EXponent Reduction for Projective Schur Algebras

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Abstract. In this paper it is proved that the “exponent reduction property” holds for all projective Schur algebras. This was proved in an earlier paper of the authors for a special class, the “radical abelian algebras”. The precise statement is as follows: let $A$ be a projective Schur algebra over a field $k$ and let $k(\mu)$ denote the maximal cyclotomic extension of $k$. If $m$ is the exponent of $A \otimes_k k(\mu)$, then $k$ contains a primitive $m$th root of unity. One corollary of this result is a negative answer to the question of whether or not the projective Schur group $PS(k)$ is always equal to $Br(L/k)$, where $L$ is the composite of the maximal cyclotomic extension of $k$ and the maximal Kummer extension of $k$. A second consequence is a proof of the “Brauer-Witt analogue” in characteristic $p$: if char$(k)=p \neq 0$, then every projective Schur algebra over $k$ is Brauer equivalent to a radical abelian algebra.

1. Introduction

The projective Schur group of a field $k$, denoted by $PS(k)$, is the subgroup of the Brauer group $Br(k)$ generated by (in fact, consisting of) all classes that may be represented by a projective Schur algebra $A$. By definition, a finite dimensional $k$-central simple algebra $A$ is projective Schur if $A^*$, the group of units of $A$, contains a subgroup $\Gamma$ which spans $A$ as a $k$–vector space and is finite modulo the center, that is, $k^*\Gamma/k^*$ is a finite group. The notions of projective Schur algebras and projective Schur group are the projective analogues of Schur algebras and the Schur group of $k$ (denoted by $S(k)$). These notions (of projective Schur algebra and of projective Schur group) were introduced in 1978 by Lorenz and Opolka [6]. One of the main points for making this generalization is that $PS(k)$ contains (by abuse of language) all symbol algebras. More precisely, any symbol algebra is a projective Schur algebra in an obvious way (indeed, let $A = (a, b)_n$ be the symbol algebra generated by $x$ and $y$ subject to the relations $x^n = a \in k^*, y^n = b \in k^*, yx = \zeta_n xy$, where $\zeta_n \in k^*$ is a primitive $n$-th root of unity. It is easy to see that $\Gamma = \langle x, y \rangle$ spans $A$ as a $k$–vector space and $k^*\Gamma/k^* \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. In particular, $k^*\Gamma/k^*$ is a finite group). Invoking the Merkurev-Suslin Theorem, one deduces that if $k$ contains all roots of unity (resp. contains the $n$th roots of unity), then $PS(k) = Br(k)$ (resp. $PS(k) \supseteq Br(k)_n$) where the subscript $n$ denotes $n$-torsion. The subgroup $PS(k)$ may be large even if roots

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of unity are not present in \( k \). Indeed, if \( k \) is a number field, then \( \text{PS}(k) = \text{Br}(k) \) as shown in [6]. Here one uses the fact that for \( k \) a number field, any element in \( \text{Br}(k) \) is split by a cyclic extension which is contained in a cyclotomic extension of \( k \). In general, the projective Schur group \( \text{PS}(k) \) is properly contained in \( \text{Br}(k) \), e.g. when \( k \) is a rational function field \( k_0(x) \) over any number field \( k_0 \). For power series fields \( k = k_0((x)) \) (over a number field \( k_0 \)) the situation depends on the field \( k_0 \). For instance if \( k_0 \) is a number field, any element in \( \text{Br}(k_0) \) is split by a cyclic extension which is contained in a cyclotomic extension of \( k_0 \).

In general, the projective Schur group \( \text{PS}(k) \) is properly contained in \( \text{Br}(k) \), e.g. when \( k \) is a rational function field \( k_0(x) \) over any number field \( k_0 \). For power series fields \( k = k_0((x)) \) (over a number field \( k_0 \)) the situation depends on the field \( k_0 \). For instance if \( k_0 \) is a number field, Kummer extensions do not split any element whose order is prime to the number of roots of unity in \( k \) (restriction-corestriction argument). As mentioned earlier, when \( k \) is a number field, every element in \( \text{PS}(k) = \text{Br}(k) \) is split by a cyclotomic extension of \( k \). In [2] the following is proved:

**Theorem 1.1.** Every element in \( \text{PS}(k) \) has a splitting field which is the composite of a cyclotomic extension of \( k \) and a Kummer extension of \( k \).

