The maximum number of edges in geometric graphs with pairwise virtually avoiding edges

Eyal Ackerman* Noa Nitzan† Rom Pinchasi†

July 27, 2011

Abstract

Let G be a geometric graph on n vertices that are not necessarily in general position. Assume that no line passing through one edge of G meets the relative interior of another edge. We show that in this case the number of edges in G is at most $2n-3$.

1 Introduction

A *geometric graph* is a graph drawn in the plane such that its vertices are distinct points and its edges are straight-line segments connecting the vertices. Two edges in a geometric graph are called *avoiding* (sometimes also *parallel*) if they are opposite sides of a convex quadrilateral. It was shown by Katchalski and Last [5] and Valtr [10] (see also [8]) that the maximum number of edges in a geometric graph on n vertices with no pair of avoiding edges is $2n-2$. Many such extremal questions about geometric graphs avoiding certain geometric patterns have been studied over the years (see [4, §9.5 and §9.6] for some other examples).

In most (perhaps even all) of these works the notion of a geometric graph usually refers to a graph on a set of vertices drawn as points in *general position*, that is, no three of the vertices are on a line. In this paper we will particularly be interested in the case where the vertex set of the graph may contain more than two points on a line, and investigate the following question: What is the maximum number of edges in a geometric graph G on n vertices in which every pair of edges are *virtually avoiding*?

Definition 1.1 (virtually avoiding edges)**.** *Two edges in a geometric graph are called* virtually avoiding *if any line containing one of them does not cross the relative interior of the other edge. Two edges will be called* virtually crossing *if they are not virtually avoiding, that is, if there is a line that contains one of the edges and crosses the relative interior of the other edge.*

See Figure 1 for examples of these notions. Notice that when the points are in general position two edges are virtually avoiding if and only if they are avoiding or have a common vertex. However, for points that are not necessarily in general position, these notions are even further apart, as it may happen that a line containing one of two virtually avoiding edges meets a vertex of the other edge (see for example e_1 and e_6 in Figure 1).

Observe that virtually avoiding edges cannot be collinear and cannot cross each other. The latter property implies that a graph with pairwise virtually avoiding edges is planar and

*Department of Mathematics, Physics, and Computer Science, University of Haifa at Oranim, Tivon 36006, Israel. `ackerman@sci.haifa.ac.il`.

†Mathematics Department, Technion—Israel Institute of Technology, Haifa 32000, Israel. `{???,room}@math.technion.ac.il`.

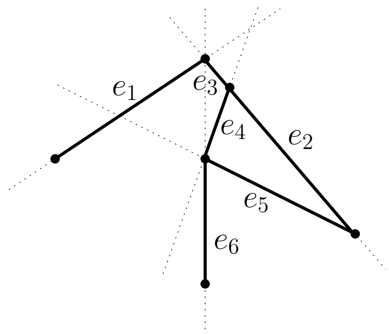


Figure 1: Both pairs of edges (e_1, e_4) and (e_2, e_6) are avoiding. Both (e_1, e_3) and (e_1, e_6) are virtually avoiding. (e_1, e_5), (e_2, e_3), and (e_1, e_7) are all virtually crossing.

hence has at most $3n - 6$ edges, for $n \geq 3$. Note also that in a triangulation of a set of n vertices in (strictly) convex position every pair of edges are virtually avoiding and there are $2n - 3$ edges. It is easy to show by induction that no other graph G with pairwise virtually avoiding edges has more edges if its vertices are in general position.

To see the inductive argument observe that we may assume that the degree of each vertex is at least 3, or else we may conclude by the induction hypothesis. Let x be a vertex of G that is also extreme on the convex hull of the set of vertices of G. Because the degree of x is at least 3, there must be an edge e adjacent to x that is not an edge of the convex hull of the set of vertices of G. The line ℓ through e divides the rest of the vertices of G into two non-empty sets of cardinalities n_1 and n_2, respectively. Notice also that no edge of G crosses ℓ, as this will contradict our assumption that every two edges in G are virtually avoiding. We can therefore conclude the theorem by the induction hypothesis on the two parts of G defined by ℓ where the vertices of e are common to both parts: The number of edges in G is therefore at most

$$(2(n_1 + 2) - 3) + (2(n_2 + 2) - 3) - 1 = 2n - 3,$$

where the -1 in the left hand side is due to the fact that e is counted in both parts of G.

