D}_1$) where the “long” sides are given by $\alpha$ and $\alpha'$, while the “short” sides are arcs from $\partial \mathcal{D}_1 - \partial \mathcal{D}_2$.

**Lemma 3.3.** Let $\mathcal{D}_1, \mathcal{D}_1' \subset H_1$ and $\mathcal{D}_2, \mathcal{D}_2' \subset H_2$ be complete decomposing systems, and assume that the pair $\mathcal{D}_1', \mathcal{D}_2'$ satisfies the rectangle condition. Then the number of intersection arc types on $\partial \mathcal{D}_1'$ with respect to $\mathcal{D}_1$ is uniformly bounded above.

**Proof.** The system $\mathcal{D}_2$ determines the number of points of $\partial \mathcal{D}_2$ on each of the $3g - 3$ components of $\partial \mathcal{D}_1$. Hence it determines their complementary components on $\partial \mathcal{D}_1$. Thus there are finitely many relative isotopy classes of arcs (in $\partial H_1$ or in $\mathcal{D}_1$) connecting them.  \[\square\]

**Lemma 3.4.** Let $\mathcal{D}_1, \mathcal{D}_1' \subset H_1$ and $\mathcal{D}_2, \mathcal{D}_2' \subset H_2$ be complete decomposing systems, and assume that the pair $\mathcal{D}_1', \mathcal{D}_2'$ satisfies the rectangle condition. Then the number of connecting arc types on $\partial \mathcal{D}_1'$ with respect to $\mathcal{D}_1$ is uniformly bounded above.

**Proof.** Every connecting arc $\alpha$ is contained in a single pair of pants $P$ from the decomposition of $\partial H_1$ with respect to $\mathcal{D}_1$. Hence its isotopy class relative endpoints is essentially determined by the choice of the boundary curves from $\partial P \subset \partial \mathcal{D}_1$ which contain the endpoints of $\alpha$. More precisely, up to relative isotopy these arcs are determined by the intervals on such a boundary curve which in turn are determined by the intersections with the system $\mathcal{D}_2$, up to possible twists around these boundary curves. Thus we need to show that there are only finitely many choices for the number of such twists:

As the connecting arcs are disjoint among themselves, if one of them spirals around a boundary component $\partial D_i$ of $P$, then so do all of those connecting arcs which have an endpoint on $\partial D_i$. This spiraling is “controlled” by the arcs from $\partial \mathcal{D}_2$ in $P$: By Lemma 2.2 (b) for each $D_i$ from $\mathcal{D}_1$ there must be at least one arc $\beta$ from $P \cap \partial \mathcal{D}_2$ which intersects $\partial D_i$.

We note that somewhere on $\partial D_i$ there must be a wave with respect to $\partial \mathcal{D}_1'$: This wave is given by two adjacent intersection points on $\partial D_i$ with two connecting arcs $x_1, x_2$ that lie on the same curve $\partial D_j' \subset \partial \mathcal{D}_1'$, such that, when running once around $\partial D_j'$, the arcs $x_1, x_2$ are traversed in opposite directions (see Fig. 4).

Now assume that $x_1$ and $x_2$ spiral around $\partial D_i$ for some time, in a parallel fashion, thus intersecting the above arc $\beta$ at least once. But then the band spanned by the spiraling arcs $x_1$ and $x_2$ intersects...
Lemma 3.5. Let \( \beta \) in a wave on \( \beta \subset \partial D_k \subset \partial D_2 \) with respect to \( \partial D'_1 \). Since the disk \( D_k \) belongs to \( \mathcal{D}_2 \) or has some wave with respect to \( \mathcal{D}_2 \), this would contradict Lemma 2.2 (a). Hence \( x_1 \) and \( x_2 \) cannot spiral around \( \partial D_i \), and hence there are only finitely many connecting arc types on any pair of pants \( P \) which are determined only by \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \). \( \square \)

We call the components of \( H_1 \), when cut along \( \mathcal{D}_1 \) (or \( \mathcal{D}'_1 \)), \textit{solid pairs of pants} and denote them by \( B_k \) (or \( B'_k \), respectively), for \( k = 1, \ldots, 2g - 2 \). Denote by \( \mathcal{B} \) (or \( \mathcal{B}' \), respectively) the collection of these solid pairs of pants. We defined above a disk piece to be a connected component of some \( \partial \mathcal{D}_1 \cap B_k \). Define a \textit{disk type} to be a class of disks pieces whose boundaries are composed of intersection arcs and connecting arcs which are parallel pairwise. It follows from the previous discussion that disk pieces which belong to the same disk type lie in one of the \( B_k \) as a parallel stack, that is, homeomorphic to horizontal disks in \( D^2 \times \mathbb{R} \).

