

A generalization of Thue's theorem to packings of non-equal discs, and an application to a discrete approximation of entropy

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Abstract

In this paper we generalize the classical theorem of Thue about the optimal circular disc packing in the plane. We are given a family of circular discs, not necessarily of equal radii, with the property that the inflation of every disc by a factor of 2 around its center does not contain any center of another disc in the family (notice that this implies that the family of discs is a packing). We show that in this case the density of the given packing is at most $\frac{\pi}{2\sqrt{3}}$, which is the density of the optimal unit disc packing.

1 Introduction

By a *disc* we will always mean a circular disc in the plane, that is, the set of points in the plane whose distance from some point, the center of the disc, is smaller than or equal to a number r , the radius of the disc. A *packing* is a family of pairwise disjoint discs in \mathbb{R}^2 . When a packing is contained in some bounded set S , then the *density* of the packing is the percentage of the volume of S that is covered by the sets of the packing. The notion of density of a packing is generalized also for unbounded sets S by exhausting them with bounded sets, usually the intersection of S with larger and larger balls (or cubes) centered, say, at the origin.

The optimal unit disc packing in the plane is a celebrated classical problem that goes back to the 1663 famous Kepler's problem about the densest unit sphere packing in \mathbb{R}^3 . In 1773 Lagrange proved that among all lattice packings in \mathbb{R}^2 , that is packing of unit discs where the sets of centers is a lattice in \mathbb{R}^2 , the densest one (the hexagonal lattice) has density $\frac{\pi}{2\sqrt{3}}$. The densest unit disc packing problem in the plane, without any additional assumption, was solved only in 1910 by Thue [9] and ever since this result is known as Thue's theorem. Thue's proof was considered incomplete and a full complete proof of the theorem was given in 1943 by L. F. Tóth [10]. Since then more proofs, each more elegant than the other, were presented, as this theorem and topic attracted quite some attention (see [8, 4, 6, 1]).

It is not hard to see that for any bounded (open) set S in the plane, one can find a packing of discs contained in S , not necessarily of the same radii, whose density is arbitrarily close to 1.

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Indeed, assume we are given a disc packing \mathcal{P} in S of density $\alpha < 1$. Consider the subset of S not covered by the discs in \mathcal{P} and find a union of pairwise disjoint squares covering at least half of its area. Inside each square we can place a disc covering at least half of the area of the square. Altogether if we add those discs to our packing \mathcal{P} we get another packing whose density is at least $\alpha' > \alpha + \frac{1}{4}(1 - \alpha)$. Equivalently, $1 - \alpha' < \frac{3}{4}(1 - \alpha)$. We see from here that by repeating this procedure many times we can get disc packings with density β such that $1 - \beta$ is arbitrarily close to 0.

Therefore, the question of optimal disc packing with no further assumptions on the family of discs in the packing is not very interesting. There are, however, some works about packing of non-congruent discs. In [11], Tóth observed that the optimal density of disc packing in the plane remains $\frac{\pi}{2\sqrt{3}}$ even if we allow the radii of the discs to be in the interval between 0.906 and 1. This interval has been extended to $[0.702, 1]$ in [2]. Likos and Henley [7] consider the optimal density of disc packing that contains only discs of radii 1 and r where $r < 1$ is given. Even this, seemingly simple, problem turns to be difficult for almost all values of r with very few exceptions (see [5]).

In this paper we consider the density of packings of circular discs with different radii in a way that generalizes Thue's theorem on one hand and does not follow from any of the known proofs of Thue's theorem on the other hand.

We say that a family \mathcal{F} of discs in the plane is *locally finite* if every bounded set in the plane may contain only finitely many discs in \mathcal{F} . Notice that when considering the density of a packing \mathcal{F} there is not much loss of generality by assuming that \mathcal{F} is locally finite. This is because we can partition the plane into say unit squares. In each unit square discard all the discs whose radius is small enough so that altogether all the discarded discs do not cover more than a very small percentage of the unit square in question. By doing this we remain with a locally finite family of discs and the overall density of our packing reduces only by arbitrarily small number.