Perhaps somewhat surprisingly the same question becomes significantly less trivial if we allow the set of vertices of G to be not in general position. The simple inductive argument will fail now if we wish to prove the same bound, because the line ℓ may contain more than just two vertices and these vertices will be taken in account when considering both parts of the graph. Other obvious strategies such as proving that the number of edges is at most $2n - 3 - k$, where k is the number of edges of the convex hull of the points that are *not* present in G, have also failed.

In what follows we aim to prove the same result in the more general setting of a geometric graph whose set of vertices may not necessarily be in general position.

Theorem 1. *Let G be a geometric graph on n vertices that are not necessarily in general position. If every two edges of G are virtually avoiding, then G has at most $2n - 3$ edges for $n \geq 2$.*

In other words, we prove that a geometric graph on n vertices with no pair of virtually crossing edges has at most $2n-3$ edges. It would be interesting to consider geometric graphs in which there are no k pairwise virtually crossing edges, for some fixed number k (put differently, every set of k edges contains a pair of virtually avoiding edges). In particular, such a graph does not contain k pairwise crossing edges and therefore has at most $O(n \log n)$ edges by a result of Valtr [9]. It is a well-known conjecture that any graph with no k pairwise crossing edges has at most $O(n)$ edges [6, Problem 3.3]. However, this conjecture was only verified for $k \leq 4$ [1, 2]. Therefore, it might be easier to prove:

Conjecture 2. *Let G be a geometric graph on n vertices (in general position). If G does not contain k pairwise virtually crossing edges, then G has at most $c_k n$ edges, where c_k is a constant that depends only on k.*

Indeed, using an idea from [7] one can show that a complete n-vertex geometric graph has $\lfloor \frac{n}{2} \rfloor$ pairwise virtually crossing edges. It is conjectured that such a graph also contains $\Omega(n)$ pairwise crossing edges, however, the best known bound is only $\Omega(\sqrt{n})$[3].

As an application of Theorem 1 we obtain the following:

Corollary 3. *Let $\mathcal{L}$ be a set of n lines in the plane, and suppose that for every line $L \in \mathcal{L}$ two consecutive intersection points on L are marked. Then at least $\frac{n+3}{2}$ intersection points are marked over all.*

Proof. Let V be the set of intersection points that are marked, and let E be the set of segments connecting two consecutive intersection points in V, that were marked on the same line. Observe that $G = (V, E)$ is a geometric graph in which every two edges are virtually avoiding. Thus, $n = |E| \leq 2|V| - 3$. □

2 Proof of Theorem 1

We prove the theorem by induction on n. The claim clearly holds for $n = 2$. Let G be a geometric graph on $n > 2$ vertices with pairwise virtually avoiding edges, and assume that the claim holds for any such graph with less than n vertices. We may assume that the minimum degree in G is at least 3 and that G is 2-connected, for otherwise, we conclude the theorem by induction.

Recall that G is planar, let m be the number of its edges, let F be the number of its faces, and let f_k be the number of its faces of size k, for $k \geq 3$. Then:

$$m = 3n - 6 - \sum_k (k-3) f_k. \tag{1}$$

Indeed, Equation (1) follows easily from Euler's Formula, since:

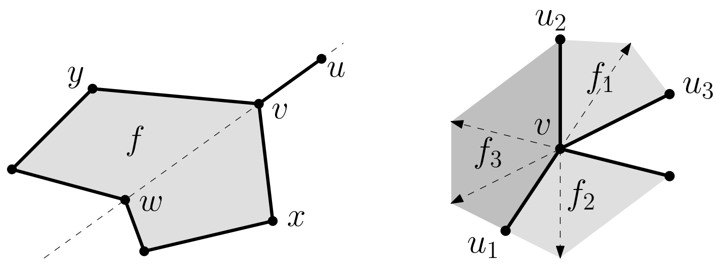
$$3m = 3n + 3F - 6 = 3n - 6 + \sum_k k \cdot f_k - \sum_k (k-3) f_k = 3n - 6 + 2m - \sum_k (k-3) f_k.$$

Therefore, it is enough to prove that $\sum_k (k-3) f_k \geq n - 3$. This is done using the *discharging method*: First, we assign a *charge* of $k - 3$ to every face of size k. Then, we redistribute the charges (*discharge*) such that every vertex has charge at least 1, the unbounded face has charge -3, and every bounded face has a non-negative charge. This implies that the initial charge $\sum_k (k-3) f_k$ is at least $n - 3$.

