A finite family of pseudo-discs must include a “small” pseudo-disc

Rom Pinchasi*

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Abstract

We show that there is an absolute constant $c \leq 156$ such that in every finite family $\mathcal{F}$ of pseudo-discs in the plane one can find a member $D \in \mathcal{F}$ such that among all of the pseudo-discs in $\mathcal{F}$ intersecting D there are at most c pairwise disjoint sets.

1 Introduction

For a compact set D in $\mathbb{R}^2$ we denote by ∂D the boundary of D. We will consider only sets D such that ∂D is a simple closed Jordan curve.

A family of *pseudo-discs* $\mathcal{F}$ is a family of sets with the following properties: for every $D \in \mathcal{F}$, ∂D is a simple closed Jordan curve, and for every pair of sets $D_1, D_2 \in \mathcal{F}$ the closed curves ∂D_1 and ∂D_2 intersect in at most two points. A family of circular discs is one natural example for a family of pseudo-discs.

Given a family of sets $\mathcal{F}$ (in $\mathbb{R}^2$ for example), we denote by $G_{\mathcal{F}}$ the *intersection graph* of $\mathcal{F}$. This is a graph whose vertices correspond to the sets in $\mathcal{F}$ and two sets are connected by an edge if and only if they intersect. One very nice property of the intersection graph of a finite family $\mathcal{F}$ of circular discs in the plane is that it always has a vertex v such that the graph induced on the set of neighbors of v does not have a large independent set. Indeed, consider the smallest circular disc D in $\mathcal{F}$. It is shown in [3] (see Lemma 5.1 in [3]) that among the discs in $\mathcal{F}$ intersecting with D there are at most five pairwise disjoint discs. The argument is simple although verifying it may be a bit technical. Denote by O the center of D. If O_1 and O_2 are the centers of two disjoint discs in $\mathcal{F}$ that intersect with D, then $\measuredangle O_1 O O_2 > \frac{\pi}{3}$. A simpler folklore argument that gives a slightly worse constant is that every disc in $\mathcal{F}$ that intersects with D, which we assume to be a unit disc, covers an area of at least π of the disc of radius 3 centered at O. It follows that there are at most $\frac{\pi 3^2}{\pi} = 9$ pairwise disjoint discs in $\mathcal{F}$ that intersect D.

In the above arguments we used strong geometric properties of the circle such as its shape and area. In this paper we show that in fact the same result, apart from the value of the constant 5, follows from much weaker considerations, namely from topological reasons.

Theorem 1. *There is an absolute constant c with the following property. Let $\mathcal{F}$ be any finite family of pseudo-discs in $\mathbb{R}^2$. Then $G_{\mathcal{F}}$ has a vertex v such that the subgraph of $G_{\mathcal{F}}$ induced by the set of neighbors of v does not have an independent set of size greater than c.*

*Mathematics Dept., Technion—Israel Institute of Technology, Haifa 32000, Israel. room@math.technion.ac.il.

Remark. The constant c in Theorem 1 that will come from our proof is $c = 156$. With some extra effort we could push the constant down a bit further. Getting a nearly best constant here will be a somewhat greater challenge.

We remark also that the result in Theorem 1 is not true for general intersection graphs. In particular, the complete bipartite graph on $n/2$ and $n/2$ vertices can be realized as an intersection graph of ellipses in the plane. In this graph the neighborhood of each vertex consists of an independent set of size $n/2$ (see Figure 1).

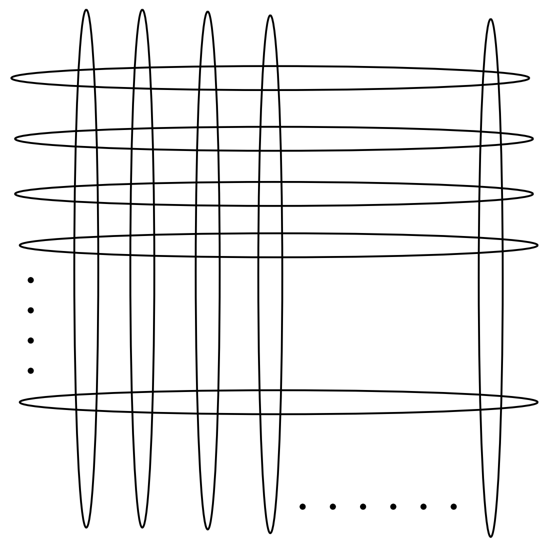

Figure 1: A family of ellipses.

2 Some preliminaries

In this section we prepare the ground for the proof of Theorem 1 with some auxiliary lemmas of more general flavor. We start with the following lemma that will be crucial for the proof.

