On the perimeter of k pairwise disjoint convex bodies contained in a convex set in the plane

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Abstract

We prove the following isoperimetric inequality in $\mathbb{R}^2$, conjectured by Glazyrin and Morić. Given a convex body S and k pairwise disjoint convex bodies $C_1, \ldots, C_k$ that are contained in S, then $\sum_{i=1}^{k} Per(C_i) \leq Per(S) + 2(k-1)Diam(S)$. Here $Per(\cdot)$ denotes the perimeter of a set and $Diam(\cdot)$ is the diameter of a set.

1 Introduction

For a bounded convex set S in $\mathbb{R}^2$ we denote by $Per(S)$ the perimeter of S and we denote by $Diam(S)$ the diameter of S.

Given a convex body S and another convex set C that is contained in S, it is a well known fact that $Per(C) \leq Per(S)$. In other words, the largest possible perimeter of a convex body that lies inside another convex body S is the perimeter of S. The following problem, raised by Glazyrin and Morić in [1], is therefore a very natural generalization of this simple isoperimetric statement:

Problem 1. Given a convex body S and $k \in \mathbb{N}$, what is the maximum possible sum of perimeters of k pairwise disjoint convex sets contained in S?

For $k = 1$ the answer to Problem 1 is $Per(S)$. For $k = 2$ it is not hard to see that the answer to Problem 1 is $Per(S) + 2Diam(S)$. Indeed, given two disjoint convex sets C_1 and C_2, contained in S, let ℓ be any separating line of C_1 and C_2. Denote by a and b the two points of intersection of ℓ with the boundary of S. The segment ab divides S into two convex sets S_1 and S_2, such that $S_1 \supset C_1$ and $S_2 \supset C_2$. We have

$$Per(C_1) + Per(C_2) \leq Per(S_1) + Per(S_2) = Per(S) + 2|ab| \leq Per(S) + 2Diam(S).$$

The fact that this bound for $k = 2$ is best possible follows from a general construction for every k that we shall now describe.

For $k > 1$ the following natural construction gives a good lower bound for the answer to Problem 1: Consider a diameter ab of the set S. Let a' and b' be two points on the boundary of S that are

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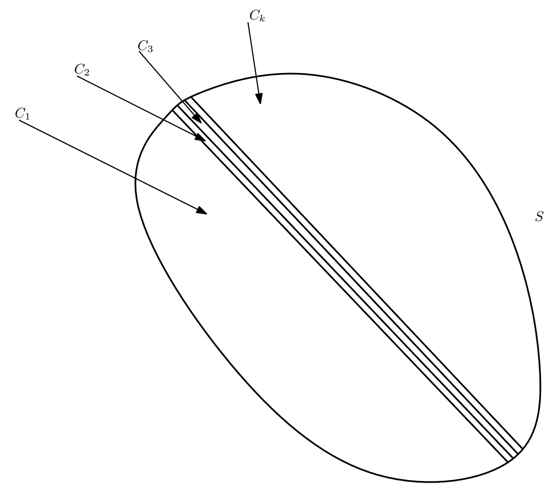

Figure 1:

very close to a and b, respectively, so that $aa'b'b$ is a convex quadrilateral. Inside the quadrilateral $aa'b'b$ take $k-2$ convex sets that are very very thin and are very close to being equal to the segment ab. The perimeter of each such set is very close to $2Diam(S)$. In addition take the two subsets of S, disjoint from the quadrilateral $aa'b'b$, that are bounded in the two half-planes determined by the line through a and b and the line through a' and b' (see Figure 1). Altogether we get a collection of k convex sets with a total perimeter that is very close to $Per(S) + 2(k-1)Diam(S)$. In fact, if we allow the sets C_i in Problem 1 to be pairwise disjoint only in their interiors and allow a convex set to be a segment, then we would get a collection of k convex sets with a total perimeter that is equal to $Per(S) + 2(k-1)Diam(S)$.

Glazyrin and Morić conjectured in [1] that $Per(S) + 2(k-1)Diam(S)$ is indeed the answer to Problem 1. (In fact in [1] there is a more general conjecture for any dimension d where $Per(S)$ is replaced by the $(d-1)$-dimensional volume of the boundary of S and $Diam(S)$ is replaced by the maximum $(d-1)$-dimensional volume of a hyper-plane section of S.)

The conjecture of Glazyrin and Morić in the plane is proved in [1] in the case where S is either a square, or any triangle. For a general convex set S in $\mathbb{R}^2$ it is shown in [1] that if $C_1, \ldots, C_k$ are k pairwise disjoint convex sets contained in S, then $Per(C_1) + \ldots + Per(C_k) \leq 1.22195Per(S) + 2(k-1)Diam(S)$.

The main result of this paper is a proof of the conjecture of Glazyrin and Morić:

Theorem 1. *Let S be a compact convex set in $\mathbb{R}^2$ and let $C_1, \ldots, C_k$ be k pairwise disjoint convex sets contained in S. Then*

$$\sum_{i=1}^{k} Per(C_i) \leq Per(S) + 2(k-1)Diam(S).$$

2 Some preliminaries and auxiliary first steps in the proof.

By relaxing a bit the conditions in Theorem 1 and assuming only that the *interiors* of $C_1, \ldots, C_k$ are pairwise disjoint, it follows from simple compactness arguments that there exist $C_1, \ldots, C_k$ such that $\sum_{i=1}^{k} Per(C_i)$ is maximum. More precisely, the space of all compact subsets of S under the Hausdorff metric is compact (Blaschke's selection theorem [2]). Moreover, the perimeter function of a convex set is continuous under the Hausdorff metric and therefore one can find $C_1, \ldots, C_k$ in S, with pairwise disjoint interiors, such that $\sum_{i=1}^{k} Per(C_i)$ is maximum. We will therefore concentrate on these $C_1, \ldots, C_k$ and give an upper bound on the sum of their perimeters.

Alternatively, and even somewhat simpler, for our proof of Theorem 1 it will be enough to assume that no convex set C_j can be extended to a larger one containing it. Therefore, it will be enough to consider convex sets $C_1, \ldots, C_k$ in S, with pairwise disjoint interiors, such that none of them can be extended to a larger convex set containing it and still satisfy the same conditions. Such maximal sequence of convex sets can easily be found using Zorn's lemma for example.

We may assume that no C_i is degenerate, that is, flat one dimensional segment. This is because otherwise we can discard this C_i, whose perimeter is clearly at most $2Diam(S)$, and conclude the proof by induction on k.

We recall that a *vertex* of a compact convex set C is a point p through which there is a half-space containing C whose bounding hyper-plane meets C only at p. For example every point on the boundary of a circular disc is a vertex of this circular disc, while the vertices of a square are only the four points at its corners.

Claim 1. *Each of $C_1, \ldots, C_k$, as a convex set, has at most k (in particular, a finite number of) vertices not on the boundary of S.*

Proof. Indeed, assume to the contrary that say C_1 has vertices $v_0, \ldots v_k$ listed in clockwise order on the boundary of C_1 and that those vertices do not lie on the boundary of S. For every $0 \leq j \leq k$ let ℓ_j be a line supporting C_1 at the vertex v_j alone. For every $0 \leq j < k$ denote by R_j the bounded region, outside of C_1, that is bounded by ℓ_j, ℓ_{j+1}, and the boundary of C_1 (see Figure 2). There must be a point of $C_2 \cup \ldots \cup C_k$ in R_j or else one can find a point p in the interior of R_j and replace C_1 by the convex hull of C_1 and p, thus obtaining a convex set C_1' with larger perimeter than C_1 and still interior-disjoint from each of $C_2, \ldots, C_k$. Observe moreover that if for some t there are points of C_t in two different regions R_j and $R_{j'}$, then the interiors of C_t and C_1 are not disjoint (see Figure 2).