This was a key result used to show that in general \( \text{PS}(k) \) is properly contained in \( \text{Br}(k) \), since one shows that (in general) \( \text{Br}(L/k) \) is properly contained in \( \text{Br}(k) \), where \( L \) denotes the composite of all cyclotomic and Kummer extensions of \( k \).

It is now natural to ask [4, p. 528]: is \( \text{PS}(k) = \text{Br}(L/k) \) in general? In this paper we show that this is false:

**Theorem 1.2.** There are fields \( k \) for which \( \text{PS}(k) \) is properly contained in \( \text{Br}(L/k) \) where \( L \) is the composite of the maximal cyclotomic extension of \( k \) and the maximal Kummer extension of \( k \).

This will follow (as shown below) from the following theorem which is the main result of this paper.

**Theorem 1.3.** Let \( k(\mu) \) be the maximal cyclotomic extension of \( k \) and let \( \text{res} : \text{PS}(k) \to \text{PS}(k(\mu)) \) be the restriction map. Then for all \( \alpha \in \text{PS}(k) \), the exponent \( \exp(\text{res}(\alpha)) \) of \( \text{res}(\alpha) \) “divides the number of roots of unity in \( k \)”. More precisely, if \( m = \exp(\text{res}(\alpha)) \), then \( k \) contains a primitive \( m \)th root of unity.

In terms of algebras the theorem says that if \( A \) is a projective Schur algebra over \( k \), then \( \exp(A \otimes_k k(\mu)) \) divides the number of roots of unity in \( k \). We say that projective Schur algebras satisfy the **exponent reduction property**.

To show how Theorem 1.2 follows from Theorem 1.3, take \( k = \mathbb{Q}(x,t) \), the rational function field in two indeterminates over the rationals. Let \( K/\mathbb{Q}(t) \) be a regular extension (over \( \mathbb{Q} \)) which is cyclic of degree four (see, for example, [7, p.224] and let \( D = (K(x)/\mathbb{Q}(x,t), \sigma, x) \) be the corresponding cyclic algebra. It is clear that the order of \( D \) is not reduced by any cyclotomic extension, that is, \( \exp(D \otimes_k k(\mu)) = 4 \), whereas
$k$ does not contain the fourth roots of unity. By Theorem 1.3, $[D] \notin PS(k)$. On the other hand, $D$ is split by a Kummer $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ extension, so $[D] \in Br(L/k)$.

In [4] it is shown that certain projective Schur algebras, so-called radical abelian algebras, satisfy the exponent reduction property. A projective Schur algebra is radical if it is isomorphic to a crossed product algebra $A = (K/k, G, \beta)$ where $K/k$ is a $(G$-Galois) radical extension ($K = k(x_1^{1/n_1}, x_2^{1/n_2}, \ldots, x_r^{1/n_r})$, $x_i \in k^*$) and the class $\beta$ is represented by a 2-cocycle $f$ whose values in $K^*$ are finite modulo $k^*$. If $G$ is abelian we call $A$ radical abelian. A radical algebra should be viewed as the “natural” way to construct projective Schur algebras. This is analogous to the construction of cyclotomic algebras (see [10]).

It is conjectured in [1] that every element in $PS(k)$ may be represented by a radical algebra, even (as conjectured in [4]) by a radical abelian algebra. In this paper we show the stronger conjecture holds if the field $k$ has positive characteristic. In fact we show that every element in $PS(k)$ may be represented by the tensor product of algebras which are “natural” examples of radical abelian algebras.