Lemma 1. *Let $\mathcal{F}$ be a family of pseudo-discs. Let $D \in \mathcal{F}$ and let $x \in D$ be any point. Then D can continuously be shrunk to the point x so that at each moment $\mathcal{F}$ is a family of pseudo-discs.*

Proof. We rely on the result in [4] by which an arrangement of pseudo-discs can be swept by a pseudo-disc. More precisely it is shown in [4] that given any family $\mathcal{F}$ of pseudo-discs one can choose any member D of $\mathcal{F}$ and define a continuous family of discs $D(t)$ for $t \in [0, 1]$ such that $D(0) = D$, and for every $s < t$ we have $D(s) \subset D(t)$, and $D(1)$ contains the union of all sets in $\mathcal{D}$. Moreover, for every $0 \leq t \leq 1$, $\mathcal{F} \setminus \{D\} \cup \{D(t)\}$ is a family of pseudo-discs.

In order to see how this result in [4] implies our lemma, let $x \in D$ be a point in a member D of $\mathcal{F}$. Apply the inversion mapping Inv to the plane with respect to the point x. That is, take a complex number z in the plane to the point $1/(z - x)$. This one to one mapping of $\mathbb{R}^2 \setminus \{x\}$ to itself takes the curves $\{\partial C \mid C \in \mathcal{F}\}$ to another family of simple closed curves every pair of which intersect at most twice (because the inversion mapping is one to one). Let $\mathcal{F}'$ be the family of all regions bounded by curves of the form $Inv(\partial C)$, where $C \in \mathcal{F}$. Therefore, $\mathcal{F}'$ is a family of pseudo-discs. Let D' denote the region in the plane bounded by $Inv(\partial D)$. One can use the result in [4] to continuously expand D' until it covers the entire union of the sets in $\mathcal{F}'$. Taking now the inverse mapping of Inv we get a shrinking of D to the point x, as desired. ∎

We will need the following simple but important corollary:

Corollary 1. *Let $\mathcal{B}$ be a family of pairwise disjoint sets in the plane and let $\mathcal{F}$ be a family of pseudo-discs. Let D be a member of $\mathcal{F}$ and suppose that D intersects exactly k members of $\mathcal{B}$ one of which is the set $B \in \mathcal{B}$. Then for every $2 \leq \ell \leq k$ there exists a set $D' \subset D$ such that D' intersects B and exactly $\ell - 1$ other sets from $\mathcal{B}$, and $\mathcal{F} \cup \{D'\}$ is again a family of pseudo-discs.*

Proof. Given $2 \leq \ell \leq k$, choose a point $x \in B$ and use Lemma 1 to continuously shrink D towards the point x until the first time that D, which we denote now by D', intersects only ℓ of the sets in $\mathcal{B}$. ∎

Another important tool in the proof is the following planarity lemma.

Lemma 2. *Let $\mathcal{B}$ be a family of pairwise disjoint closed connected sets in $\mathbb{R}^2$. Let $\mathcal{F}$ be a family of pseudo-discs. Define a graph G whose vertices correspond to the sets in $\mathcal{B}$ and connect two sets $B, B' \in \mathcal{B}$ if there is a set $D \in \mathcal{F}$ such that D intersects B and B' but not any other set from $\mathcal{B}$. Then G is planar.*

For the proof we will need the following elementary lemma from [1]:

Lemma 3 (Lemma 1 in [1])**.** *Let D_1 and D_2 be two pseudo-discs in the plane. Let x and y be two points in D_1 but not in D_2. Let a and b be two points in D_2 but not in D_1. Let e be any Jordan arc connecting x and y that is fully contained in D_1. Let f be any Jordan arc connecting a and b that is fully contained in D_2. Then e and f cross an even number of times.*