In particular, for every $t > 1$ there is at most one region R_j that may contain a point of C_t. We obtain a contradiction as the number of regions R_j is greater than $k-1$. ∎

It follows from Claim 1 that each of $C_1, \ldots, C_k$ is a convex body whose boundary is composed of a finite number of straight line segments and possibly some portions of the boundary of S.

We next consider the union of the boundaries of the sets $C_1, \ldots, C_k$, and the boundary of S. This is a collection of arcs composing the boundary of S and a union of straight line segments that are the edges of the sets $C_1, \ldots, C_k$, not lying on the boundary of S. We thus get a *planar map* that we denote by M. To be more precise in the definition of M, the faces of M are the connected components of the plane after removing the boundaries of S and $C_1, \ldots, C_k$. Observe that among these faces there is also the unbounded face. The vertices of the map M are the vertices of the sets $C_1, \ldots, C_k$, considered as convex sets, that do not lie on the boundary of S, as well as those points that belong to three or more boundaries of faces in M (as we shall see later, all the vertices in M are in fact of the latter type). The edges of M are the connected components of the union of the boundaries of S and $C_1, \ldots, C_k$ after removing the vertices of M.

If we consider only those edges in M that are not contained in the boundary of S, then we

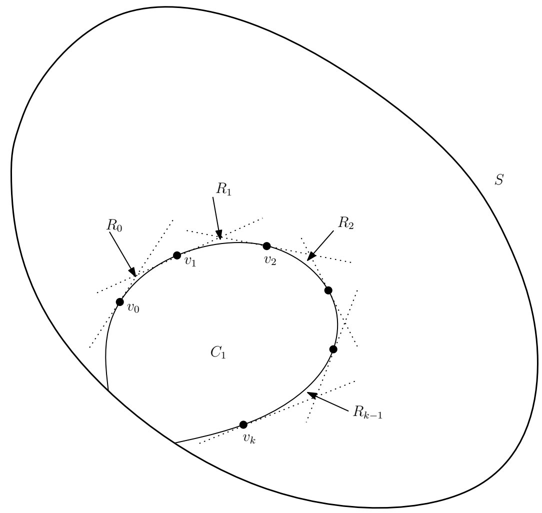

Figure 2: Proof of Claim 1

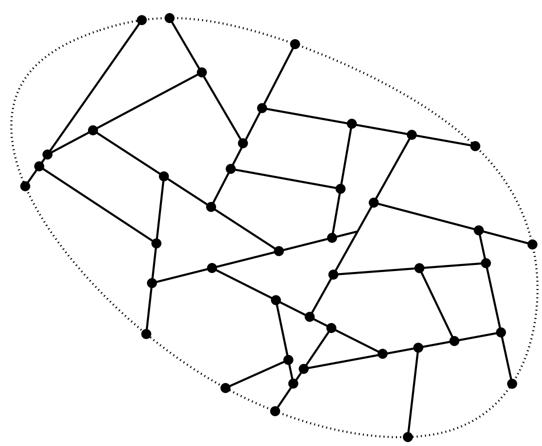

Figure 3: The geometric graph G. The map M includes also the dotted edges.

get in fact a geometric graph, drawn in the plane, that we denote by G (see Figure 3). Given any planar map or geometric graph we use $V(\cdot)$ to denote the set of its vertices. With a slight abuse of notation for a polygon H we will denote by $V(H)$ the set of vertices of H, as a convex set.

We call a vertex v of a geometric graph *convex* if the clockwise angle between any two consecutive edges adjacent to v is smaller than or equal to π. A convex vertex v is said to be *flat* if one of the angles between two consecutive edges adjacent to v is equal to π. A vertex of a geometric graph that is not convex is called a *reflex* vertex. Notice that every vertex of G that lies on the boundary of S must be a reflex vertex of the geometric graph G.

Definition 1. We call a bounded face of M that is not contained in any of $C_1, \dots, C_k$, a *hole*.

As we shall see next every hole in M is in fact a convex face of M. There is no loss of generality, however, to assume that every hole is simply connected, and therefore its boundary is a cycle (possibly not simple) in M. This is because if a hole in M is not simply connected, then M is not connected and this contradicts the maximality of $C_1, \dots, C_k$.

Let us emphasize at this point that when we refer to the vertices of a face F of M, and this

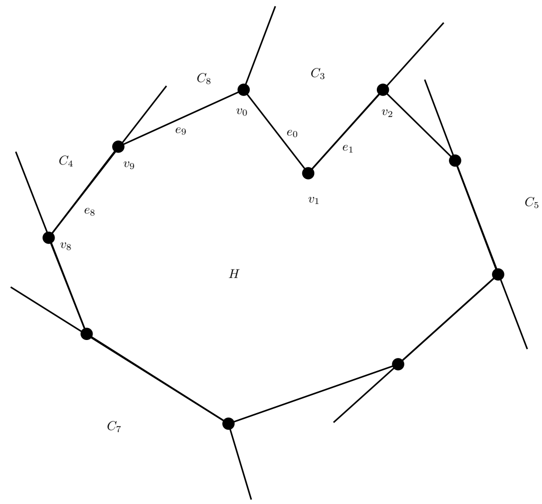

Figure 4: v_1 is a reflex vertex of H. v_9 is an exposed vertex of e_9 but it is not an exposed vertex of e_8. v_2 is not an exposed vertex of e_1.

includes also the case where F is a hole, then we mean the vertices of F in the planar map M and not the vertices of F as a convex set in the plane (as we shall see all the bounded faces of M are indeed convex).

Claim 2.

1. *Let H be a face of M that is a hole. Then no vertex of H may lie on the boundary of S.*
2. *Every vertex of G that does not lie on the boundary of S must be a convex vertex of G.*
3. *Let H be a hole in M whose vertices (as vertices of the map M) are $v_0, \ldots, v_{t-1}$, in this clockwise cyclic order on its boundary. For every $j = 0, \ldots, t-1$ let C_{i_j} be the convex set whose boundary contains the segment $v_j v_{j+1}$. Then either for every $0 \leq j \leq t-1$ v_j lies in the relative interior of a straight edge of C_{i_j}, or for every $0 \leq j \leq t-1$ v_j lies in the relative interior of a straight edge of $C_{i_{j-1}}$ (here, of course, the indices j are taken modulo t). In the former case we say that H is* oriented clockwise *and in the latter case we say that H is* oriented counterclockwise.

Remark. Notice that Claim 2 implies that every hole (and therefore every bounded face) of M is convex. Notice also that, by the third part of Claim 2, every vertex of a hole is flat.

Proof of Claim 2. Let H be a hole and let e be an edge of H (here H is viewed as a face in the map M) that is not part of the boundary of S. Because every edge in M that is not contained in the boundary of S is a portion of the boundary of one of the sets $C_1, \ldots, C_k$, then e must be contained in an edge of a convex set C_i for some $1 \leq i \leq k$. A vertex x of the edge e, viewed as an edge of the map M, is said to be an *exposed* vertex of e in H, if x is also a vertex of C_i, viewed as a convex set (see Figure 4).

Notice that no edge e of H, that is not part of the boundary of S, can have both of its vertices exposed. This is because in this case e is also an edge of one of the convex sets $C_1, \ldots, C_k$. One can therefore take a point y in the interior of H very close to e and replace C_i by the convex hull of C_i and y. The resulting set C_i' is still contained in S and is interior-disjoint from each of

$C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_k$. In addition C_i' has strictly larger perimeter than C_i. This contradicts the maximality of $C_1, \ldots, C_k$.

Notice also that if u is any vertex of H and the two edges of H adjacent to u are g and g', then u must be an exposed vertex of at least one of g and g' (this is again because the sets $C_1, \ldots, C_k$ are interior-disjoint).

