On coloring points with respected to rectangles

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Abstract

In a coloring of a set of points P with respect to a family of regions one requires that in every region containing at least two points from P, not all the points are of the same color. Perhaps the most notorious open case is coloring of n points in the plane with respect to axis-parallel rectangles, for which it is known that $O(n^{0.368})$ colors always suffice, and $\Omega(\log n/\log^2 \log n)$ colors are sometimes necessary.

In this note we give a simple proof showing that every set P of n points in the plane can be colored with $O(\log n)$ colors such that every axis-parallel rectangle that contains at least *three* points from P is non-monochromatic.

1 Introduction

A *hypergraph* (or *range space*) H consists of a vertex set V and a (hyper)edge set $E \subseteq 2^V$. A (valid) *coloring* of H assigns a color to every vertex in H such that in every edge of H that contains at least two vertices not all vertices are of the same color (that is, the edge is *non-monochromatic*). If in every edge there is a color that appears (at least once and) at most k times, then we say that the coloring is *k-conflict-free* (1-conflict-free coloring is called *conflict-free* coloring).

Coloring and and conflict-free coloring of hypergraphs that stem from geometric regions have attracted some attention lately, due to their applications to frequency assignment in wireless networks, scheduling in RFID networks, and decompositions of multiple coverings (see the recent survey of Smorodinsky on conflict-free coloring [11] and the references therein). In a typical geometric setting, the vertex set of the hypergraph is a set of points P, and its edge set is defined by a family of regions $\mathcal{F}$, such that every region $F \in \mathcal{F}$ defines an edge that consists of the points of P that belong to F.

Perhaps the most challenging open question concerning coloring of geometric hypergraphs is to find tight asymptotic bounds for the minimum number of colors that suffice for coloring any set P of n points in the plane such that any axis-parallel rectangle that contains at least two points from P contains points of different colors. Har-Peled and Smorodinsky [8] were the first to consider this problem and gave an $O(\sqrt{n})$ upper bound. Soonafter several others pointed out that this can be slightly improved to $O(\sqrt{n/\log n})$ [8, 9]. Ajwani et al. [1] significantly improved the bound to $O(n^{0.382})$, and very recently Chan [6] obtained the currently best upper bound of $O(n^{0.368})$. A lower bound of $\Omega(\log n/\log^2 \log n)$ was proved by Chen et al. [7], and it is conjectured that the true bound should also be polylog(n). In this note we give a simple proof for such an upper bound when considering only rectangles that contain at least *three* points. In fact, we prove the following more general result, that relates to the notion of *k-colorful* coloring, introduced in [3].

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Theorem 1. *For every integer $k > 0$, every set P of n points in the plane can be colored with $O(k^4 \log n)$ colors such that every axis-parallel rectangle that contains at least $2k-1$ points from P contains points of at least k different colors.*

Next we describe two consequences of Theorem 1 (for $k = 2$). For a set of points P and an integer $d > 1$, let $\alpha_d(P)$ be the maximum size of a (d-*independent*) subset $Q \subseteq P$ such that there is no axis-parallel rectangle R such that $R \cap P \subseteq Q$ and $|R \cap P| = d$. Chen et al. [7] proved that if P is a set of n points that is randomly and uniformly selected from the unit square, then almost surely $\Omega(n/\log^{1/(d-1)} n) \leq \alpha_d(P) \leq O(dn \log^2 \log n / \log^{1/(d-1)} n)$. Theorem 1 implies that $\alpha_3(P) = \Omega(n/\log n)$ for every set P of n points. This bound can be further improved using a result of Alon, see a remark at the end of Section 2.

Corollary 2. *For every set P of n points $\alpha_3(P) = \Omega(\frac{n \log \log n}{\log n})$.*

Clearly this bound holds for $\alpha_d(P)$ when $d \geq 3$. It would be interesting to show a bound that increases with d.

The following general framework [11] guarantees a d-conflict-free coloring with relatively small number of colors, provided that $\alpha_d(P)$ is large: Find a large d-independent set of vertices, color it with a new color, remove it from the set of vertices, and continue in the same manner. Applying the same approach and using Corollary 2 we get:

Corollary 3. *Every set of n points in the plane can be 2-conflict-free colored with respect to axis-parallel rectangles using $O(\frac{\log^2 n}{\log \log n})$ colors.*

2 Proof of Theorem 1

Let k be a positive integer, and let P be a set of n points in the plane. We may assume that $k \geq 2$, since the theorem trivially holds for $k = 1$. A *q-rectangle* is an axis-parallel rectangle that contains exactly q points from P. A rectangle is *k-colorful* if it contains points of at least k different colors. Since any q-rectangle such that $q > (2k-1)$ contains a $(2k-1)$-rectangle, it is enough to prove that P can be colored with $O(k^4 \log n)$ colors such that every $(2k-1)$-rectangle is k-colorful.

We may assume without loss of generality that no two points in P share the same x- or y-coordinate, since otherwise a slight perturbation of the points can only extend the set of $(2k-1)$-rectangles. Moreover, only the relative position of the points rather than their actual coordinates matter when it comes to the set of $(2k-1)$-rectangles they induce. Thus, we may assume that the points in P lie on an $n \times n$ portion of the integer grid.