Before describing the (dis)charging scheme, we first make some observations (we postpone for later the proofs of some of which). Observe that since G is 2-connected, the boundary of each of its faces is a simple polygon. Recall that a vertex v of a face f is called a *reflex* vertex of f if the internal angle at v is greater than π. Otherwise v is called a *convex* vertex of f. (Note that an internal angle that equals π implies two non-avoiding edges.)

Definition 2.1 (needy vertex)**.** *Let v be a convex vertex of a bounded face f and let x and y be its two neighbors on the boundary of f. If v has a neighbor $u \neq x, y$ such that x and y are separated by the line through v and u, then we say that v is a* needy *vertex of f and that u is a* witness *for that (see Figure 2(a)).*

Proposition 2.2. *Let f be a bounded face of G with exactly three convex vertices. Then f has neither needy nor reflex vertices (in particular f must be a triangle in this case).*



(a) v is a needy vertex of f, u is a witness for that.

(b) A non-reflex vertex which is not on the unbounded face is a needy vertex of at least three faces.

Figure 2: Needy vertices.

Proposition 2.3. *Let f be a bounded face of G with exactly four convex vertices.*

1. *If f has a reflex vertex, then it has no needy vertices and no other reflex vertices (in particular f is a pentagon in this case).*
2. *If f has a needy vertex, then f is a convex quadrilateral and has at most one other needy vertex.*

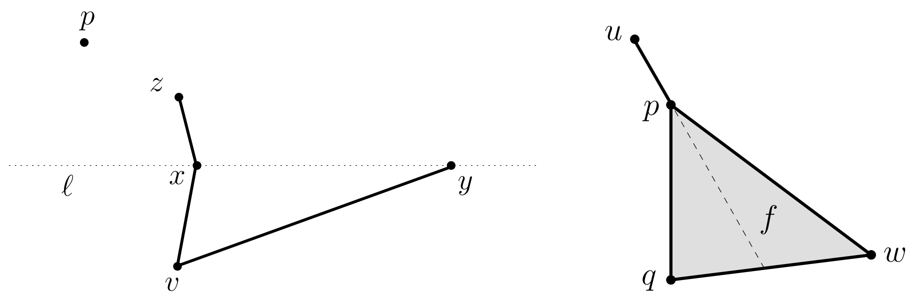
Charging and discharging. As mentioned above, we assign a *charge* of $k-3$ to every face of size k in G, for any $k \geq 3$. In the discharging phase, the unbounded face sends one unit of charge to each vertex on its boundary. Every bounded face sends one unit of charge to every reflex vertex on its boundary, and $\frac{1}{3}$ units of charges to each of its needy vertices.

Proposition 2.4. *After applying the discharging rules the following holds:*

1. *The charge at every vertex is at least* 1*;*
2. *The charge of the unbounded face of G is -3; and*
3. *The charge of every bounded face is nonnegative.*

Proof. Let v be a vertex in G. If v is on the unbounded face or v is a reflex vertex of some bounded face, then its charge is clearly at least 1. Suppose that v is not on the unbounded face and is not a reflex vertex of any bounded face. Then v lies in the convex hull of its neighbors, and therefore (by Carathéodory's theorem) in the convex hull of three of its neighbors which we denote by u_1, u_2, and u_3. For $i = 1, 2, 3$ let f_i denote the face to which the ray $\overrightarrow{u_i v}$ enters at v. Notice that f_1, f_2, and f_3 are distinct faces (because G is 2-connected) and v is a needy vertex with respect to each f_i (having u_i as a witness to this, see Figure 2(b)). Therefore, the final charge at v is at least one unit, as each f_i donates $\frac{1}{3}$ of a unit of charge to v. This proves (1).