To see the first part of the claim, assume to the contrary that H has a vertex on the boundary of S and let $e_1, e_2, \ldots, e_t$ be a maximal sequence of consecutive edges of H such that for every $1 \leq i < t$ the vertex adjacent to both e_i and e_{i+1} does not lie on the boundary of S. Therefore, both e_1 and e_t are adjacent to vertices (possibly the same vertex) that lie on the boundary of S. Notice that the vertex of e_1 on S and the vertex of e_t on S (possibly they are the same vertex) are exposed vertices of their respective edges. Since for every $1 \leq i < t$, the vertex adjacent to e_i and e_{i+1} is exposed with respect to at least one of e_i and e_{i+1}, then at least $t+1$ times a vertex of H is exposed with respect to one of $e_1, \ldots, e_t$. It follows now from the pigeonhole principle that one of the edges $e_1, \ldots, e_t$ must have both of its vertices exposed which is a contradiction.

To see the second part of the claim, assume to the contrary that v is a reflex vertex in M that does not lie on the boundary of S (consider for example the vertex v_1 in Figure 4). Let $g_0, \ldots, g_d$ denote the edges adjacent to v in clockwise order such that the clockwise angle between g_d and g_0 is strictly greater than π (in Figure 4 $g_0 = e_0$ and $g_d = e_1$).

Let H denote the face of M that has v as a vertex and the internal angle of H at v is greater than π. Notice that H must be a hole. g_0 and g_d are therefore two adjacent edges of H meeting at v. By the first part of the claim, no edge of H is part of the boundary of S. Notice, therefore, that every edge of H is either an edge or a portion of an edge of one of the convex sets $C_1, \ldots, C_k$.

Observe further that because v is a reflex vertex it is exposed both as a vertex of g_0 and as a vertex of g_d, where g_0 and g_d are viewed as edges of H. Because every vertex of H is exposed for at least one of the two edges in H adjacent to it, it follows from the pigeonhole principle that there must be an edge of H such that both of its vertices are exposed. As we have seen already, this is a contradiction.

To see the third part of the claim let H be a hole as assumed with vertices $v_0, \ldots, v_{t-1}$. Recall that every vertex of H is exposed with respect to one of the edges adjacent to it and no edge of H has both of its vertices exposed with respect to it. It follows that every vertex of H is exposed with respect to precisely one edge adjacent to it (rather than two). Otherwise, from the pigeonhole principle, there is an edge of H whose both vertices are exposed, which is a contradiction.

If v_0 is not exposed with respect to v_0v_1, then it is exposed with respect to the edge $v_{t-1}v_0$. But then v_{t-1} is not exposed with respect to $v_{t-1}v_0$ and so v_{t-1} is exposed with respect to the edge $v_{t-2}v_{t-1}$. We can continue with this argument and conclude that in general every v_j cannot be exposed with respect to to v_jv_{j+1} and therefore it must be exposed with respect to the edge $v_{j-1}v_j$. In this case H is oriented counterclockwise.

If v_0 is not exposed with respect to $v_{t-1}v_0$, then it is exposed with respect to the edge v_0v_1. But then v_1 is not exposed with respect to v_0v_1 and so v_1 is exposed with respect to the edge v_1v_2. We can continue with this argument and conclude that in general every v_j cannot be exposed with respect to the edge $v_{j-1}v_j$ and therefore it must be exposed with respect to the edge v_jv_{j+1}. In this case H is oriented clockwise (see Figure 5).

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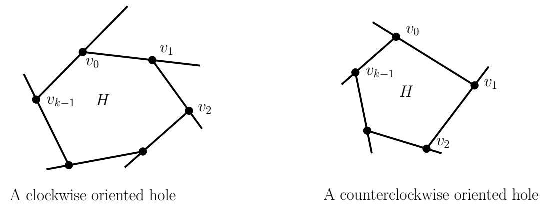

Figure 5: Typical clockwise and counterclockwise oriented holes.

In the next step, that we take just for convenience and the simplification of the presentation, we show that we may assume that every vertex of M has degree equal to 3. We do this by deforming slightly the map M in such a way that has only a minor effect on the perimeter of the sets C_i and the set S. There is more than one way to do this. We note that after the deformation of the map M we can no longer claim the maximality of $\sum_i^k Per(C_i)$. However, we will not use this maximality in the sequel. The upper bound we will derive for the sum $\sum_i^k Per(C_i)$ after the deformation is valid also for the sum $\sum_i^k Per(C_i)$ before the deformation, as the two sums are arbitrarily close.

The way we suggest to deform the map M is as follows. Assume we have a vertex v of M of degree greater than 3. Suppose first that v does not lie on the boundary of S. Therefore, by Claim 2, v is a convex vertex of G.

We split into two cases.

Case 1. v is a flat vertex. Let $e_0, \ldots, e_{d-1}$ denote the edges adjacent to v listed according to their clockwise cyclic order around v. Because v is flat, we may assume without loss of generality that e_0 and e_{d-1} are collinear. On the edge e_0 we designate $d-3$ vertices $z_1, \ldots, z_{d-3}$ (listed according to their decreasing distance from v) very (very) close to v. We then, close to the vertex v, slightly (very slightly) bend each of the edges $e_1, \ldots, e_{d-3}$ so that they end at the vertices $z_1, \ldots, z_{d-3}$, respectively (see Figure 6).

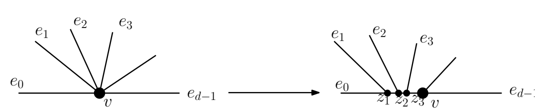

Figure 6: The deformation close to a flat vertex.

Case 2. v is a convex vertex but not flat. v lies in the convex hull of its neighbors and therefore in the convex hull of three of its neighbors. Let e_1, e_2, e_3 denote the edges connecting v to three such neighbors, listed according to their clockwise cyclic order around v.

We show how to deform the edges adjacent to v between e_1 and e_2 and similar treatment should be given to the edges adjacent to v between e_2 and e_3 and between e_3 and e_1.

Let $g_1, \ldots, g_d$ denote the edges adjacent to v strictly between e_1 and e_2, listed according to their clockwise cyclic order around v.

On the edge e_1 we designate d vertices $z_1, \ldots, z_d$ (listed according to their decreasing distance from v) very (very) close to v. We then, close to the vertex v, slightly (very slightly) bend each of the edges $g_1, \ldots, g_d$ so that they end at the vertices $z_1, \ldots, z_d$, respectively (see Figure 7).

If v is a vertex on the boundary of S, we do something very similar. Let $e_0, \ldots, e_{d-1}$ denote the edges adjacent to v, listed according to their clockwise cyclic order around v. Without loss of

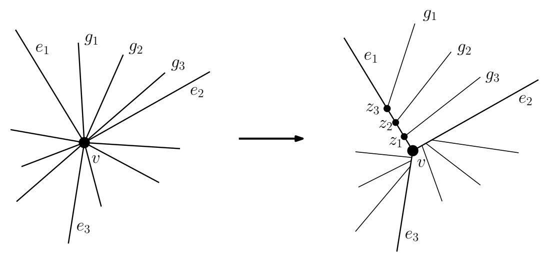

Figure 7: The deformation close to a vertex not on the boundary of S.

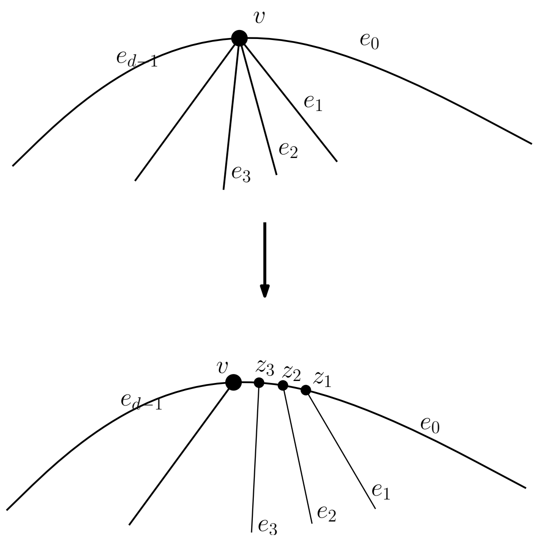

Figure 8: The deformation close to a vertex on the boundary of S.

generality assume that e_0 and e_{d-1} are the two edges adjacent to v that are part of the boundary of S.