Let $A = (K/k, Gal(K/k), \alpha)$ be a crossed product algebra over the field $k$. It is well known that if $K/k$ is cyclic, one may represent the class $\alpha \in H^2(Gal(K/k), K^*)$ by a 2-cocycle whose values are in $k^*$ (rather than $K^*$). If in addition the field $K$ can be embedded in a radical extension $L$ of $k$, then we may represent the algebra $A$ by the crossed product algebra $(L/k, Gal(L/k), inf(\alpha))$ where $inf(\alpha)$ denotes the image of $\alpha$ under the inflation map. The point of this is that also $inf(\alpha)$ may be represented by a 2-cocycle with values in $k^*$ (in fact, the same set of values), so we see that $(L/k, Gal(L/k), inf(\alpha))$ is a radical algebra. Note that if the extension $L/k$ is abelian, then $(L/k, Gal(L/k), inf(\alpha))$ is a radical abelian algebra. We see that if an element $[A]$ in $Br(k)$ is split by a cyclic extension of $k$ which is contained in a radical (resp. radical abelian) extension of $k$ then $A$ is equivalent to a radical (resp. radical abelian) algebra. Here we consider two types of such algebras.

Type I: Symbol algebras. These are split by a cyclic Kummer extension of $k$.

Type II: Algebras that are split by a cyclic extension of $k$ which is contained in a cyclotomic extension of $k$. Note that algebras of type I and type II are radical abelian.

**Theorem 1.4.** Let $k$ be a field of positive characteristic. Then every element in $PS(k)$ may be represented by an algebra which is the tensor product of algebras of types I and II.

Using the fact that a tensor product of radical (radical abelian) algebras is equivalent to a radical (radical abelian) algebra (see [3,Lemma 2.4], where it is proved for radical algebras, but the same proof applies for radical abelian algebras), we obtain:

**Corollary 1.5.** If $char(k) \neq 0$, every element in $PS(k)$ is represented by a radical abelian algebra.

2. Proofs of the results

We are given a projective Schur algebra $A$ over $k$. By definition, $A^*$ (the group of units of $A$) contains a subgroup $\Gamma$ which spans $A$ as a $k$-vector space and which is finite modulo $k^*$. We write $A = k(\Gamma)$. For any subgroup $H$ of $\Gamma$ we may consider the
subalgebra spanned by $H$ and denote it by $k(H)$. Recall that $\Gamma$ is center by finite, so by a theorem due to Schur, $\Gamma'$, the commutator subgroup of $\Gamma$, is finite. The subalgebras $k(H)$ may not be simple, but if we restrict ourselves to subgroups $H$, with $\Gamma' \subseteq H \subseteq \Gamma$ we have the following reduction [1]:

**Theorem 2.1.** Let $k(\Gamma)$ be a projective Schur algebra over $k$. Then it is Brauer equivalent to a projective Schur algebra $k(U)$, $U/k^*$ finite and such that for every subgroup $H$ of $U$ with $U' \subseteq H \subseteq U$, $k(H)$ is a simple algebra.

We say the algebra $k(U)$ is reduced. Thus, for the proof of Theorem 1.3, we shall assume that $k(\Gamma)$ is a reduced algebra. Consider the subalgebra $k(\Gamma')$ and let $K \supseteq k$ be its center. Since the group $\Gamma'$ is finite, it follows that $k(\Gamma')$ is a Schur algebra over $K$. Furthermore it is a simple component of the group algebra $k\Gamma'$ and therefore $K \subseteq k(\zeta)$ where $k(\zeta)$ is a cyclotomic extension of $k$ ($\zeta$ a root of unity) [2]. In particular, the family $\Omega := \{ k(H) : \Gamma' \subseteq H \subseteq \Gamma, \, k(H) \text{ is a Schur algebra over its center } L_H, \text{ and } L_H \text{ is contained in a cyclotomic extension of } k \}$ is not empty.

Let $k(H)$ be a maximal element in $\Omega$. Observe that such an element exists because $k(\Gamma)$ is finite dimensional over $k$. Let $k(\zeta)$ be a cyclotomic extension of $k$ which contains $L_H = Z(k(H))$. By Brauer’s Splitting Theorem [5, pp. 385,418], $k(H)$ is split by a cyclotomic extension of $L_H$, so we may assume that $k(\zeta)$ splits $k(H)$; that is, $k(\zeta) \otimes_{L_H} k(H) \cong M_r(k(\zeta))$ for some integer $r$. We are to show that $\exp(k(\Gamma) \otimes_k k(\zeta))$ divides the number of roots of unity in $k$.