Observe that (2) holds by definition, therefore it remains to show (3) that the final charge of every bounded face is non-negative. Let f be a bounded face of size k with r reflex vertices. Thus, f sends at most $r + (k-r)/3$ units of charge to its vertices. If $r \leq k-5$, then f sends at most $k - 5 + 5/3 < k - 3$ units of charges and thus remains with a positive charge. Since every face has at least three convex vertices, it remains to consider the cases $r = k-3$ and $r = k-4$. If f has exactly three convex vertices, then by Proposition 2.2 it is a triangle and it does not send any charge to its vertices and therefore ends up with its initial charge (zero). Finally, suppose that f has exactly four convex vertices and it sends some charge to its vertices. Then by Proposition 2.3 either f is a convex quadrilateral with at most two needy vertices, or f is a pentagon with exactly one reflex vertex and no needy vertices. In



(a) (b)

Figure 3: Illustrations for Propositions 2.5 and 2.2.

both cases f sends at most one unit of charge to its vertices and ends up with a non-negative charge. □

It remains to prove Propositions 2.2 and 2.3. To this end we first prove the following.

Proposition 2.5. *Let p and q be two adjacent vertices on a boundary of a simple polygon P. If P contains exactly one convex vertex v different from p and q, then either P is a triangle, or there are two edges of P different than pq that are virtually crossing.*

Proof. Let ℓ be the line passing through the two neighbors x and y of v on the boundary of P. Assume without loss of generality that p, q, v is the clockwise order of p, q and v along the boundary of P. Assume ℓ is horizontal and v lies below ℓ. If P is not a triangle, then one of p and q must lie above ℓ, or else P has another convex vertex (the highest vertex of P). Without loss of generality assume that p lies above ℓ and that x is the neighbor of v on the clockwise path from v to p along the boundary of P. Let z denote the other neighbor of x on the boundary of P. z must also lie above ℓ or else P has another convex vertex (the lowest vertex of P on the clockwise path from x to p along the boundary of P). Now it is easy to see that the edges xz and vy are virtually crossing (see Figure 3(a)). □

Proof of Proposition 2.2: Let p, q, and w denote the three convex vertices of f and assume that this their clockwise order on the boundary of f. Replace $f_{p,q}$ with an edge and notice that we are left with a simple polygon. Indeed, otherwise there are at least 4 convex vertices, namely p, q, and the two vertices of f furthest from the line pq in each of the two half-planes bounded by it. Keeping in mind that no two edges of G are virtually crossing, we conclude from Proposition 2.5 that both p and q are neighbors of w. Similarly one shows that p is a neighbor of q. Hence, f is a triangle. Suppose that p is a needy vertex of f and that u is a witness for that (see Figure 3(b)). Then pu and qw are virtually crossing, a contradiction. □

Proof of Proposition 2.3: Let a, b, c, and d denote the fours convex vertices of f and assume they appear in this cyclic clockwise order along the boundary of f. If f does not have any reflex vertex, then f is a convex quadrilateral. In this case only one of every pair of opposite vertices of f may be a needy vertex. Indeed, let x and y be two opposite vertices of f. If both are needy, let u be the witness for x and v be the witness for y. Then ux and vy must be collinear edges, which is a contradiction.

For two vertices u, v on the boundary of f we denote by $f_{u,v}$ the clockwise path (polygonal chain) from u to v on the boundary of f. Suppose that f has a reflex vertex p, and assume without loss of generality that that p lies on $f_{a,b}$. We may assume that p is a neighbor of b. Let $\vec{r}$ be the ray through p with apex at b, and let x be the first point on $\vec{r}$ in which it crosses $f_{b,a}$.