On the edge e_0 we designate $d-3$ vertices $z_1, \ldots, z_{d-3}$ (listed according to their decreasing distance from v) very (very) close to v. We then, close to the vertex v, slightly (very slightly) bend each of the edges $e_1, \ldots, e_{d-3}$ so that they end at the vertices $z_1, \ldots, z_{d-3}$, respectively (see Figure 8).

After completing the above step we have a map M such that the degree of each vertex is equal to 3. The edges in M that are not contained in the boundary of S are arbitrarily close to being straight line segments. The bounded faces of M are arbitrarily close (in the Hausdorff metric) to being convex. Hence, in the sequel, we can treat the bounded faces of M as convex and the edges of M, not contained in the boundary of S, as straight line segments. By doing so, our estimates on the total perimeter of $C_1, \ldots, C_k$ will be correct up to arbitrarily small error.

This is a convenient point to make one important remark about holes in M. We notice that after the deformation of M no two holes may share a vertex. Indeed, suppose that v is a vertex of two different holes H and H'. By the third part of Claim 2, v lies in the relative interior of an edge of C_i (that overlaps with an edge of H) as well as in the relative interior of an edge of C_j (that overlaps with an edge of H') for some $i, j \in \{1, \ldots, k\}$. It is not possible that $i \neq j$ because this would imply that C_i and C_j are not interior-disjoint. Therefore, it must be that $i = j$. This

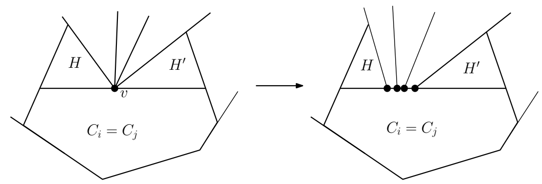

Figure 9: On the left: the only possibility for two holes to share a vertex before the deformation. On the right: the situation after the deformation.

situation is described in the left side of Figure 9. However, notice that v is a flat vertex and after the deformation H and H' are vertex-disjoint (see the right side of Figure 9).

We denote by F_B the number of bounded faces of the map M.

Claim 3. *The number of vertices of M is equal to $2(F_B - 1)$.*

Proof. Notice that because of the maximality of $C_1, \dots, C_k$ we know that M is connected. Denote as usual by V, E, and F the number of vertices, edges, and faces (including the unbounded face), respectively, in the map M. By Euler's formula we have

$$V - E + F = 2. \tag{1}$$

Because the degree of each vertex in M is equal to 3, we have

$$3V = 2E. \tag{2}$$

We also have $F = F_B + 1$. This, together with (1) and (2), gives $V = 2(F_B - 1)$. ∎

3 The proof in the case where S is contained in a ball of diameter $Diam(S)$ and M has no holes.

In this section we show how to prove the Theorem 1 in the case where S is contained in a disc of diameter $Diam(S)$ and M has no holes. We do this mainly for didactic reasons as this case is more elegant while at the same time the argument contains all the important ideas in the proof of the general statement. Therefore, M has exactly k bounded faces and it follows from Claim 3 that M has precisely $2(k-1)$ vertices.

Assume without loss of generality that S is contained in the disc of diameter $Diam(S)$ centered at the origin O.

Those edges of M that are not part of the boundary of S we call *internal edges*. In fact, the internal edges of M are precisely the edges of the geometric graph G. Every internal edge of M is a straight line segment. We denote the set of internal edges of M by $I(M)$.

The total perimeter of the sets $C_1, \dots, C_k$ is equal to the perimeter of S plus twice the total length of the internal edges of M. This is because every internal edge of M is a portion of the boundary of two of the sets $C_1, \ldots, C_k$ (due to the fact that no face in M is a hole). Hence, it remains to show that the total length of all internal edges of M does not exceed $(k-1)Diam(S)$.

We shall now give an overview of the rest of the argument before introducing it formally. The idea of the proof is to observe that the length of a segment is proportional to (in fact half of) the integrated value of its projection on the x-axis while rotated at angle that varies from 0 to π. Given any rotation of the map M we will show that its set of edges can be decomposed into $k-1$ x-monotone polygonal paths, where each path is delimited by two vertices of M and each vertex is an endpoint of exactly one path. We can therefore estimate from above the total length of the projections of all edges of M on the x-axis by the values of the x-coordinates of the vertices of M at any moment of the rotation. We will then show how to obtain such an upper bound that will be good enough to prove the theorem under our assumptions.

In order to formalize this idea we introduce some notation. For every point $a = (a_1, a_2)$ we define $D_x(a) = a_1$, that is, the projection of a on the x-axis.

For a set C in the plane and an angle θ we denote by C^θ the set obtained from C by rotating it counterclockwise about the origin at angle θ. We denote by $f(C)$ the length of the projection of C on the x-axis. For a line segment e we denote by $|e|$ the Euclidean length of the segment e.

We will need the following well known fact about bounded convex sets in the plane, whose proof we omit. Notice that for a set C in the plane $f(C^\theta)$ is the length of the projection of C on the x-axis after rotated counterclockwise at angle θ.

Lemma 1. *For a bounded convex set C in the plane we have*

$$\int_0^\pi f(C^\theta)d\theta = Per(C).$$

In particular, if we take C in Lemma 1 to be equal to a line segment e, then we have $\int_0^\pi f(e^\theta)d\theta = 2|e|$. We remark that the principle of Lemma 1 will be used several times in the rest of this text, possibly without a reference.

Recall that we need to show that $\sum_{e \in I(M)} |e| \leq (k-1)Diam(S)$. We have:

$$\sum_{e \in I(M)} |e| = \frac{1}{2} \sum_{e \in I(M)} \int_0^\pi f(e^\theta)d\theta = \frac{1}{2} \int_0^\pi \sum_{e \in I(M)} f(e^\theta)d\theta. \tag{3}$$

The sum $\sum_{e \in I(M)} f(e^\theta)$ is the sum of the lengths of the projections of the internal edges of M after M is rotated counterclockwise at angle θ. Fix an angle θ and consider the map M^θ obtained from M by a rotation at angle θ around the origin O in the counterclockwise direction.

Denote by $I(M^\theta)$ the set of internal edges of M^θ. $I(M^\theta)$ is a collection of straight line segments. We decompose the edges in $I(M^\theta)$ into x-monotone polygonal paths in the following way. Set $A = I(M^\theta)$. Pick any edge $e \in A$ and let P_1 be any maximal (under containment) x-monotone path containing e, composed of edges in A (see Figure 10).

Then remove the edges in P_1 from A and continue in the same manner with the updated set of edges A.

We thus get a collection of x-monotone polygonal paths $P_1, \ldots, P_t$.

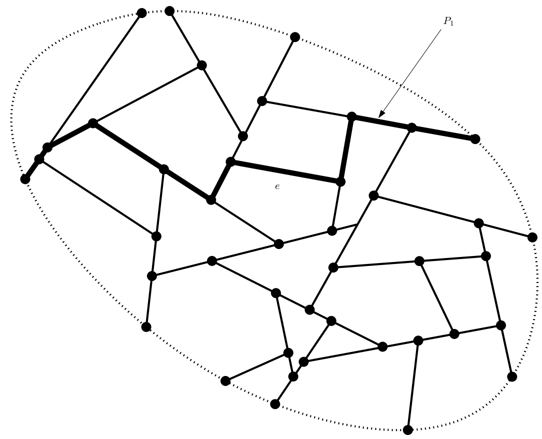

Figure 10: Decomposing into x-monotone paths: the path P_1.