A priori a disk piece can have in its boundary distinct connecting arcs or intersection arcs that belong to the same arc type. However, this turns out to be impossible, if the rectangle condition is imposed:

**Lemma 3.5.** Let \( \mathcal{D}_1, \mathcal{D}'_1 \subset H_1 \) and \( \mathcal{D}_2, \mathcal{D}'_2 \subset H_2 \) be complete decomposing systems, and assume that the pair \( \mathcal{D}'_1, \mathcal{D}'_2 \) satisfies the rectangle condition. Then any intersection arc type or connecting arc type can occur in the boundary of a given disk piece at most once.

**Proof.** Given a disk piece \( \Delta' \subset \mathcal{D}'_1 \), orient its boundary \( \partial \Delta' \) and assume that some arc type appears more than once in \( \partial \Delta' \). Hence there are two distinct arcs \( x_1, x_2 \) in \( \partial \Delta' \) which belong to the same arc type.

Let \( B_k \) be the solid pair of pants that contains \( \Delta' \). Note that \( \partial B_k \) is a 2-sphere and \( \partial \Delta' \) is a simple closed curve on this sphere. Hence, if the orientation induced on \( x_1 \) and \( x_2 \) by the choice of orientation on \( \partial \Delta' \) induces on them the same orientation as parallel intersection or connecting arcs, then there must be a third arc \( x_3 \) in \( \partial \Delta' \) of the same arc type, such that \( x_3 \) runs between \( x_1 \) and \( x_2 \), but with the opposite orientation: Otherwise \( \partial \Delta' \) would either not be simple or not be a closed curve.

Hence we can assume by a standard innermost argument that \( x_1 \) and \( x_2 \) are adjacent arcs in the same arc type, and that \( \partial \Delta' \) traverses them in opposite directions. Let \( \partial D_i \subset \partial \mathcal{D}_1 \) be the curve which contains an endpoint of this arc type i.e., \( D_i \subset \mathcal{D}_1 \) is one of the three boundary disks of \( B_k \). Furthermore let \( \beta \) be the subarc on \( \partial D_i \) which joins the endpoints of \( x_1 \) and \( x_2 \). Since the two arcs are adjacent in the arc type, and are traversed by \( \partial \Delta' \) in opposite directions, it follows that \( \beta \) is a wave on \( \partial D_i \subset \partial \mathcal{D}_1 \) with respect to \( \mathcal{D}'_1 \). In particular, \( \beta \) does not meet \( \mathcal{D}_2 \) in its interior. As we assume that \( \mathcal{D}'_1 \) and \( \mathcal{D}'_2 \) satisfy the rectangle condition, this contradicts Lemma 2.2(b). \( \square \)

**Proposition 3.6.** Let \( \mathcal{D}_1, \mathcal{D}'_1 \subset H_1 \) and \( \mathcal{D}_2, \mathcal{D}'_2 \subset H_2 \) be complete decomposing systems, and assume that the pair \( \mathcal{D}'_1, \mathcal{D}'_2 \) satisfies the rectangle condition. Then there is a finite set of disk types in any of the solid pair of pants \( B_k \) from \( H_1 - \mathcal{D}_1 \), such that any of the disk pieces of \( \mathcal{D}'_1 - \mathcal{D}_1 \) belongs to one of the disk types in the above finite set. Furthermore, the number of disk types in this finite set is uniformly bounded above.

**Proof.** We can apply Lemmas 3.3–3.5 to conclude that \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) determine a finite set of intersecting arc types, and a finite set of connecting arc types, which can possibly appear in the boundary of
a disk type $A$. Furthermore, each of those appears in the boundary of $A$ at most once. Hence there are only finitely many possible disk types for $A$, and they are dependent only on $D_1$ and $D_2$. 