For an integer i we define the graph $G_i(P)$ as follows. The vertices of $G_i(P)$ are the points of P. Two points $p, q \in P$ form an edge in $G_i(P)$ if there exists a k-rectangle that contains both of them and whose aspect ratio is $1/2^i$ (the aspect ratio of a rectangle is the ratio between its height and its width). For a set I of integers let $G_I(P) = \bigcup_{i \in I} G_i(P)$. (Therefore, $G_I(P)$ is a subgraph of the *Delaunay graph* [7].)

Lemma 2.1. *$G_I(P)$ has $O(k^4|I|n)$ edges.*

Proof. Rectangles of the same aspect ratio intersect as pseudo-disks, that is, their boundaries intersect at most twice. Therefore, by [5, Thm. 13] the number of (combinatorially different) k-rectangles of a given aspect ratio is $O(k^2 n)$. Since we consider $|I|$ different aspect ratios and every k-rectangle induces $O(k^2)$ edges, the number of edges of $G_I(P)$ is $O(k^4|I|n)$. □

From Lemma 2.1 we can conclude:

Lemma 2.2. *$G_I(P)$ is $O(k^4|I|)$-colorable.*

Proof. Let $c \geq 1$ be an integer constant such that $G_I(P)$ has at most $ck^4|I|n$ edges (c exists by Lemma 2.1). It is easy to show by induction on n that $G_I(P)$ is $(ck^4|I|+1)$-colorable. For $n \leq k$ the claim trivially holds. Let P be a set of $n > k$ points. Since $G_I(P)$ contains at most $ck^4|I|n$ edges, it has a vertex v of degree at most $ck^4|I|$. Let $P' = P \setminus \{v\}$. By the induction hypothesis $G_I(P')$ is $(ck^4|I| + 1)$-colorable. If two points $p, q \in P'$ are both in a rectangle R of aspect ratio x that contains exactly k points from P, then there is also a rectangle of aspect ratio x that contains them both and $k - 2$ other points from P': it could be R if $v \notin R$ or a rectangle obtained from R by extending it while maintaining its aspect ratio until hitting a new point (there is such a point since $n > k$). Therefore, $G_I(P) \setminus \{v\}$ is a subgraph of $G_I(P')$ and hence $G_I(P)\setminus\{v\}$ is $(ck^4|I|+1)$-colorable. Since v has at most $ck^4|I|$ neighbors, the $(ck|I|+1)$-coloring of $G_I(P) \setminus \{v\}$ can be extended to a $(ck^4|I| + 1)$-coloring of $G_I(P)$. □

With Lemma 2.2 in hand we can now complete the proof of Theorem 1. Consider $G = G_I(P)$ for $I = \{-\lfloor \log n \rfloor, \ldots, \lceil \log n \rceil\}$. By Lemma 2.2 there is a coloring of G with $O(k^4 \log n)$ colors. We will show that under this coloring every $(2k - 1)$-rectangle is k-colorful.

Let R be a $(2k - 1)$-rectangle, let h and w be its height and width, respectively, and assume without loss of generality that $h \leq w$. Since we assumed that the points in P lie on an $n \times n$ grid we have $h \geq 1$ and $w \leq n$. Therefore, there is an integer $1 \leq t \leq \lceil \log n \rceil$ such that $2^{t-1}h \leq w \leq 2^t h$. Thus, we can cover R (and no point in $\mathbb{R}^2 \setminus R$) by two rectangles of height h and width $2^{t-1}h$. One of these rectangles contains at least k points of P. If it contains more points, then we can shrink it continuously while maintaining its aspect ratio until it contains exactly k points of P. The k points inside the rectangle (whose aspect-ratio is $1/2^{t-1}$) form a k-clique in G, and therefore are colored by k different colors. □

Remarks. For the case $k = 2$ it is not necessary to use [5, Thm. 13] for the proof of Lemma 2.1, since it is not hard to show that $G_i(P)$ is planar in this case (see, e.g., [5]). Moreover, it is also easy to see that for every vertex v in G the subgraph $G[N(v)]$ is 4-colorable, where $N(v)$ denotes the neighbors of v and $G[U]$ denotes the subgraph induced by a vertex subset U. (The Southeast neighbors of v can be colored alternately with two colors. The same two colors can be used to color the Northwest neighbors of v. Two additional colors can be used to color the Southwest and Northeast neighbors of v.) Alon [2] proved that if G is a graph with average degree d and $G[N(v)]$ is r-colorable for every vertex v, then G has an independent set of size at least $c_r \frac{n \log d}{d}$, where c_r depends only on r. It follows that $G_I(P)$ has an independent set of size $\Omega(\frac{n \log \log n}{\log n})$, which implies Corollary 2.

Bar-Noy et al. [4] gave a general framework for *online*[1] conflict-free coloring *k-degenerate* hypergraphs. Their approach and the proof of Theorem 1 imply:

Corollary 4. *There is a randomized algorithm that online colors every set of n points in the plane with respect to axis-parallel 3-rectangles and uses $O(\log^2 n)$ colors with high probability.*

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