Claim 4. *We must have $t = k - 1$ and moreover, every vertex of M^θ is an endpoint (either left endpoint or right endpoint) of precisely one of the paths $P_1, \ldots, P_t$.*

Proof. We consider the geometric graph obtained from M^θ by ignoring the boundary of (the rotation of) S. The resulting geometric graph G^θ has $2(k-1)$ vertices. Some of the vertices have degree 1. These are precisely those vertices that lie on the boundary of S. The rest of the vertices in G^θ have degree 3 and they are convex vertices, that is, they lie in the convex hull of their neighbors. We first show that every vertex of G^θ (and therefore also of M^θ) is an endpoint of at least one polygonal path P_i. Consider a vertex v. If v is not an endpoint of any of the polygonal paths $P_1, \ldots, P_t$, then the degree of v in each of those paths is even and as those paths are edge-disjoint, this implies that the degree of v is even in the graph G^θ, which is not the case. To see that v is an endpoint of precisely one of the paths $P_1, \ldots, P_t$, assume to the contrary that v is an endpoint of two chains P_i and P_j. Observe that it is not possible that v if the left endpoint for one path and the right endpoint for the other because then those two paths could be unified to be one longer path contradicting the rules of our construction.

Assume, therefore, without loss of generality that v is a left endpoint of both P_i and P_j (the case where v is the right endpoint of both paths is symmetric). Because v belongs to two paths, the degree of v in G^θ cannot be equal to 1 and it is therefore equal to 3. Consider the vertical line ℓ through v. There are two edges adjacent to v that lie in the half-plane bounded to the right of ℓ. These are the two edges adjacent to v in the paths P_i and P_j. Because v is a convex vertex of G there must be an edge e adjacent to v the lie on the half-plane bounded to the left of v. Observe that v must be the right endpoint of the path that contains e. This is impossible, as we have seen already, as v is already a left endpoint of another path (even two). ∎

Notice that the sum $\sum_{e \in I(M)} f(e^\theta)$ in (3) is nothing else but the sum of the lengths of the projections of the paths $P_1, \ldots, P_{k-1}$ on the x-axis. This is because each path P_i is x-monotone and the paths $P_1, \ldots, P_{k-1}$ form a partitioning of the set of edges in $I(M^\theta)$. Consider a path P_i. It has a left endpoint v_i and a right endpoint u_i. The length of the projection of P_i on the x-axis is equal to $D_x(u_i) - D_x(v_i)$. This, in turn, is bounded from above by $|D_x(u_i)| + |D_x(v_i)|$. Because every vertex of G^θ is an endpoint of precisely one path and the endpoints of every path are vertices of G^θ, we have proved the following inequality:

$$\sum_{e\in I(M)} f(e^{\theta}) \le \sum_{v\in V(G^{\theta})} |D_x(v)| = \sum_{v\in V(M^{\theta})} |D_x(v)|. \tag{4}$$

Going back to equality (3), we need to estimate from above the integral $\int_0^{\pi} \sum_{e\in I(M)} f(e^{\theta})d\theta$. In view of (4), we have:

$$\int_0^{\pi} \sum_{e\in I(M)} f(e^{\theta})d\theta \le \int_0^{\pi} \sum_{v\in V(M^{\theta})} |D_x(v)|d\theta. \tag{5}$$

Recall that M^{θ} is a rotation of M around the origin in the counterclockwise direction at angle θ. By our assumption, M is contained in the disc of diameter $Diam(S)$ around the origin.

Therefore, every vertex v of M contributes to the right hand side of (5) precisely $\int_0^{\pi} |Ov| \sin\theta d\theta \le \frac{Diam(S)}{2}\int_0^{\pi} \sin\theta d\theta = \frac{1}{2}Diam(S)2 = Diam(S)$.

Therefore, we have

$$\sum_{e\in I(M)} |e| = \frac{1}{2}\int_0^{\pi} \sum_{e\in I(M)} f(e^{\theta})d\theta \le \frac{1}{2}\int_0^{\pi} \sum_{v\in V(M^{\theta})} |D_x(v)|d\theta \le \frac{1}{2}2(k-1)Diam(S) = (k-1)Diam(S). \tag{6}$$

This completes the proof of the theorem in the case where S is contained in a disc of diameter $Diam(S)$ and the map M does not have any holes.

4 The proof in the case where M contains no holes.

In this section we show how to get rid of the assumption, made in Section 3, that S is contained in a ball of diameter $Diam(S)$. We will prove Theorem 1 under the only assumption that M contains no holes (in the next section we will prove the theorem in the most general case).

The main ideas in the proof already appear in the more restricted case above where S is contained in a disc of diameter $Diam(S)$. We start with some auxiliary notation and observations.

Given a set P of $2n$ points in the plane, a *halving line* of P is any line ℓ such that there are at most n points of P in each of the two open half-planes bounded by ℓ.

Given an angle $0 \le \theta < \pi$ we define the function $h(P,\theta)$ to be the sum of the distances of the points of P from a halving line for the set P whose slope is $\tan\theta$. Observe that $h(P,\theta)$ is well defined even when there is more than one choice of a halving line for the set P.

Theorem 2. *Let P be any set of $2n$ points in the plane. Then $\int_0^{\pi} h(P,\theta)d\theta \le 2nDiam(P)$.*

Proof. We prove the theorem by induction on the number of points of P that lie in the interior of the convex hull of P. If this number is 0, then P is in convex position, that is, all the points in P are on the boundary of the convex hull of P. Let $p_1 \ldots, p_{2n}$ denote the points of P as they appear in a clockwise cyclic order on the boundary of the convex hull of P. Notice that for every $0 \le i < n$ and every halving line ℓ of the set P the points p_i and p_{i+n} are separated by ℓ (unless ℓ is the unique line that passes through both p_i and p_{i+n}). Therefore the sum of the distances of

p_i and p_{i+n} to any halving line ℓ is equal to the length of the orthogonal projection of the segment p_ip_{i+n} on a line perpendicular to ℓ.

Therefore,

$$\begin{aligned}\int_0^{\pi} h(P,\theta)d\theta &= \sum_{i=1}^{n}\int_0^{\pi}|p_ip_{i+n}||\cos t|dt \\ &= \sum_{i=1}^{n} 2|p_ip_{i+n}| \leq 2nDiam(P).\end{aligned}$$

This proves the theorem in the case where the points of P are in convex position. Assume therefore that there is a point $p \in P$ that does not lie on the boundary of the convex hull of P. Let ℓ be any line through p that does not pass through any other point of P. Assume without loss of generality that ℓ is the x-axis. We will show that there is a direction such that if we push p in this direction along ℓ, then the integral $\int_0^{\pi} h(P,\theta)d\theta$ does not decrease. Once we establish this fact, then we move p along ℓ, while not decreasing $\int_0^{\pi} h(P,\theta)d\theta$, until p reaches the boundary of the convex hull of P and conclude the theorem from the induction hypothesis (notice that $Diam(P)$ depends only on the convex hull of P).

Fix θ between 0 and π and consider the function $T_\theta : \ell \rightarrow \mathbb{R}$ defined by $T_\theta(x) = h(P \setminus \{p\} \cup \{x\}, \theta)$. We claim that T_θ is a convex function of x as a point on ℓ. Recall that we identify ℓ with the x-axis. Let m be a line with slope $\tan\theta$ through a point of $P \setminus \{p\}$ such that there are precisely $n-1$ points of $P \setminus \{p\}$ in the open half-plane that is bounded to the left of m (that is, the half plane that contains points x on the x-axis where x is arbitrarily small). Such a line m exists except for only finitely many values of θ (that do not influence $\int_0^{\pi} T_\theta(x)d\theta$).

Observe that for every point x on the x-axis the line m is a halving line for the set $P \setminus \{p\} \cup \{x\}$. This is because there are precisely (or more accurately, at most) $n-1$ points of $P \setminus \{p\}$ in each of the two open half-planes bounded by m and therefore, there are at most n points of $P \setminus \{p\} \cup \{x\}$ in each of the two open half-planes bounded by m. Therefore, the function $T_\theta(x)$ is equal to a constant (which is the sum of the distances of the points in $P \setminus \{p\}$ from m) plus the distance of the point x from m, which is a convex function of x.

