Adequate field extensions and Frattini subgroups

Let \( L/F \) be a finite field extension. We always assume that \( F \) is a global field. The field \( L \) is \( F \)-adequate, if there exists a division algebra with center \( F \) and a maximal subfield isomorphic to \( L \). This definition is due to M. Schacher (1968), who proved, among many other results, that if \( F \leq K \leq L \), then \( K \) is also \( F \)-adequate, and that if \( L/F \) is separable, then \( L \) is \( F \)-adequate iff the Galois closure of \( L \) is \( F \)-adequate. D. B. Leep-T. L. Smith-R. Solomon (2002) proved a partial converse: if \( L/F \) is a Galois extension with Galois group \( G \), \( K \) is the fixed field of the Frattini subgroup \( \Phi(G) \) of \( G \), and if \( G \) has a certain property, which the authors call Frattini closed, and \( K \) is \( F \)-adequate, then \( L \) is \( F \)-adequate.

Following Leep-Smith-Solomon, we first define, for each finite group \( G \), a certain characteristic subgroup, which we call the local Frattini subgroup of \( G \), and denote by \( \Psi(G) \). This subgroup is contained in the Frattini subgroup, and using it we can remove the assumption of frattini closure, proving

**Theorem 1.** Let \( F \) be a global field, let \( L/F \) be a Galois extension with Galois group \( G \), and let \( K \) be the fixed field of \( \Psi(G) \). Then \( L \) is \( F \)-adequate if and only if \( K \) is \( F \)-adequate.

We provide several characterization of the local Frattini subgroup, and then return to the Frattini closed groups. These turn out to be those groups for which \( \Psi(G) = \Phi(G) \), and there are group theoretical motivations to study them. That was done by J. S. Rose (1980), who gave several classes of such groups, including the ones that were given later independently by Leep-Smith-Solomon. Here we generalize these results, finding several new classes. Combined with the aforementioned results, and results of D. Chillag-J. Sonn (1981) we obtain, e.g.

**Corollary 2.** Let \( L/F \) be a Galois extension with group \( G \), where either \( F = \mathbb{Q} \), the field of rationals, or \( F \) is a global field of characteristic \( p > 0 \), and \( |L:F| \) is prime to \( p \), and let \( K \) be the fixed field of \( \Phi(G) \). Then \( L \) is \( F \)-adequate if and only if the Sylow subgroups of \( G \) are metacyclic, and \( K \) is \( F \)-adequate.

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