Proof of Lemma 2. We will draw G in the plane in such a way that every pair of edges in G that do not share a common vertex cross an even number of times. The Hanani-Tutte Theorem ([2, 5]) then implies the planarity of G. For every $B \in \mathcal{B}$ choose a point $v_B \in B$. If B and B' are neighbors in G draw an edge between v_B and $v_{B'}$ as follows. Let $D \in \mathcal{F}$ be such that D intersects both B and B' but no other set from $\mathcal{B}$. Draw an arc A_B^D inside B from v_B to the closest point v_B^D in $B \cap D$ (possibly $v_B \in D$ in which case $v_B = v_B^D$). Similarly, draw an arc $A_{B'}^D$ inside B' from $v_{B'}$ to the closest point $v_{B'}^D$ in $B' \cap D$. Then draw another arc $A_{BB'}^D$ connecting v_B^D and $v_{B'}^D$ that is fully contained in D. The concatenation of all three arcs is the drawing of the edge between v_B and v_B. We claim that in this drawing of the graph G every pair of edges with no common vertex cross an even number of times. Indeed, let $B, B', K, K' \in \mathcal{B}$. Suppose $e = A_B^D \cup A_{BB'}^D \cup A_{B'}^D$ is the arc connecting v_B and $v_{B'}$ and suppose $f = A_K^C \cup A_{KK'}^C \cup A_K^C$ is the arc connecting v_K and $v_{K'}$. Notice that A_B^D is disjoint from f as $A_B^D \subset B$ while $f \subset K \cup C \cup K'$ and none of C, K, and K' intersects with B. Similarly, $A_{B'}^D \cap f = \emptyset$, $A_K^C \cap e = \emptyset$, and $A_{K'}^C \cap e = \emptyset$. Therefore, all the crossings between e and f are the crossings between $A_{BB'}^D$ and $A_{KK'}^C$. Lemma 3 implies that these two arcs cross an even number of times. ∎

3 Proof of Theorem 1

Let $\mathcal{B} \subset \mathcal{F}$ be a subfamily of maximum cardinality of pairwise disjoint sets from $\mathcal{F}$. Denote $n = |\mathcal{B}|$ and let $\mathcal{H} = \mathcal{F} \setminus \mathcal{B}$. Let $\mathcal{H}_1 \subset \mathcal{H}$ be all the sets in $\mathcal{H}$ that intersect precisely one of the sets in $\mathcal{B}$.

Let $\mathcal{H}_2 \subset \mathcal{H}$ be all the sets in $\mathcal{H}$ that intersect precisely two of the sets in $\mathcal{B}$.

For every $B \in \mathcal{B}$ let $\alpha_1(B)$ denote the maximum size of an independent set in the intersection graph of all the sets in $\mathcal{H}_1$ that intersect with B. Observe that for every $B \in \mathcal{B}$ we have $\alpha_1(B) \leq 1$. This is because if $\alpha_1(B) \geq 2$, then one can find two disjoint sets D, D' in $\mathcal{H}_1$ that intersect only with B and not with any other set in $\mathcal{B}$. This is a contradiction to the maximality of $\mathcal{B}$, as $\mathcal{B} \cup \{D, D'\} \setminus \{B\}$ is a larger family of pairwise disjoint sets from $\mathcal{F}$.

For every $B \in \mathcal{B}$ let $\alpha_2(B)$ denote the maximum size of an independent set in the intersection graph of all the sets in $\mathcal{H}_2$ that intersect with B. We next aim to show an $O(n)$ upper bound for $\sum_{B \in \mathcal{B}} \alpha_2(B)$. To this end define a graph G whose vertices correspond to the sets in $\mathcal{B}$. Connect two sets B and B' in $\mathcal{B}$ by an edge if and only if there is a set in $\mathcal{H}_2$ that intersects with both B and B'. Let $e(G)$ denote the number of edges in G. By Lemma 2, G is planar and therefore $e(G) < 3n$. For every $B \in \mathcal{B}$ let $d(B)$ denote the degree of B in the graph G. Therefore, $\sum_{B \in \mathcal{B}} d(B) = 2e(G) \leq 6n$. We claim that for every B we have $\alpha_2(B) \leq 2d(B)$. This is because if for some $B \in \mathcal{B}$ we have $\alpha_2(B) > 2d(B)$, then, by the pigeonhole principle, there must be at least three pairwise disjoint sets D_1, D_2, D_3 from H_2 that intersect with B and with some neighbor B' of B in the graph G. This is a contradiction to the maximality of $\mathcal{B}$ because the family $\mathcal{B} \cup \{D_1, D_2, D_3\} \setminus \{B, B'\}$ is a larger family of pairwise disjoint sets from $\mathcal{F}$.

We conclude that

$$\sum_{B \in \mathcal{B}} \alpha_2(B) \leq 2 \sum_{B \in \mathcal{B}} d(B) \leq 12n. \tag{1}$$

Next, we consider the family $\mathcal{H}_3 = \mathcal{H} \setminus (\mathcal{H}_1 \cup \mathcal{H}_2)$. Every set $D \in \mathcal{H}_3$ intersects with three or more of the sets in $\mathcal{B}$. For every $B \in \mathcal{B}$ let $\alpha_3(B)$ denote the maximum size of an independent set in the intersection graph of all the sets in $\mathcal{H}_3$ that intersect with B. We next aim to show an $O(n)$ upper bound for $\sum_{B \in \mathcal{B}} \alpha_3(B)$. Once this is established, then we have $\sum_{B \in \mathcal{B}} \alpha_1(B) + \alpha_2(B) + \alpha_3(B) = O(n)$. This implies that there exists a set $B \in \mathcal{B}$ such that $\alpha_1(B) + \alpha_2(B) + \alpha_3(B)$ is bounded by a constant independent of n and this immediately implies the desired result.

