Geometric Graphs with no Two Parallel Edges

Rom Pinchasi
Israel Institute of Technology, Technion[1]

April 30, 2007

Abstract

We give a simple proof for a theorem of Katchalski, Last, and Valtr, asserting that the maximum number of edges in a geometric graph G on n vertices with no pair of parallel edges is at most $2n-2$. We also give a strengthening of this result in the case where G does not contain a cycle of length 4. In the latter case we show that G has at most $\frac{3}{2}(n-1)$ edges.

1 Introduction

A *geometric graph* is a graph drawn in the plane with its vertices as points and its edges as straight line segments connecting corresponding points. A topological graph is defined similarly except that its edges are simple Jordan arcs connecting corresponding points. (for additional theory and problems on geometric and topological graphs consult [5]).

Two edges in a geometric graph are said to be *parallel*, if they are two opposite edges of a convex quadrilateral. In [2] and [6] Katchalski, Last, and Valtr proved a conjecture of Kupitz (see [3]) and obtained the following result:

Theorem 1.1. *A geometric graph on n vertices with no pair of parallel edges has at most $2n-2$ edges.*

Remark: This result is tight for $n \geq 4$, by a construction of Kupitz ([3]).

Here we show that Theorem 1.1 can be easily derived from another theorem of Cairns and Nikolayevsky ([1]) about *generalized thrackles*.

Definition 1.2. *A topological graph G is called a* generalized thrackle*, if every pair of edges of G meet an odd number of times.*

The notion of a generalized thrackle was first introduced by Lovász, Pach, and Szegedy in [4], in connection with Conway's thrackle conjecture. We will make use of the following theorem of Cairns and Nikolayevsky ([1]), usually referred to as the generalized thrackle theorem.

[1]e-mail: room@math.technion.ac.il

Theorem 1.3. *The maximum number of edges in a generalized thrackle on n vertices is $2n-2$.*

In the same paper ([1], Theorem 2) Cairns and Nikolayevsky show that a generalized thrackle on n vertices that does not contain a cycle of length 4 has at most $\frac{3}{2}(n-1)$ edges.

2 Proof of Theorem 1.1

We simply show that if G is a geometric graph with no pair of parallel edges, then G can be drawn in the plane as a generalized thrackle.

Indeed, position a sphere S which touches the plane and project the plane onto the lower hemisphere of S through the center of S. Then the vertices of G project to points on S, and the edges of G project to great arcs connecting corresponding points. Now replace each great arc on S by the relative closure of its complement on the great circle that contains it.

We claim that the resulting drawing of G on the sphere S has the property that any two edges that do not share a common vertex, meet exactly once and every two edges sharing a common vertex meet exactly twice. This follows because there are no two parallel edges in G. Specifically, consider first two edges e_1, e_2 in G which do not share a common vertex. Since e_1 and e_2 are not parallel, then there is a line which contains one of them, say e_1, and crosses the other (e_2). Let C_1 and C_2 be the great circles on S which contain the projections I_1 and I_2 of e_1 and e_2 respectively on S. Then C_1 crosses I_2 at a point A in the lower hemisphere of S. It follows that the complements of I_1 and I_2 cross exactly once at the point A' that is opposite to A on S.

If e_1 and e_2 share a common vertex V, then I_1 and I_2 meet at a point A (the projection of V on S) which is an endpoint for both subarcs. Therefore the relative closures of the complements of I_1 and I_2 on C_1 and C_2 respectively, meet at A and also cross at the opposite point on S, A'.

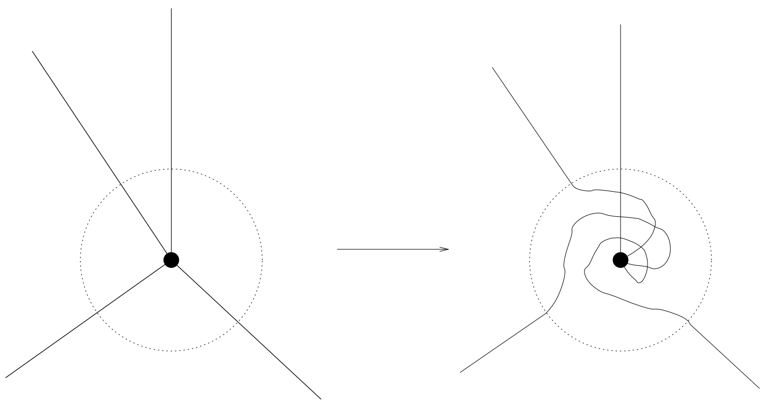

Figure 1: the local change of the drawing near a vertex

To make the resulting drawing on S a generalized thrackle, perform a small modification of the drawing in a small neighborhood of each vertex. Reverse the cyclic order of the edges adjacent to the vertex, and get an additional intersection of every two of them. See Figure 1. This way every two edges in the drawing of G on S cross either once or three times, depending on whether they do not share a common vertex or they do.

We conclude the proof by applying Theorem 1.3 to the resulting drawing which is a generalized thrackle.

As an immediate corollary from the result in [1], we get that if in addition G does not contain a cycle of length 4, then G has at most $\frac{3}{2}(n-1)$ edges. In a similar manner, we can get further improvement if we know on other forbidden short cycles in the graph G. More explicitly, if the shortest even cycle in a generalized thrackle G has length $2r$, then G has at most $\frac{r}{r-1}(n-1)$ edges. This follows from a certain doubling technique (described also in [1]) by which G is transformed to a bipartite planar graph with twice as many vertices and edges. The result then follows from a simple application of Euler's formula.

Acknowledgments

The author would like to thank János Pach for very helpful discussions.

References

[1] G. Cairns and Y. Nikolayevsky, Bounds for generalized thrackles, *Discrete Comput. Geom.* **23** (2000), no. 2, 191–206.

[2] M. Katchalski and H. Last, On geometric graphs with no two edges in convex position, *Discrete Comput. Geom.* **19** (1998), no. 3, Special Issue, 399–404.

[3] Y. Kupitz, On pairs of disjoint segments in convex position in the plane, *Ann. Discrete Math.* **20** (1984), 203–208.

[4] L. Lovász, J. Pach, and M. Szegedy, On Conway's thrackle conjecture, *Discrete Comput. Geom.* **18** (1997), no. 4, 369–376.

[5] J. Pach, Geometric graph theory, Chapter 10 in J.E. Goodman and J. O'Rourke: Handbook of Discrete and Computational Geometry, 2nd ed., CRC Press, 2004, pp. 219–238.

[6] P. Valtr, On geometric graphs with no k pairwise parallel edges, *Discrete Comput. Geom.* **19** (1998), no. 3, Special Issue, 461–469.