**Remark 3.7.** Note that in all of Lemmas 3.2–3.5 and Proposition 3.6 we require that only $D_1'$ and $D_2'$ satisfy the rectangle condition, but not necessarily $D_1$ and $D_2$.

### 4. Thick and thin regions

In the last section we considered the solid pairs of pants $B_k \in \mathcal{B}$ obtained from cutting the handlebody $H_1$ along the complete decomposing disk system $\mathcal{D}_1$. In this section we change our point of view and consider the solid pairs of pants $B'_l$, obtained from cutting $H_1$ along the disk system $\mathcal{D}'_1$. The collection of these solid pairs of pants will be called $\mathcal{B}'$. The connected components of the intersection $B_k \cap B'_l$ of any $B_k \in \mathcal{B}_1$ with any $B'_l \in \mathcal{B}'_1$ are called *parts*, and we distinguish two kinds of them:

**Definition 4.1.** A connected component of $B_k \cap B'_l$ is called a *thin part* if its intersection with $D'_1$ consists of two disk pieces which belong to the same disk type in $B_k \in \mathcal{B}$. Otherwise the connected component is called a *thick part*.

In any solid pair of pants $B_k$ a *stack* is a maximal collection of thin parts. The boundary of the stack is composed of disk pieces from $D'_1$ all belonging to the same disk type. Notice that the complementary components in $B_k$ of the union of all stacks are precisely the thick parts of $B_k$.

We now want to group together the parts in one solid pair of pants $B'_l$ into larger units, called *regions*:

**Definition 4.2.** For each $B'_l \in \mathcal{B}'_1$, a *thick region* is a maximal union of thick parts in $B'_l$ which is connected. The region is *thick peripheral* if it is disjoint from at least one of the three boundary disks of $B'_l$ from the system $\mathcal{D}'_1$ (see Fig. 7). The region is called *central* if all three boundary disks are met (see Fig. 5). A *thin region* is a maximal connected union of thin parts contained in $B'_l$ (see Fig. 6). The *volume* of any region is the number of parts contained in that region. Finally, the *diameter* of a region is given via the distance between parts, where adjacent parts are defined to have distance $1$.

In Fig. 7 below we display a schematic picture of a thick peripheral region. Note that in general they can be more complicated.

**Lemma 4.3.** There are finitely many possibilities for the central and the thick peripheral regions in $H_1$ which are completely determined by $\mathcal{D}_1$ and $\mathcal{D}_2$ only. In particular, their number, as well as the volume and the diameter of any of them, are uniformly bounded above by constants $N \geq 0$, $K \geq 0$ and $d \geq 0$, respectively.

**Proof.** We observed above that, in any solid pair of pants $B_k$, the complementary components of the union of all stacks are precisely the thick regions in $B_k$. Since the stacks are in one to one correspondence with the disk types, the claim follows directly from Proposition 3.6. □
Lemma 4.4.  (a) Every solid pair of pants \( B_i' \) has a unique central region.

(b) For every disk \( D_j' \) from \( D_1' \) which lies on the boundary of \( B_i' \) the intersection of \( D_j' \) with the central region of \( B_i' \) is connected.

Proof. (a) For every disk \( D_i \) from \( D_1 \) any connected component \( \Delta \) of \( D_i \cap B_i' \) cuts \( B_i' \) into two distinct connected components. Hence, if \( \Delta \) misses one of the three disks from \( D_1' \) which lie on the boundary of \( B_i' \), say \( D_j' \), then this disk \( D_j' \) can intersect only one of the two connected components.

Now, note that by Definition 4.2 any two distinct thick regions in \( B_i' \) are connected by a path \( \gamma \) which crosses at least one thin region, and hence, in the boundary of this thin region, \( \gamma \) crosses a component \( \Delta \) as above. This shows that at most one of the two thick regions can be central.
To show the existence of a central region we first consider a connected component $\Delta$ of $D_i \cap B'_i$ which meets all three disks from $\mathcal{D}'_1$ that lie on the boundary of $B'_i$. Such a $\Delta$ can not be contained in a thin or in a thick peripheral region, so that a central region must exist: if there is no such $\Delta$, then, as shown above, each $\Delta$ cuts $B'_i$ into a connected component that meets only two of the three boundary disks from $\mathcal{D}'_1$, and a second connected component that meets all three boundary disks. It follows directly that the intersection of these second connected components, for all $\Delta$, is a single thick part which must meet all three boundary disks. Hence there exists a central region to which this part belongs.