It follows now that also the function $\int_0^{\pi} T_\theta(x)d\theta = \int_0^{\pi} h(P \setminus \{p\} \cup \{x\}, \theta)d\theta$ is a convex function of x. This implies that there is a direction such that if we push p in this direction along ℓ, then the integral $\int_0^{\pi} h(P,\theta)d\theta$ does not decrease. ∎

Theorem 2 implies now Theorem 1 in the case where the map M contains no holes. Indeed, recall that two times the sum of the lengths of the internal edges of M, that we denoted by $I(M)$, is equal to $\sum_{e\in I(M)}\int_0^{\pi} f(e^\theta)d\theta = \int_0^{\pi}(\sum_{e\in I(M)} f(e^\theta))d\theta$. Recall that when M has no holes then it has precisely $2(k-1)$ vertices and for every θ we can decompose $I(M^\theta)$ into $(k-1)$ x-monotone paths $P_1, \ldots, P_{k-1}$. For every $1 \leq i \leq k$ we denote by v_i and u_i the left endpoint and right endpoint, respectively, of P_i. Recall that every vertex of M^θ appears precisely once either as u_i or as v_i for some $1 \leq i \leq k-1$. The length of the projection of the edges of P_i on the x-axis is equal to $D_x(u_i) - D_x(v_i)$. This, in turn, is bounded from above by the sum of the distances of u_i and v_i from any vertical line, and in particular from the vertical halving line of the set of vertices of M^θ. Therefore, $\sum_{e\in I(M)} f(e^\theta) \leq h(V(M),\theta)$. Consequently,

$$\int_0^{\pi}\sum_{e\in I(M)} f(e^\theta)d\theta \leq \int_0^{\pi} h(V(M),\theta)d\theta \leq 2(k-1)Diam(V(M)) \leq 2(k-1)Diam(S).$$

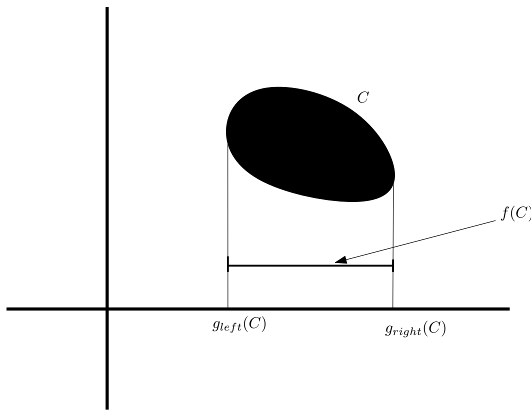

Figure 11: Some notation.

5 Adding the *holes* into the proof

In this section we show how to extend our argument above to the most general case where the map M may contain holes.

We now introduce two auxiliary functions:

Definition 2. Let C be a compact convex set in the plane. We denote by $g_{right}(C)$ the largest x-coordinate of a point in C. Similarly, we denote by $g_{left}(C)$ the smallest x-coordinate of a point in C.

Therefore, $g_{right}(C) - g_{left}(C)$ is equal to $f(C)$, the length of the projection of C on the x-axis (see Figure 11).

We recall that a hole is a face of M that is not covered by any of the convex sets $C_1, \ldots, C_k$. As we have already seen in Claim 2, every hole must be convex and no vertex of a hole may lie on the boundary of S. Recall moreover that no two holes can share a vertex.

Denote by m the number of holes in the map M and let $H_1, \ldots, H_m$ denote these m holes. Therefore the number of bounded faces in the map M is $k + m$. We follow the same line of arguments as in the case of no holes and therefore we reduce to the case where every vertex of M has degree equal to 3. By Claim 3, M has $2(k + m - 1)$ vertices.

The place where we act differently than in the proof in the case of no holes is when we decompose the internal edges of M^θ into x-monotone paths. Before we decompose $I(M^\theta)$ into x-monotone paths, we consider each hole H_i^θ. The boundary of H_i^θ is a union of two x-monotone polygonal paths connecting the rightmost vertex of H_i^θ with the leftmost vertex of H_i^θ. What we do is that for every hole H_i^θ we ignore one of these paths in the sense that we delete its edges from M^θ and decompose only the remaining edges of $I(M^\theta)$ (see Figure 12).

Notice that the rightmost and leftmost vertices of H_i^θ are no longer endpoints of any of the paths in the decomposition. Therefore, we obtain again $k-1$ x-monotone paths $P_1, \ldots, P_{k-1}$ such that every vertex of M^θ that is not the rightmost or the leftmost vertex of any hole must be an endpoint of exactly one of the paths $P_1, \ldots, P_{k-1}$. Notice, however, that when we now sum up the lengths of the projections of the chains $P_1, \ldots, P_{k-1}$ on the x-axis this sum is no longer equal to the sum of the lengths of the projections of all edges in $I(M^\theta)$ but it equals precisely $T(\theta) = \sum_{e \in I(M)} f(e^\theta) - \sum_{i=1}^{m} f(H_i^\theta)$. When we integrate this sum over $0 \leq \theta \leq \pi$ we get:

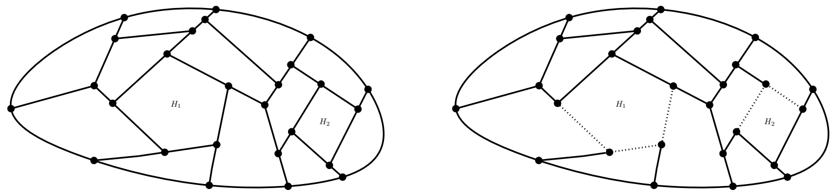

Figure 12: Modifying the proof when there are holes.

$$\int_0^\pi T(\theta)d\theta = \int_0^\pi \sum_{e\in I(M)} f(e^\theta)d\theta - \int_0^\pi \sum_{i=1}^m f(H_i^\theta)d\theta = \sum_{e\in I(M)} 2|e| - \sum_{i=1}^m Per(H_i). \quad (7)$$

Notice that the right hand side of (7) is precisely the sum of the lengths of the boundary edges of $C_1, \ldots, C_k$ that do not lie on the boundary of S. We will be done once we show that this expression is bounded from above by $2(k-1)Diam(S)$.

Let $q : [0, \pi] \to \mathbb{R}$ be any (continuous) function. We should think of $q(\theta)$ as representing the vertical line $x = q(\theta)$ at time θ. Fix θ between 0 and π. Consider now the decomposition of $I(M^\theta)$ into $k-1$ x-monotone paths and let u_i and v_i denote the right and the left endpoints of P_i, respectively. As in the analysis in the case where there are no holes, the sum of the lengths of the projections of the paths $P_1, \ldots, P_{k-1}$ is equal to $\sum_{i=1}^{k-1} D_x(u_i) - D_x(v_i)$. This expression is bounded from above by $\sum_{i=1}^{k-1} |D_x(u_i) - q(\theta)| + |D_x(v_i) - q(\theta)|$, regardless of the value of $q(\theta)$.

Therefore, the sum of the lengths of the projections of all paths $P_1, \ldots, P_{k-1}$ on the x-axis is bounded from above by

$$\sum_{v\in V(M^\theta)} |D_x(v) - q(\theta)| - \sum_{i=1}^m (|g_{right}(H_i^\theta) - q(\theta)| + |g_{left}(H_i^\theta) - q(\theta)|).$$

Notice that here we subtracted the contribution of the leftmost and rightmost vertices of the holes $H_1, \ldots, H_m$ to the sum $\sum_{v\in V(M^\theta)} |D_x(v) - q(\theta)|$, as those vertices are never endpoints of any of the paths $P_1, \ldots, P_{k-1}$.