Using repeatedly Corollary 1 with $\mathcal{F} = \mathcal{H}_3$ and with $\ell = 3$, we can find, for every $D \in \mathcal{H}_3$ and every $B \in \mathcal{B}$ that is intersected by D, a (new) pseudo-disc $D^B \subset D$ that intersects with B and with exactly two more sets from $\mathcal{B}$. Moreover, the collection of all the new sets D^B obtained in this way is a family of pseudo-discs. We denote this family by $\mathcal{D}$.

In what follows some of the arguments are very similar to those used in the proof of Theorem 5 in [1]. We denote by P the collection of all pairs of sets from $\mathcal{B}$ that appear together in some triple in T.

Claim 1. $|P| < 12n$.

Proof. Pick every set in $\mathcal{B}$ with probability $\frac{1}{2}$. We denote the resulting subset of $\mathcal{B}$ by $\mathcal{B}^*$. Call a pair (B, B') in P *good* if both $B, B' \in \mathcal{B}^*$ and there exists a set in $\mathcal{D}$ that intersects B and B' and no other set from $\mathcal{B}^*$. Observe that every pair in P is good with probability of at least $\frac{1}{2} \cdot \frac{1}{2} \cdot (1 - \frac{1}{2}) = \frac{1}{8}$. Denote by P^* the collection of all good pairs in P. By Lemma 2, P^* is the set of edges of a planar graph on the set of vertices $\mathcal{B}^*$. Therefore, $|P^*| < 3|\mathcal{B}^*|$. Taking expectations we see that

$$\frac{1}{8}|P| \leq Ex(|P^*|) < 3Ex(|\mathcal{B}^*|) = 3 \cdot \frac{1}{2}n = \frac{3}{2}n.$$

This implies $|P| < 12n$. ∎

We consider the graph G on the set of vertices $\mathcal{B}$ with the set of edges P. For every $B \in \mathcal{B}$ denote by $d(B)$ the degree of B in this graph. Notice that, in view of Claim 1,

$$\sum_{B \in \mathcal{B}} d(B) = 2|P| < 24n. \tag{2}$$

Fix $B \in \mathcal{B}$. Define a graph G^B on the set of neighbors of B in G where we connect two neighbors B_1, B_2 of B in G by an edge in G^B if and only if $\{B, B_1, B_2\}$ is a triple in T. This is equivalent to that there is $D \in \mathcal{D}$ that intersects with B, B_1, and with B_2. Denote by $e(G^B)$ the number of edges in G^B. By ignoring the set B and applying Lemma 2, we see that G^B is planar. G^B has $d(B)$ vertices and is planar and therefore $e(G^B) < 3d(B)$.

We claim that for every $B \in \mathcal{B}$ we must have $\alpha_3(B) \leq 2e(G^B)$. Indeed, observe that every set $D \in \mathcal{H}_3$ that intersects B must intersect also two sets $B_1, B_2 \in \mathcal{B}$ such that B_1 and B_2 are connected by an edges in G^B. Assume to the contrary that $\alpha_3(B) > 2e(G^B)$, then by the pigeonhole principle there exists three pairwise disjoint sets $D_1, D_2, D_3 \in H_3$ and two sets $B_1, B_2 \in \mathcal{B}$ different from B such that each of the sets D_1, D_2, and D_3 intersects each of the sets B_1, B_2, and B. This is impossible as it gives an embedding of the graph $K_{3,3}$ in the plane (recall that also the sets B, B_1, B_2 are pairwise disjoint).

We conclude that for every $B \in \mathcal{B}$ we have $\alpha_3(B) \leq 2e(G^B) < 6d(B)$. Inequality (2) yields

$$\sum_{B \in \mathcal{B}} \alpha_3(B) < \sum_{B \in \mathcal{B}} 6d(B) < 144n. \tag{3}$$

Inequality (1), inequality (3), and the fact that $\alpha_1(B) \leq 1$ for every $B \in \mathcal{B}$ imply that

$$\sum_{B \in \mathcal{B}} \alpha_1(B) + \alpha_2(B) + \alpha_3(B) < n + 12n + 144n = 157n. \tag{4}$$

The proof is complete as (4) implies the existence of $B \in \mathcal{B}$ such that $\alpha_1(B) + \alpha_2(B) + \alpha_3(B) \leq 156$. ∎

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