(b) We observe that the subdisk $\Delta$ on the boundary of a thin region, as above, intersects a disk $D'_j$ in at most one arc. Hence we can apply the same arguments as in case (a) to any of the disks $D'_j$ on the boundary of $B'_i$. $\square$

A maximal connected union $P'$ of thin or thick peripheral parts of $B'_i$ is called peripheral component of $B'_i$. Notice that any such peripheral component $P'$ meets precisely two disks $D'_i$ and $D'_j$ from the collection $\mathcal{D}'_1$. It follows from the proof of Lemma 4.4 that the intersections $P' \cap D'_i$ and $P' \cap D'_j$ are subdisks, and that $P'$ meets the closure of its complement $B'_i - P'$ in a subdisk $\Delta$ of some $D_i$ from $\mathcal{D}_1$, where $\Delta$ belongs to a thin part of $P'$. Hence the boundary $\partial P'$ consists of $\Delta$, of $P' \cap D'_i$ and $P' \cap D'_j$, and of a band $A$ that has as boundary two “long” arcs $\alpha_i \subset \partial D'_i$, $\alpha_j \subset \partial D'_j$, and two “short” arcs $\beta, \beta' \subset \partial \Delta$.

Lemma 4.5. (a) The arcs $\alpha_i = A \cap \partial D'_i$ and $\alpha_j = A \cap \partial D'_j$ meet exactly the same sequence of disks from $\mathcal{D}_1$.

(b) The number of disk pieces in the subdisks $P' \cap D'_i$ and $P' \cap D'_j$ is equal.
Proof. (a) The band $A$ is topologically a disk (since $P'$ is a subball of the 3-ball $B'_1$), and we work with the assumption that $\mathcal{D}_1$ and $\mathcal{D}'_1$ are tight, so that their boundary curves intersect only essentially. Hence $\partial\mathcal{D}_1$ meets $A$ in a collection of parallel arcs with one endpoint on $z_i$ and the other on $z_j$.

(b) We observe that $P'$ may very well contain thick peripheral regions, so that the pattern of intersection arcs on $P' \cap D'_1$ and on $P' \cap D'_j$ may be quite different. However, it follows directly from (a) that the number of intersection arcs on $P' \cap D'_1$ and on $P' \cap D'_j$ must agree, which implies the claim. 

Imagine the disk $D'_j$ in a horizontal position, so that it is part of the boundary of an adjacent solid pair of pants above it, and a second adjacent solid pair of pants below it. Both of these solid pairs of pants are from the collection $\mathcal{B}'$ defined above. We call the intersection of $D'_j$ with the central region from the top solid pair of pants the top central subdisk, and the one from the bottom the bottom central subdisk. We measure the distance between them by counting the number of transverse intersections with the disk system $\mathcal{D}_1$ of any path in $D'_j$ connecting the two central subdisks, and taking the minimum over all such paths.

We define the extended top central region (and similarly the extended bottom central region) to be the central region of the top solid pair of pants together with all parts from the bottom solid pair of pants which are adjacent to the top central subdisk.

Proposition 4.6. If $\mathcal{D}'_1$ and $\mathcal{D}'_2$ satisfy the double rectangle condition, then the distance between top and bottom central subdisks on any of the disks $D'_j$ in $\mathcal{D}'_1$ is uniformly bounded from above by a constant $c \geq 0$.

Proof. Since $\mathcal{D}'_1$ and $\mathcal{D}'_2$ satisfy the double rectangle condition, we can apply Lemma 2.5 to show that every disk $E_i$ from the system $\mathcal{D}_2$ must intersect every adjacent triple from the system $\mathcal{D}'_1$ in some arc $z_h \subset \partial E_i$. We consider in particular the four adjacent triples which are contained in the union of the two solid pairs of pants $B'_1, B'_m$ adjacent to the disk $D'_j$ on the top and on the bottom.