We say that \mathcal{F} has *sub-linear radii growth* if as n goes to infinity the maximum radius of a disc of \mathcal{F} whose center is contained in a ball of radius n around the origin is $o(n)$. Notice in particular that if the radii of the discs in \mathcal{F} are bounded, then clearly \mathcal{F} has sub-linear radii growth. It will be convenient for us to assume that our packing has sub-linear radii growth in order to avoid discussing "boundary effects" when considering the density of the packing \mathcal{F} restricted to a large ball. We remark that for any packing \mathcal{F} , the maximum radius of a disc in \mathcal{F} whose center is contained in a ball B of radius n around the origin (assuming B contains at least two such centers, which is true when n is large enough) is clearly at most $2n$.

Theorem 1. *Let \mathcal{F} be a locally finite collection of circular discs in the plane with the property that the inflation of every disc around its center by a factor of 2 does not contain any of the centers of the other discs in \mathcal{F} (notice that such \mathcal{F} is necessarily a packing). Assume that \mathcal{F} has sub-linear radii growth. Then the density of the packing \mathcal{F} is not larger than the density of the optimal unit disc packing, namely $\frac{\pi}{2\sqrt{3}}$.*

Notice that the factor of 2 in Theorem 1 is best possible and cannot be replaced by a smaller number. Indeed, observe that a unit disc packing satisfies the condition in Theorem 1, as the distance between any two centers in a unit disc packing is at least 2. Therefore, Theorem 1 generalizes Thue's theorem. If we take an optimal unit disc packing, with density $\frac{\pi}{2\sqrt{3}}$, and inflate each disc by a factor of $1 + \epsilon$ (for small positive ϵ) around its center, then the density of the union of all discs in the family (which is not a packing anymore) is strictly greater than $\frac{\pi}{2\sqrt{3}}$. Observe

that the family of these inflated discs satisfies the condition in Theorem 1 once we replace 2 with $\frac{2}{1+\epsilon}$. It could be, however, that one could replace the 2 in Theorem 1 by a smaller number, with the additional assumption that \mathcal{F} is a packing.

We remark that the condition in Theorem 1 that the inflation of every discs in \mathcal{F} by a factor of 2 does not contain a center of any other disc in \mathcal{F} is equivalent to that the radius of every disc D in \mathcal{F} is at most $\frac{1}{2}$ times the smallest distance from the center of D to a center of another disc in \mathcal{F} . We could therefore assume, without loss of generality, that for every disc $D \in \mathcal{F}$ the radius of D is *equal* to $\frac{1}{2}$ times the smallest distance from the center of D to a center of another disc in \mathcal{F} .

2 Entropy approximation

In this section we introduce another motivation for Theorem 1, which is beyond the scope of combinatorial geometry. Here we describe the claim and sketch a proof. The interested reader may consult [3] for a detailed discussion and further results. The non-interested reader may skip this section, since no part of it is needed in the rest of this text. However, to understand this section we need the definition of $cell(D)$ corresponding to a disc D , as defined in the first paragraph of section 3.

Consider the set $\mathcal{B}(\Omega)$ of Borel probability measures on a "nice", compact set $\Omega \subset \mathbb{R}^2$ (we may assume it is a disc, or square). The *entropy* of a measure $\mu \in \mathcal{B}(\Omega)$ is defined as the Lebesgue integral

$$E(\mu) := \int_{\Omega} \rho \ln \rho ,$$

where $\mu := \rho dx$ if such a density exists, or $E(\mu) = +\infty$ if such a density does not exist.

Our object is to find a proper approximation of the entropy on the class of *N-empirical measures*:

$$\mathcal{B}_N(\Omega) := \left\{ \mu_N = N^{-1} \sum_{i=1}^N \delta_{x_i} ; \quad x_i \in \Omega , \quad x_i \neq x_j \text{ for } i \neq j , \quad 1 \leq i, j \leq N \right\} \subset \mathcal{B}(\Omega)$$

where $N \in \mathbb{N}$. Let $\mathcal{B}_{\infty}(\Omega) := \cup_{N \in \mathbb{N}} \mathcal{B}_N(\Omega)$.