To start with, note that the group $\Gamma$ acts on $k(H)$ by conjugation, hence on its center $L_H$.

**Lemma 2.2.** The fixed field $L_H^\Gamma$ of $L_H$ under $\Gamma$ is $k$.

**Proof.** This is clear, since $L_H^\Gamma \subseteq Z(k(\Gamma)) = k$.

Let $\hat{H} = C_\Gamma(L_H)$, the centralizer of $L_H$ in $\Gamma$. By the definition of $\hat{H}$ it follows that $\Gamma' \subseteq H \subseteq \hat{H}$ and $L_H \subseteq Z(k(\hat{H})) = L_{\hat{H}}$. We claim: $L_H = L_{\hat{H}}$. Indeed, the group $\Gamma/\hat{H}$ acts faithfully on $L_H$, and since $L_H^{\Gamma/\hat{H}} = k$ we see that $|\Gamma/\hat{H}| = [L_H : k]$. But $\Gamma/\hat{H}$ acts faithfully on $L_{\hat{H}}$ as well, so by a similar argument, we have $L_{\hat{H}}^{\Gamma/\hat{H}} = k$. This implies $[L_{\hat{H}} : k] = |\Gamma/\hat{H}| = [L_H : k]$, proving the claim.

In order to simplify notation let $L = L_H = L_{\hat{H}}$.

**Proposition 2.3.** $k(\Gamma) \otimes_k L$ is Brauer equivalent to $k(\hat{H})$.

**Proof.** Consider $k(\Gamma)$ as a module over $k(\hat{H})$. Let us show it is free of rank $|\Gamma/\hat{H}|$. Let $\{ g_1, ..., g_n \}$ be representatives of the left cosets of $\hat{H}$ in $\Gamma$. Clearly, they span $k(\Gamma)$ as a left $k(\hat{H})$ module. To show they are independent we let $w = g_1 z_1 + g_2 z_2 + ... + g_n z_n = 0$ be a nontrivial relation of shortest length. Clearly $s > 1$ since the $g_i$ are invertible. Take $x \in L$ with $g_1 x g_1^{-1} \neq g_2 x g_2^{-1}$ (such an element exists since $\Gamma/\hat{H}$ acts faithfully on $L$), and consider

$xw - g_1(x)w = xw - g_1 x g_1^{-1} w$

$= [z_1 g_1(x) g_1 + z_2 g_2(x) g_2 + ... + z_n g_n(x) g_n]$

$-[z_1 g_1(x) g_1 + z_2 g_2(x) g_2 + ... + z_n g_n(x) g_n]$
\[
L \otimes_k k(\Gamma) \cong \text{End}_{k(\tilde{H})}(k(\Gamma)) = \text{Hom}_{k(\tilde{H})}(k(\Gamma), k(\Gamma)) \cong M_\mu(k(\tilde{H}))
\]

(k(\Gamma) with the left \(k(\tilde{H})\) structure). Indeed, the algebra \(k(\Gamma)\) acts on \(k(\Gamma)\) from the right and \(L\) acts on \(k(\Gamma)\) from the left. Clearly the actions commute and both commute with the \(k(\tilde{H})\) left action. (Of course, here we are using that \(L = Z(k(\tilde{H}))\).

This gives a nontrivial homomorphism \(\eta\) from \(L \otimes_k k(\Gamma)\) into \(\text{End}_{k(\tilde{H})}(k(\Gamma))\). The map \(\eta\) is 1-1 (since \(L \otimes_k k(\Gamma)\) is simple) and therefore surjective (the dimensions over \(k\) are equal). This completes the proof of the proposition.

Observe that the algebras \(k(H)\) and \(k(\tilde{H})\) have the same center \(L\), and so (by the double centralizer theorem) we have \(k(\tilde{H}) \cong k(H) \otimes_L A\) (\(A\) central simple over \(L\)).