If the top central subdisk and the bottom central subdisk intersect in $D'_j$, then their distance is by definition 0. In the case where the top and the bottom central subdisks of $D'_j$ are disjoint, we observe that the extended top and bottom regions in $B'_1 \cup B'_m$ are separated by pairs of parts, one on the top, one on the bottom, which belong to peripheral components of $B'_1$ and of $B'_m$. In particular, the union of these pairs of parts meets only two of the four disks from $\mathcal{D}'_1$ which lie on the boundary of $B'_1 \cup B'_m$.

Hence for at least one of the above four adjacent triples, the corresponding arc $z_h$ intersects both, the top and bottom extended central regions. As a consequence, the distance between the top and bottom central subdisks on $D'_j$ is bounded above by the minimal number of intersections with $\mathcal{D}_1$ of any curve from $\mathcal{D}_2$. We will denote this upper bound which depends only on $\mathcal{D}_1$ and $\mathcal{D}_2$ by $c$. 

5. Dual trees

For every disk $D'_j$ from $\mathcal{D}'_1$ we consider a graph whose vertices are in one to one correspondence with the disk pieces of $D'_j$, and whose edges are in one to one correspondence with the intersection arcs $z_i \subset D'_j \cap \mathcal{D}_1$. Each $z_i$ cuts $D'_j$ into two distinct connected components. Hence the above graph is a tree, called the dual tree $T'_j$. 

We measure the distance in $T'_j$ by the usual simplicial metric, i.e., by associating to every edge the length 1. The volume of a subtree of $T'_j$ is given by the number of vertices contained in the subtree. The area of a subdisk is the number of disk pieces in the subdisk, which is equal to the volume of the corresponding dual subtree.

The top and bottom central subdisks of $D'_1$ define top and bottom central subtrees of $T'_j$. The complementary components are called top or bottom peripheral subtrees of $T'_j$. Similarly, any thick peripheral region in the adjacent top or bottom solid pair of pants defines, by way of intersection with $D'_j$, top or bottom thick peripheral subtrees of $T'_j$.

**Remark 5.1.** Proposition 4.6 shows that the distance in $T'_j$ between the top and the bottom central subtrees is uniformly bounded by the constant $c > 0$, if the double rectangle condition is satisfied by $D'_1$ and $D'_2$.

**Lemma 5.2.** For any real number $b > 0$ in any of the $T'_j$ the volume and the number of complementary components of the $b$-neighborhood of the top or of the bottom central subtree, or of any thick peripheral subtree, are uniformly bounded above by some constant $k = k(b) > 0$.

**Proof.** The valence of a given vertex in the tree is exactly the number of intersection arcs of the corresponding disk piece with $D_1$, which in turn is fixed for all the disk pieces from the same disk type. Since there are only finitely many disk types and finitely many thick parts, which are all determined a priori by $D_1$ and $D_2$, see Proposition 3.6, it follows directly that there is an upper bound $b_0$ on the valence of any vertex in $T'_j$. But then the volume as well as the number of complementary components of the $b$-neighborhood of any finite subtree is clearly bounded above by $b_0^b$ times the volume of that subtree, where the latter is bounded uniformly in terms of the constant $K$ from Lemma 4.3.

To continue the proof we need to define the following class of subtrees of any $T'_j$:

A subtree $R'_j$ of $T'_j$ will be called a red subtree, if it satisfies the following conditions:

(a) There is only one vertex, the root of $R'_j$, which is adjacent to some edge contained in $T'_j$ but not in $R'_j$, and this edge is unique. In other words, $R'_j$ is obtained from $T'_j$ as connected component after removing a single edge.

(b) The subtree $R'_j$ is disjoint from the top or from the bottom central subtree of $T'_j$. In the first case $R'_j$ is called a top red subtree, and in the second a bottom red subtree.

We now describe a two-tiered method, called the disk pushing procedure, of how to pass

(I) from a bottom red subtree in one of the $T'_j$ to a particular top red subtree in the same $T'_j$, and
(II) from a top red subtree in $T'_j$ to a particular bottom subtree in an adjacent $T'_k$.

It is this procedure that allows us to uniformly bound the size of the thin parts of the disks in $D'_1$ and thus it is a crucial tool for the proof of our main result.