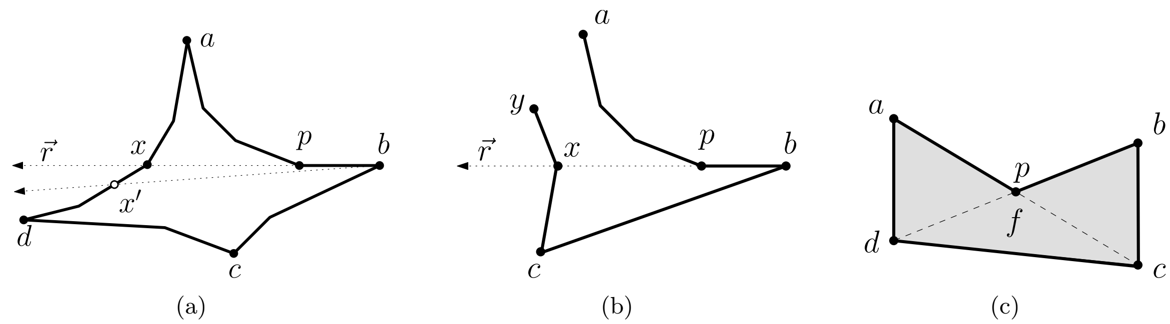


Figure 4: Illustrations for Proposition 2.3.

We claim that $x = d$. Suppose for contradiction that $x \neq d$. Clearly $x \notin f_{b,c}$, for otherwise there must be a convex vertex on $f_{b,c}$ different from b and c. Suppose that $x \in f_{d,a}$. If we slightly rotate $\vec{r}$ counterclockwise, then it will cross $f_{d,a}$ at a point x' which slightly precedes x on $f_{d,a}$ (see Figure 4(a)). By applying Proposition 2.5 on the polygon consisting of the chain $f_{x',b}$ and the segment $\overline{bx'}$, we conclude that b and a are neighbors in this polygon, which is a contradiction (note that in this case x is a reflex vertex and remains so in the new polygon).

Therefore, $x \in f_{c,d}$. If $f_{b,x}$ touches (but does not cross) $\vec{r}$ at a point y that is before x on $\vec{r}$, then the open chains $f_{b,y}$ and $f_{y,x}$ must both contain a convex vertex. However, c is the only convex vertex in the open chain $f_{b,x}$. Thus, the polygon consisting of $f_{b,x}$ and $\overline{xb}$ is simple, and it follows from Proposition 2.5 that its vertices are b, c, and x. In a similar way we can deduce that a and d are neighbors in f.

Denote by y the other neighbor of x in f. The vertices y and c lie on different sides of $\vec{r}$, since it crosses $f_{b,a}$ at x. If $x \neq d$ then x must be a reflex vertex, and then yx and bc are virtually crossing (see Figure 4(b)), which is a contradiction. Therefore, $x = d$ or in other words c and d are neighbors in f. Now it is evident that f must be a pentagon such that the edge ap is collinear with c and the edge bp is collinear with d (see Figure 4(c)). Notice that in this case no vertex of f may be needy. □

References

[1] E. Ackerman, On the maximum number of edges in topological graphs with no four pairwise crossing edges, *Discrete Comput. Geom.* **41** (2009), 365-375.

[2] P. K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir, Quasi-planar graphs have a linear number of edges, *Combinatorica* **17** (1997), no. 1, 1–9.

[3] B. Aronov, P. Erdős, W. Goddard, D.J. Kleitman, M. Klugerman, J. Pach, L.J. Schulman, Crossing families, *Combinatorica* **14** (1994), 127-134.

[4] P. Brass, W. Moser, J. Pach, *Research Problems in Discrete Geometry*, Springer, 2005.

[5] M. Katchalski and H. Last, On geometric graphs with no two edges in convex position, *Discrete Comput. Geom.* **19** (1998), no. 3, 399-404.

[6] J. Pach, Notes on geometric graph theory, In J. E. Goodman, R. Pollack and W. Steiger, editors, *Discrete and Computational Geometry: Papers from DIMACS special year*, volume 6 of *DIMACS series*, 273–285, AMS, Providence, RI, 1991.

[7] J. Pach, R. Pinchasi, and M. Sharir, A tight bound for the number of different directions in three dimensions, *J. Combinatorial Theory, ser. A.* **108** (2004), 1–16.

[8] R. Pinchasi, Geometric graphs with no two parallel edges, *Combinatorica* **28** (2008), no. 1, 127–103.

[9] P. Valtr, Graph drawings with no k pairwise crossing edges, In G. D. Battista, editor, *Graph Drawing*, volume 1353 of *Lecture Notes in Computer Science*, 205–218, Springer, 1997.

[10] P. Valtr, On geometric graphs with no k pairwise parallel edges, *Discrete Comput. Geom.* **19** (1998), no. 3, 461-469.