Since Ω is a compact set, $\mathcal{B}(\Omega)$ is compact with respect to the weak ($C_b^*(\Omega)$) topology, that is, for every sequence $\{\mu_j\} \in \mathcal{B}(\Omega)$ there exists a subsequence μ_{j_k} and a measure $\mu \in \mathcal{B}(\Omega)$ such that $\mu_{j_k} \rightharpoonup \mu$ as $k \rightarrow \infty$, that is:

$$\lim_{k \rightarrow \infty} \int_{\Omega} \phi d\mu_{j_k} = \int_{\Omega} \phi d\mu$$

for any bounded continuous ϕ on Ω . It is also evident that $\mathcal{B}_{\infty}(\Omega)$ is dense in $\mathcal{B}(\Omega)$ with respect to the weak topology.

Let $\Omega^{\otimes} := \cup_{N \in \mathbb{N}} \Omega^{\otimes N}$ where $X \in \Omega^{\otimes N}$ if and only if $X := (x_1, \dots, x_N)$ is an *unordered* sequence of N distinct points in Ω . Let $N(X) := N$ if and only if $X \in \Omega^{\otimes N}$. We first note that each $\mu \in \mathcal{B}_{\infty}(\Omega)$ can be identified with a point $X \in \Omega^{\otimes N}$. Thus, a measure $\mu \in \mathcal{B}_N(\Omega)$ can be identified with $X \in \Omega^{\otimes N}$ via $\mu := \delta_X \equiv N(X)^{-1} \sum_{i=1}^{N(X)} \delta_{x_i}$.

Definition 1. A Γ -approximation of the entropy E is a function $\mathcal{E} : \Omega^{\otimes} \rightarrow \mathbb{R} \cup \{\infty\}$ such that

i) For *any* sequence $\{X_N\}$ such that $X_N \in \Omega^{\otimes N}$ and $\delta_{X_N} \rightarrow \mu \in \mathcal{B}(\Omega)$,

$$\liminf_{N \rightarrow \infty} \mathcal{E}(X_N) \geq E(\mu) .$$

ii) For any $\mu \in \mathcal{B}(\Omega)$ such that $E(\mu) < \infty$ there exists a sequence $\{\tilde{X}_N\} \in \Omega^{\otimes}$ such that $\delta_{\tilde{X}_N} \rightarrow \mu$ and

$$\lim_{N \rightarrow \infty} \mathcal{E}(\tilde{X}_N) = E(\mu) .$$

Given $X = (x_1, \dots, x_N) \in \Omega^{\otimes N}$, let $r_i(X)$ be half the minimal distance of x_i to the rest of the points in X :

$$r_i(X) := \frac{1}{2} \min_{j \neq i} |x_i - x_j| .$$

We now pose the following result:

Theorem 2.

$$\mathcal{E}(X) := -2 \sum_{i=1}^{N(X)} \ln(r_i(X)) - \ln(2\sqrt{3}N(X))$$

is a Γ -approximation of the entropy.

We now sketch the proof of Theorem 2.

Let $D_i(X)$ be the disc of radius $r_i(X)$ centered at x_i .