We can further improve on this upper bound as follows. We define an auxiliary function $Err(v, q(\theta), \theta)$ where v is any vertex of M^θ. We consider the geometric graph G^θ and recall that the degree of v in G^θ is either 1 or 3. If v is not a rightmost or leftmost vertex of any hole H_i^θ, then we define $Err(v, q(\theta), \theta) = 2|D_x(v) - q(\theta)|$ in two cases. One case is if $D_x(v) > q(\theta)$ and v has (one) more edges adjacent to it in the graph G^θ going to the right than to the left. The other case where we define $Err(v, q(\theta), \theta) = 2|D_x(v) - q(\theta)|$ is when $D_x(v) < q(\theta)$ and v has (one) more edges adjacent to it in the graph G^θ that go to the left than to the right. In all other cases we set $Err(v, q(\theta), \theta) = 0$.

We claim that the expression

$$\sum_{v\in V(M^\theta)} |D_x(v) - q(\theta)| - \sum_{i=1}^m (|g_{right}(H_i^\theta) - q(\theta)| + |g_{left}(H_i^\theta) - q(\theta)|) - \sum_{v\in V(M^\theta)} Err(v, q(\theta), \theta) \quad (8)$$

is equal to the sum of the lengths of the projections of all paths $P_1, \ldots, P_k$ on the x-axis.

To see the reason for this, consider a path P_i with leftmost vertex v_i and rightmost vertex u_i. In particular, u_i and v_i are not the leftmost or rightmost vertices of any hole. As we have seen before u_i is an extreme vertex of precisely one of the paths and this path is P_i. This implies that u_i has one more edge going to the left in G^θ than to the right. If $D_x(u_i) < q(\theta)$, then $D_x(u_i) - D_x(v_i)$ is strictly smaller than $|D_x(u_i) - q(\theta)| + |D_x(v_i) - q(\theta)|$ and in fact it is equal to $|D_x(u_i) - q(\theta)| + |D_x(v_i) - q(\theta)| - Err(u_i, q(\theta), \theta)$. Notice that in this case $Err(v_i, q(\theta), \theta) = 0$ because in M^θ v_i has two edges going to the right and $D_x(v_i) < D_x(u_i) < q(\theta)$. A similar reasoning holds when $v_i > q(\theta)$. In this case $D_x(u_i) - D_x(v_i)$ will be equal to $|D_x(u_i) - q(\theta)| + |D_x(v_i) - q(\theta)| - Err(v_i, q(\theta), \theta)$ while $Err(u_i, q(\theta), \theta) = 0$.

The rest of the proof goes as follows. We will prove the following key lemma

Lemma 2. *For every hole H_i one can find two vertices of H_i, that will be denoted by a_i and b_i, such that for every (integrable) function $q(\theta)$ we have*

$$\int_0^\pi (|D_x(a_i^\theta) - q(\theta)| + |D_x(b_i^\theta) - q(\theta)|) d\theta \quad \leq$$

$$\int_0^\pi \big(|g_{right}(H_i^\theta) - q(\theta)| + |g_{left}(H_i^\theta) - q(\theta)| + \sum_{v \in V(H_i^\theta)} Err(v, q(\theta), \theta)\big) d\theta.$$

Once Lemma 2 is established, let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$. Notice that because no two holes share a vertex then $A \cup B$ consists of exactly $2m$ elements. We define for every θ the function $q(\theta)$ to be the x-coordinate of the vertical halving line of the set of $2(k-1)$ vertices $(V(M) \setminus (A \cup B))^\theta$.

Recall that the total length of all boundary edges of $C_1, \ldots, C_k$ that do not lie on the boundary of S is bounded from above by

$$\int_0^\pi \Big(\sum_{v \in V(M^\theta)} |D_x(v) - q(\theta)| - \sum_{i=1}^{m} (|g_{right}(H_i^\theta) - q(\theta)| + |g_{left}(H_i^\theta) - q(\theta)|) - \sum_{v \in V(M^\theta)} Err(v, q(\theta), \theta) \Big) d\theta.$$

By Lemma 2, this expression is bounded from above by

$$\int_0^\pi \sum_{v \in (V(M) \setminus (A \cup B))^\theta} |D_x(v) - q(\theta)| d\theta.$$

However, because $q(\theta)$ is the x-coordinate of the vertical halving line for the set $(V(M) \setminus (A \cup B))^\theta$, then the sum $\sum_{v \in (V(M) \setminus (A \cup B))^\theta} |D_x(v) - q(\theta)|$ is nothing else but $h(V(M) \setminus (A \cup B), \theta)$. Hence, by Theorem 2,

$$\int_0^\pi \sum_{v \in (V(M) \setminus (A \cup B))^\theta} |D_x(v) - q(\theta)| d\theta \leq 2(k-1) Diam(V(M) \setminus (A \cup B)) \leq 2(k-1) Diam(S).$$

which completes the proof of Theorem 1.

It remains to prove Lemma 2 that we state again for convenience:

Lemma 2 *For every hole H in M one can find two vertices of H, a and b, such that for every (integrable) function $q(\theta)$ we have*

$$\int_0^\pi |D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| d\theta \leq$$

$$\int_0^\pi \big(|g_{right}(H^\theta) - q(\theta)| + |g_{left}(H^\theta) - q(\theta)| + \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta)\big) d\theta. \tag{9}$$

Proof. We can assume that for every θ we have $g_{left}(H^\theta) \leq q(\theta) \leq g_{right}(H^\theta)$. This is because if for example $q(\theta) < g_{left}(H^\theta)$, then by increasing $q(\theta)$ to be equal to $g_{left}(H^\theta)$ we decrease both expressions $|D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)|$ and $|g_{right}(H^\theta) - q(\theta)| + |g_{left}(H^\theta) - q(\theta)|$ by the same amount while we can only decrease the term $\sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta)$. A similar argument is valid if $q(\theta) > g_{right}(H^\theta)$ and we decrease $q(\theta)$ to be equal to $g_{right}(H^\theta)$.

Under the assumption $g_{left}(H^\theta) \leq q(\theta) \leq g_{right}(H^\theta)$ we have

$$|g_{right}(H^\theta) - q(\theta)| + |g_{left}(H^\theta) - q(\theta)| = g_{right}(H^\theta) - g_{left}(H^\theta) = f(H^\theta)$$

and consequently

$$\int_0^\pi (|g_{right}(H^\theta) - q(\theta)| + |g_{left}(H^\theta) - q(\theta)|) d\theta = \int_0^\pi f(H^\theta) d\theta = Per(H).$$

Therefore, showing (9) reduces to showing the existence of two vertices a and b of H such that:

$$\int_0^\pi \big(|D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta)\big) d\theta \leq Per(H). \tag{10}$$

We will show that there exist such two vertices that are the two vertices of the same edge of H. Without loss of generality assume that the hole H is clockwise oriented. Let b and c be two vertices of H at maximum distance from each other. Let b' and b'' be the two neighbors of b on the boundary of H and let c' and c'' be the two neighbors of c on the boundary of H such that both triples b', b, b'' and c', c, c'' appear in those clockwise cyclic orders on the boundary of H. Without loss of generality assume that $\measuredangle bcc'' \geq \measuredangle cbb''$ and take $a = b'$.

Without loss of generality assume that ab is vertical and H lies to the left of the line through a and b. It is important to note that there are no vertices of H below the line through b that is parallel to cc''.

Let $\alpha = \measuredangle abc$ and let β be the measure of the angle between bb'' and the ray going from b'' upwards in the vertical direction (equivalently, $\beta = \pi - \measuredangle abb''$). We let e be the edge ab and let $e' = bc$. (See Figure 13.)

Claim 5. *For every $0 \leq \theta \leq \beta$ we have*

$$|D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta) \leq f(e^\theta). \tag{11}$$

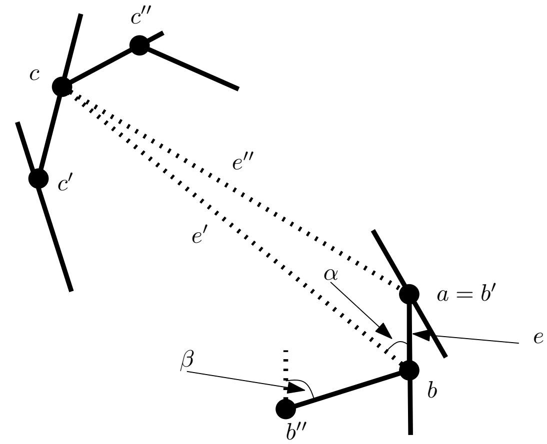

Figure 13: The proof of Lemma 2.