(I). Let $R'_j$ be a bottom red subtree of $T'_j$. We define an adjacent top red subtree $R''_j$ as follows: If $R'_j$ is disjoint from the top central subtree of $T'_j$, then we set $R''_j = R'_j$. If the top central subtree intersects $R'_j$, then we consider the $(c + d)$-neighborhood $C \subset T'_j$ of the top central subtree, for $c$ as in Proposition 4.6 and Remark 5.1, and $d$ denoting the maximal diameter of the top or bottom central subtree in any of the dual trees $T'_j$ (see Lemma 4.3). Note that by Lemma 4.3 the bound $d$
Proposition 5.3. For each pair of decomposing disk systems \( \mathcal{D}_1 \subset H_1 \) and \( \mathcal{D}_2 \subset H_2 \) there is an upper bound \( a > 0 \) such that, for any second pair \( \mathcal{D}_1' \subset H_1 \) and \( \mathcal{D}_2' \subset H_2 \) which satisfy the double rectangle condition, the area of any disk from \( \mathcal{D}_1' \) with respect to \( \mathcal{D}_1 \) is bounded above by \( a \).

Proof. By Lemma 4.3 the systems \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) determine finitely many possibilities for the thick regions in any solid pair of pants \( \mathcal{B}'_i \), and hence in particular for the central subtrees for any of the adjacent disks \( D_j \) from \( \mathcal{D}_1' \). But, since any peripheral subtree in the corresponding dual tree \( T_j \) is a red subtree as defined above, our claim will be proved if we show that the volume of any red subtree \( R_j' \subset T_j' \) is bounded in terms of \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \).

Using the disk pushing procedure above we iteratively define a sequence of red subtrees \( R_n \), starting with \( R_1 = R_j' \), as follows:

(i) If \( R_n \) is a bottom red subtree, then \( R_{n+1} \) is the adjacent top red subtree.
(ii) If \( R_n \) is a top red subtree, then \( R_{n+1} \) is the subsequent bottom red subtree.

Consider the sequence of volumes \( r_n \) of the red subtrees \( R_n \). This sequence is monotonically decreasing (not necessarily strictly) for increasing \( n \). This follows directly from the definition of
Fig. 8.

$R_{n+1}$ from $R_n$ by the disk pushing procedure. In particular if $r_n = r_{n+1} = r_{n+2}$, then the roots of the corresponding trees $R_n$, $R_{n+2}$ are vertices in the corresponding dual trees with the following property:

The corresponding disk pieces (subdisks from the collection $D'_1$) belong to neither

(a) a thick peripheral region in both the top and bottom adjacent solid pairs of pants (from the system $B'$), nor

(b) to the $(c + d)$-neighborhood of the central region of the top adjacent solid pair of pants.

As a consequence for any stationary subsequence $r_n, r_{n+1}, r_{n+2}, \ldots, r_{n+2k}$, all root vertices of the corresponding trees $R_n, R_{n+1}, R_{n+2}, \ldots, R_{n+2k}$ belong to distinct disk pieces $\Delta_n, \Delta_{n+1}, \Delta_{n+2}, \ldots, \Delta_{n+2k}$ which lie in the stack of parallel disk pieces defined by a fixed disk type. As all such stacks are finite (though not uniformly bounded by $D_1, D_2$), it follows that any such stationary subsequence must be finite.

On the other hand, any time the value $r_{n+1}$ is strictly smaller than $r_n$, then the disk pushing procedure for deriving $R_{n+1}$ from $R_n$ guarantees that $R_{n+1}$ coincides with a complementary component in some of the $T'_j$ of either one of the bottom thick peripheral subtrees, or of the $(c + d)$-neighborhood of one of the central subtrees. The maximal number of such complementary components is bounded above, by Lemma 5.2, by some $k = k(c + d)$, which only depends on $D_1$ and $D_2$. Hence the number of values of the decreasing sequence of areas $r_n$ is uniformly bounded.

It remains to observe that the quotient between two distinct values $r_n$ and $r_{n+1}$ is bounded above in terms of $D_1$ and $D_2$ only: In fact, since in the definition of the adjacent top red subtree, or of the subsequent bottom red subtree, we always chose a complementary subtree of maximal volume, the inequality $r_n - k/(r_{n+1}) \leq k$ is valid for the value $k$ specified above by Lemma 5.2. Recall here from
Lemma 5.2 that $k$ also bounds the volume of any thick peripheral or of the $(c + d)$-neighborhood of any central subtree in any of the $T'_j$.