Theorem 1.3 will follow if we show that \(\text{exp}(A)\) (in \(\text{Br}(L)\)) divides the number of roots of unity in \(k\). We start with the following key lemma.

**Lemma 2.4.** Let \(z \in \tilde{H}\) and let \(n\) be its order modulo \(k(H)^*\). Then \(n\) divides the number of roots of unity in \(k\).

**Proof.** We assume the lemma is false and get a contradiction. Taking a power of \(z\) we may, for some prime \(p\), assume the order of \(z\) modulo \(k(H)^*\) is \(p^{r+1}\) and \(k\) contains the \(p^r\)th roots of unity but not the \(p^{r+1}\)th root of unity, \(r \geq 0\).

Consider the action of \(z\) on \(k(H)\) induced by conjugation. Since this action centralizes \(L\), by the Skolem-Noether Theorem there is an element \(a \in k(H)^*\) such that \(w = za^{-1}\) centralizes \(k(H)\). In particular, it centralizes \(a\), so \(z\) and \(a\) commute in \(k(\tilde{H})\). It follows that the element \(w\) is in the center of the subalgebra \(B = k(< H, z >) = k(w)(H) = L(w)(H)\) (in \(k(\tilde{H})\)). Clearly \(L(w)\) is contained in the center of \(B\). In fact \(L(w)\) is precisely the center of \(B\) since there is an obvious map of algebras from \(L(w) \otimes_L k(H)\) (simple with center \(L(w)\)) onto \(B\).

**Claim:** The element \(w\) satisfies an equation of the form \(X^{p^{r+1}} - e = 0\) where \(e \in L^*\). Furthermore \(p^{r+1}\) is the smallest possible. To see this, recall that \(w\) centralizes \(a\) so that \(a, z, w\) commute. We therefore have \(w^{p^{r+1}} = (za^{-1})^{p^{r+1}} = z^{p^{r+1}}a^{p^{r+1}} \in k(H)^*\).

But \(w^{p^{r+1}}\) centralizes \(k(H)\), hence \(w^{p^{r+1}} \in L = Z(k(H))\). Finally, it is easy to see that the order of \(w\) modulo \(L^*\) equals the order of \(z\) modulo \(k(H)^*\) (\(= p^{r+1}\)). This proves the claim. Observe that the algebra \(B\) is a Schur algebra over its center \(L(w)\) since \(k(H)\) is, and moreover it is of the form \(k(\tilde{H})\), where \(\tilde{H} = < H, z >\). Set \(B_0 = B\) and for \(i = 1, ..., r\) let \(B_i = k(< H, z^{p^i} >) = k(w^{p^i})(H) = L(w^{p^i})(H)\). As for \(B_i\), one shows easily that \(L(w^{p^i}) = Z(B_i)\). Also, note that \(B_i\) strictly contains \(k(H)\), so we will reach a contradiction to the maximality of \(k(H)\) if we show that \(L(w^{p^i})\) is contained in a cyclotomic extension of \(k\). Let us show that this is indeed the case.

First recall that the field \(L(w)\) is the center of the algebra \(B = k(< H, z >)\). By [2, Corollary 2.3] \(L(w)\) is contained in the composite of the maximal cyclotomic extension \(k(\mu)\) of \(k\) and a (finite) Kummer extension \(k(U)\) of \(k\). Since \(k\) contains no primitive \(p^{r+1}\) root of unity, the \(p\)-primary component of the Galois group \(\text{Gal}(k(U)/k)\) has exponent \(\leq p^r\).
It follows that the $p$-primary component of $\text{Gal}(k(\mu, U)/k(\mu))$ has exponent at most $p^r$. On the other hand the field $k(\mu, w)$ is a Kummer extension of $k(\mu)$ and is contained in $k(\mu, U)$. Hence if $w^{p^r} \notin k(\mu)$, $\text{Gal}(k(\mu, w)/k(\mu))$ is cyclic of order $p^{r+1}$. This is impossible of course. It follows that $w^{p^r} \in k(\mu)$ and hence $L(w^{p^r}) \subseteq k(\mu)$. This completes the proof of the lemma.