Proof. We give most of the details, leaving some verifications to the reader. Because H is oriented clockwise, for every $0 \le \theta \le \beta$, in the geometric graph G^θ there are two edges going to the right from a^θ and two edges going to the left from b^θ (notice moreover that $D_x(b^\theta) > D_x(a^\theta)$ whenever $0 \le \theta \le \beta$).

If $D_x(a^\theta) \le q(\theta) \le D_x(b^\theta)$, then

$$\begin{aligned} |D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in (H^\theta)} Err(v, q(\theta), \theta) &\le |D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| \\ &= D_x(b^\theta) - D_x(a^\theta) = f(e^\theta). \end{aligned}$$

If $q(\theta) < D_x(a^\theta)$, then notice that a^θ is neither the leftmost or the rightmost vertex of H^θ. In this case we have $Err(a^\theta, q(\theta), \theta) = 2(D_x(a^\theta) - q(\theta))$.

Therefore,

$$\begin{aligned} &|D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta) \\ &\quad \le |D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - Err(a^\theta, q(\theta), \theta) \quad = \quad D_x(b^\theta) - D_x(a^\theta) = f(e^\theta). \end{aligned}$$

Finally, notice that the case $q(\theta) > D_x(b^\theta)$ is impossible because for $0 \le \theta \le \beta$ the vertex b^θ is the rightmost vertex of H^θ. ∎

Claim 6. *For every $\beta \le \theta \le \pi - \alpha$ we have*

$$|D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta) \le f(e^\theta). \tag{12}$$

Proof. Here too we give most of the details, leaving some verifications to the reader. Because H is oriented clockwise, for every $\beta \le \theta \le \pi - \alpha$, in the geometric graph G^θ there are two edges going to the right from a^θ and two edges going to the right from b^θ (notice also that $D_x(b^\theta) > D_x(a^\theta)$ whenever $\beta \le \theta \le \pi - \alpha$). However, there are two edges, in G^θ, going to the left from c^θ, since $\measuredangle bcc'' \ge \measuredangle cbb''$. Also $D_x(c^\theta) \le D_x(b^\theta)$.

If $D_x(a^\theta) \le q(\theta) \le D_x(b^\theta)$, then

$$\begin{aligned} |D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta) &\le |D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| \\ &= D_x(b^\theta) - D_x(a^\theta) = f(e^\theta). \end{aligned}$$

If $q(\theta) < D_x(a^\theta)$, then notice that a^θ is neither the leftmost nor the rightmost vertex of H^θ. We have $Err(a^\theta, q(\theta), \theta) = 2(D_x(a^\theta) - q(\theta))$. Therefore,

$$\begin{aligned} &|D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta) \\ &\quad \le |D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - Err(a^\theta, q(\theta), \theta) = D_x(b^\theta) - D_x(a^\theta) = f(e^\theta). \end{aligned}$$

Finally, if $q(\theta) > D_x(b^\theta)$, then $q(\theta) > D_x(b^\theta) \ge D_x(c^\theta)$. Notice that c^θ is neither the leftmost nor the rightmost vertex of H^θ, because b^θ is to the left of it and c''^θ is to the right of it. We have

$$Err(c^\theta, q(\theta), \theta) = 2(q(\theta) - D_x(c^\theta)) \ge 2(q(\theta) - D_x(b^\theta))$$

and therefore:

$$\begin{aligned} &|D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta) \\ &\quad \le |D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - Err(c^\theta, q(\theta), \theta) \\ &\quad \le |D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - 2(q(\theta) - D_x(b^\theta)) = D_x(b^\theta) - D_x(a^\theta) = f(e^\theta). \end{aligned}$$

∎

Claim 7. *For every $\pi - \alpha \le \theta \le \pi$ we have*

$$|D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta) \le f(e^\theta) + 2f(e'^\theta). \tag{13}$$

Proof. Here again we leave some verifications to the reader. Because H is oriented clockwise, for every $\pi - \alpha \le \theta \le \pi$, in the geometric graph G^θ there are two edges going to the right from each of the vertices a^θ, b^θ. However, there are two edges going to the left from c^θ, since bc is the maximal segment and therefore $\measuredangle bcc'' < \pi - \alpha$. Notice also that for every $\pi - \alpha \le \theta \le \pi$ we have $D_x(a^\theta) \le D_x(b^\theta) \le D_x(c^\theta)$.

If $q(\theta) > D_x(c^\theta)$, then notice that c^θ is neither the leftmost or the rightmost vertex of H^θ. We have $Err(c^\theta, q(\theta), \theta) = 2(q(\theta) - D_x(c^\theta))$. Hence,

$$\begin{aligned}
&|D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta) \\
&\leq |D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - Err(c^\theta, q(\theta), \theta) \\
&= (q(\theta) - D_x(a^\theta)) + (q(\theta) - D_x(b^\theta)) - 2(q(\theta) - D_x(c^\theta)) \\
&= 2D_x(c^\theta) - D_x(b^\theta) - D_x(a^\theta) \\
&= (D_x(b^\theta) - D_x(a^\theta)) + 2(D_x(c^\theta) - D_x(b^\theta)) \quad = \quad f(e^\theta) + 2f(e'^\theta).
\end{aligned}$$

If $D_x(a^\theta) \leq q(\theta) < D_x(c^\theta)$, then

$$\begin{aligned}
&|D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta) \\
&\leq |D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| \\
&\leq |D_x(a^\theta) - D_x(c^\theta)| + |D_x(b^\theta) - D_x(c^\theta)| \\
&= (D_x(b^\theta) - D_x(a^\theta)) + 2(D_x(c^\theta) - D_x(b^\theta)) \quad = \quad f(e^\theta) + 2f(e'^\theta)
\end{aligned}$$

Finally, if $q(\theta) < D_x(a^\theta)$ then notice that a^θ is neither the leftmost or the rightmost vertex of H^θ. We have $Err(a^\theta, q(\theta), \theta) = 2|D_x(a^\theta) - q(\theta)|$.

Therefore,

$$\begin{aligned}
&|D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta) \\
&\leq |D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - Err(a^\theta, q(\theta), \theta) \quad = \quad D_x(b^\theta) - D_x(a^\theta) \\
&\qquad = \quad f(e^\theta) \leq f(e^\theta) + 2f(e'^\theta)
\end{aligned}$$

∎

Claim 5, Claim 6 and Claim 7 imply

$$\begin{aligned}
\int_0^\pi (|D_x(a^\theta) - q(\theta)| + |D_x(b^\theta) - q(\theta)| - \sum_{v \in V(H^\theta)} Err(v, q(\theta), \theta)) d\theta \quad &\leq \quad \int_0^\pi f(e^\theta) d\theta + \int_{\pi-\alpha}^\pi 2f(e'^\theta) d\theta \\
&= \quad 2|e| + 2|e'|(1 - \cos\alpha). \qquad (14)
\end{aligned}$$

Let $e'' = ac$. Clearly,

$$Per(H) \geq |e| + |e'| + |e''|. \tag{15}$$

In view of (14) and (15), a and b will satisfy (10) if we show that

$$|e| + |e'| - 2|e'|\cos\alpha \leq |e''|. \tag{16}$$

Notice that $\measuredangle abc = \alpha$ and therefore, by the cosine theorem, $|e''|^2 = |e|^2 + |e'|^2 - 2|e||e'|\cos\alpha$. Hence, after some easy manipulations (16) is equivalent to

$$|e|/2 \leq |e'|\cos\alpha,$$

which in turn is equivalent to $|e'| \geq |e''|$. The last inequality is true because of the maximality of $|e'|$. ∎

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