This shows that the volume of any red subtree $R'_j$ is uniformly bounded above. □

From Proposition 5.3 we immediately obtain a proof of our main result Theorem 2.6 as stated in the Introduction. Notice that our proof is actually constructive, in that it describes a finite procedure which computes all complete decomposing systems which satisfy the double rectangle condition.

**Proof.** We pick an arbitrary pair of complete decomposing systems of disks $D_1 \subset H_1$ and $D_2 \subset H_2$. By Proposition 3.6 there are finitely many disk types with respect to these decomposing systems, which we can easily compute from the intersection pattern of $D_1$ and $D_2$ (see Section 3). By Lemma 4.3 there are finitely many possibilities for the thick regions, with an upper bound $N$ that only depends on the already computed finite set of disk types.

We compute the upper bound $d$ for their diameter, the maximal length $c$ for any curve from $D_2$, and the bound $k = k(c + d)$ as specified in the last proof. Then the formula $r_n - k/(r_{n+1}) \leq k$ from the last proof gives us the possibility to compute the largest possible area of any disk of the system $D'_1$, as in the decreasing sequence $r_1; r_2; \ldots$ the number of distinct values is bounded above by $kN$.

By symmetry we obtain a similar bound for the area of the disks from $D'_2$, so that there is only a finite number of candidates for these systems, which can be directly computed from the arbitrary chosen systems $D_1$ and $D_2$. □

**Remark 5.4.** As mentioned in the Introduction it can be shown that the Casson–Gordon rectangle condition is generic, in a precise meaning that uses Thurston’s measure on the boundary of Teichmüller space $\partial T_g$. (Roughly speaking, every system $D_2$ which does not satisfy the rectangle condition with respect to a fixed system $D_1$ has boundary curves that determine, when interpreted as measured lamination on the Heegaard surface $\Sigma_g$, a point in a closed subset of measure 0 of a finite part $\mathcal{H} D_g$ of $\partial T_g$. The part $\mathcal{H} D_g$ is determined by $D_1$, has measure $> 0$, and consists only of points given by decomposing systems $D_2$ of $H_2$.) The analogous statement for the double rectangle condition, introduced in this paper, is not so clear. This is because one can define and impose an anti double rectangle condition as follows: The adjacent disk pairs from one side do not meet all four adjacent disk triples in a double pair of pants from the other side, but only three of them, and in place of the fourth one there is a repetition of one of the earlier triples, namely the one which is non-adjacent. It is clear that the two conditions cannot be satisfied simultaneously. This anti double rectangle condition seems to be just as (non-)generic as the double rectangle condition.

A possible way to circumvent this difficulty is to consider the genericity of the set of systems $D_1, D_2$ which (a) satisfy the Casson–Gordon rectangle condition, and (b) have the property that $D_1$ can be modified into a “better” system $D'_1$ so that $D'_1, D_2$ satisfy the double rectangle condition.

An alternative resolution of the difficulty, which has implications into other directions as well, is outlined as follows:

The role of the double rectangle condition is only to give an upper bound $c \geq 0$ on the maximal distance $c(D'_1, D'_2)$ between the two central regions of any two adjacent pairs of pants (compare Proposition 4.6). If we replace the double rectangle condition by directly imposing such an upper bound on $c(D'_1, D'_2)$ (defined in proper terms, so that the hypothesis becomes independent of the reference systems $D_1, D_2$ which are used to measure the quantity $c(D'_1, D'_2)$), then the finiteness
Conclusion in our main Theorem 2.6 remains correct, and the proof stays virtually the same. In this way we can define (despite Example 2.7) for every Heegaard splitting which satisfies the Casson–Gordon rectangle condition for some disk systems $D_1, D_2$ finitely many “preferred” such systems, namely those which have $c(D_1, D_2)$ smaller than a given (sufficiently large) bound $c \geq 0$.

Acknowledgements

We would like to thank Hyam Rubinstein for pointing out Corollary 1.1 to us, and Saul Schleimer for many inspiring discussions. We also thank the Université d’Aix-Marseille III and the Technion, where most of this work was done. Finally, special thanks go to Café Parisien in Marseille for its hospitality.

References