Our next step will be to decompose the algebra $k(\hat{H})$ as a tensor product $k(H) \otimes_L A$, with exp($A$) dividing the number of roots in $k$.

To do this recall that $\text{PA}_b(F)$ is the subgroup of $\text{PS}(F)$ consisting of classes which may be represented by a projective Schur algebra $F(\Gamma)$ and $\Gamma/F^* = G$ abelian. In this case we say that $F(\Gamma)$ is of abelian type. Obviously, the natural examples are the symbol algebras. We recall from [3] the following

**Proposition 2.5.** If $F(\Gamma)$ is a projective Schur algebra of abelian type, then it is Brauer equivalent to the tensor product of symbol algebras. Furthermore, if exp($G$) $\equiv n$, then the exponent of $F(\Gamma)$ in $\text{Br}(F)$ divides $n$.

**Lemma 2.6.** Let $k(H)$ and $k(\hat{H})$ be as above. Then $k(\hat{H}) \cong k(H) \otimes_L L(\Lambda)$ where $L(\Lambda)$ is a projective Schur algebra of abelian type and $\text{exp}(\Lambda/L^*)$ divides the number of roots of unity in $k$. Furthermore, $L(\Lambda)$ is isomorphic to a product of symbol algebras.

Let us postpone the proof of the lemma and complete the proof of Theorem 1.3.

By Lemma 2.6, exp($L(\Lambda)$) (as an element in $\text{Br}(L)$) divides exp($\Lambda/L^*$), hence exp($L(\Lambda)$) divides the number of roots of unity in $k$. But $k(\Gamma) \otimes_k k(\mu) \sim k(\hat{H}) \otimes_L k(\mu)$ (as an element in $\text{Br}(L)$) and Proposition 2.3; $\sim$ denotes Brauer equivalence)

$\cong k(H) \otimes_L L(\Lambda) \otimes_L k(\mu)$
$\sim L(\Lambda) \otimes_L k(\mu)$ (k(H)

is Schur over $L$, hence is split by $k(\mu)$).

This implies that exp($k(\Gamma) \otimes_k k(\mu)$) = exp($L(\Lambda) \otimes_L k(\mu)$) which divides exp($L(\Lambda)$).

It follows that $k(\Gamma) \otimes_k k(\mu)$ divides the number of roots of unity in $k$ and the theorem is proved.

**Proof of Lemma 2.6.** (See [3, Lemma 2.3].) Recall that $\hat{H}$ normalizes $k(H)^*$ hence we may consider the subgroup $\hat{H}k(H)^*$ of the units of $k(\hat{H})$. Let $\Lambda$ be the centralizer of $k(H)$ in $\hat{H}k(H)^*$. Since $\Lambda \cap k(H)^* = L^*$, it follows that $\Lambda/L^* \subset \hat{H}k(H)^*/k(H)^*$ which is a quotient of $\hat{H}/L^* \cap \hat{H}$. This implies $\Lambda/L^*$ is finite. Furthermore, the commutator subgroup $\hat{H}'$ of $\hat{H}$ is contained in $k(H)^*$ (since $\hat{H}' \subset 1\cap H$), so $\Lambda/L^*$ is abelian.

Finally, we recall from Lemma 2.4 that the order of any element in $\hat{H}$ modulo $k(\mu)$, hence exp($\Lambda/L^*$), divides the number of roots of unity in $k$. By Proposition 2.5 it follows that the order of $L(\Lambda)$ in $\text{Br}(L)$ divides exp($\Lambda/L^*$), hence it also divides the number of roots of unity in $k$. In the proof of Proposition 2.5, it is shown that $L(\Lambda)$ is isomorphic to a product of symbol algebras. To complete the proof of Lemma 2.6, we need to show that $k(\hat{H}) = k(H) \otimes_L L(\Lambda)$. By the double centralizer theorem, we have $k(\hat{H}) \supseteq k(H) \otimes_L L(\Lambda) = k(H)L(\Lambda)$. In order to prove the reverse inclusion, take an element $z \in \hat{H}$. $z$ normalizes $k(H)$ and centralizes its center $L$. Therefore, there is an element $e(z) \in k(H)^*$ such that $z^{-1}e(z)$ centralizes $k(H)$. This puts $z^{-1}e(z)$ in $\Lambda$, and the lemma is proved. \(\square\)
Corollary 2.7. If the field $k$ contains all roots of unity then every element in $PS(k)$ is equivalent to a product of symbol algebras.

(This is of course a direct consequence of the Merkurev-Suslin Theorem.)

Proof. Let $k(\Gamma)$ be a projective Schur algebra. We may assume it is reduced. If $k$ contains all roots of unity then the field $L$ above must be $k$, so the algebra $k(H)$ coincides with the entire algebra $k(\Gamma)$. From the proof above, it follows that $k(\Gamma)$ is the tensor product of a Schur algebra $k(H)$ over $k$ and symbol algebras. But $k(H)$ must be split (Brauer’s theorem), hence $k(\Gamma) \sim$ product of symbol algebras.

We turn now to the proof of Theorem 1.4.

Let $k$ be a field of characteristic $p > 0$ and let $A = k(\Gamma)$ be a projective Schur algebra over $k$. We are to show that $A$ is represented by the tensor product of symbol algebras and an algebra which is split by a cyclotomic extension of $k$ (note that in positive characteristic every finite cyclotomic extension is cyclic). Since the tensor product of such algebras (tensor product of symbol algebras and an algebra which is split by a cyclotomic extension of $k$) is again such an algebra, we may assume $\text{exp}(A) = q^t$, where $q$ is a prime number and $t > 0$. If $k$ contains no non trivial $q$-th roots of unity (in particular if $q = p$), $A$ is split by a cyclotomic extension by Theorem 1.1 ($k$ has no Kummer $q$-extension). On the other hand if $k$ contains all $q$-power roots of unity, then $A$ is similar to a tensor product of symbol algebras by [8] or Corollary 2.7. We therefore assume $k$ contains a primitive $q^t$th root of unity ($r > 0$) but not a primitive $q^{t+1}$th root of unity. By exponent reduction (Theorem 1.3) there is a finite cyclotomic extension $k(\zeta)$ of $k$ such that $\text{exp}(A \otimes_k k(\zeta)) = q^s$, $0 \leq s \leq t$. It follows that $A^{\otimes q^s}$ is split by $k(\zeta)$ and is therefore equivalent to a cyclic algebra $(k(\zeta)/k, \sigma, a)$ where (by abuse of notation) the $2$-cocycle is determined by the relation $\sigma^n = a$, $n = [k(\zeta) : k]$. The key observation here is that the algebra $A^{\otimes q^s}$ has a $q^s$th root in $\text{Br}(k)$ which is split by a cyclotomic extension of $k$. In other words, there is a cyclic algebra $B = (k(\zeta_1)/k, \tau, b)$ with $B^{\otimes q^s} \sim A^{\otimes q^s}$. Of course $B$ is a radical abelian algebra. Let us complete the proof of the theorem assuming such a $B$ exists.

Consider the algebra $C = B^{-1} \otimes_k A$. Clearly $[C]^{q^s} = 1$ in $\text{Br}(k)$. Furthermore $k$ contains a primitive $q^s$th root of unity and so by the Merkurev-Suslin Theorem, $C$ may be represented by a product of symbol algebras. The theorem is now proved since $A \sim B \otimes_k C$. It remains to show the existence of $B$. Keeping in mind that $\text{char}(k) = p$, we see that the cyclic cyclotomic extension $k(\zeta)/k$ can be embedded into a cyclic cyclotomic extension $k(\zeta_1)/k$ of degree $q^n$. By [9, p. 262], the algebra $B = (k(\zeta_1)/k, \tau, a)$ (same $a$ as in $A$) has the desired property. $\square$

Remark. From the proof we see that if $\text{char}(k) \neq 0$, then a central simple $k$-algebra has the exponent reduction property if and only if it is a projective Schur algebra over $k$.

References


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