A *partition rule* W is defined as a mapping between Ω^{\otimes} to a partition of \mathbb{R}^2 into an essentially disjoint sets. For $X \in \Omega^{\otimes}$ the partition rule associate a collection of $N(X)$ measurable sets $W_i := W_i(X) \subset \mathbb{R}^2$ such that

- a) $W_i(X) \supset D_i(X)$ for any $i \in \{1, \dots, N(X)\}$
- b) $\cup_{i=1}^{N(X)} W_i(X) \supseteq \Omega$.
- c) $|W_i(X) \cap W_j(X)| = 0$ for any $i \neq j \in \{1, \dots, N(X)\}$. Here $|\cdot|$ stands for the Lebesgue measure of a measurable set in \mathbb{R}^2 .
- d) If $\{X_N\}$ is a density set in Ω , i.e for any open set $U \subset \Omega$, $\#(U \cap X_N) > 0$ for all N large enough, then $\lim_{N \rightarrow \infty} \max_{i \in \{1, \dots, N\}} \text{diameter}(W_i(X_N) \cap \Omega) = 0$.

Given a partition rule W we may associate with every $\delta_X \in \mathcal{B}_{\infty}(\Omega)$ another measure $\mu \in \mathcal{B}(\Omega)$ which admits a density

$$\rho_X^W(x) := N^{-1}(X) \sum_{i=1}^{N(X)} \frac{1_{W_j(X) \cap \Omega}(x)}{|W_j(X) \cap \Omega|} .$$

Here $1_A(x)$ is the characteristic function for a set A , i.e $1_A(x) = 1, 0$ if $x \in A, x \notin A$ respectively.

We now define the approximation entropy corresponding to the partition rule \vec{W} ,

$$\mathcal{E}^W : \Omega^{\otimes} \rightarrow \mathbb{R} \cup \{\infty\}$$

as the entropy of $\rho_X^W dx$ associated with $\delta_X \in \mathcal{B}_\infty(\Omega)$:

$$\mathcal{E}^W(X) := E(\rho_X^W dx) = \sum_{i=1}^{N(X)} \ln \left(\frac{1}{|W_i(X) \cap \Omega|} \right) - \ln N(X) . \quad (1)$$

We show (cf. [3]):

Proposition 1. *For any partition rule W verifying (a-d) above, \mathcal{E}^W is a Γ -approximation of the entropy.*

It is not too difficult to find partition rules. Recall the definition of Voronoi tessellation corresponding to $X = (x_1, \dots) \in \Omega^\otimes$:

$$V_i(X) := \{y \in \mathbb{R}^2 ; |y - x_i| \leq |y - x_j| \quad \forall 1 \leq j \leq N(X)\} . \quad (2)$$

Indeed, one can easily show that $X \rightarrow \vec{V}$ is a partition rule.

We now define another partition rule: Let $cell(D_i(X))$ as defined in the first paragraph of section 3 below. We know that $cell(D_i(X))$ are pairwise disjoint, while $\cup_{i=1}^N cell(D_i(X)) \cap \Omega \subset \Omega$ with a (possibly) strict inclusion. Thus, the partition X into $\{cell(D_i(X))\}$ is *not* a partition rule, since it may violate condition (b) above.

Let $Cell(X) := \cup_{i=1}^N cell(D_i(X))$. Define

$$W_i(X) := cell(D_i(X)) \cup (V_i(X) - Cell(X)) . \quad (3)$$

It is now easy to see that $W(X)$ so defined is a partition rule, verifying (a-d) above. In particular, \mathcal{E}^W as defined in (1) under the partition rule (3) is a Γ -approximation of the entropy, via Proposition 1.

By the proof of Theorem 1 (section 3 below) we find out that the area of $cell(D_i(X))$ is not smaller than $2\sqrt{3}r_i^2(X)$, i.e $|W_i(X)| \geq 2\sqrt{3}r_i^2(X)$. In addition we can show that for $X \in \Omega^\otimes$ for which δ_X approximates (in $C^*(\Omega)$) a measure $\mu \in \mathcal{B}(\Omega)$ satisfying $E(\mu) < \infty$, it follows that $W_i(X) \not\subset \Omega$ for only $o(N(X))$ of the points. So, we replace $\ln(|W_i(X) \cap \Omega|)$ in (1) by $\ln(2\sqrt{3}r_i^2(X))$, taking advantage of the monotonicity of the function \ln , and obtain that \mathcal{E} defined in Theorem 2 is not smaller, asymptotically, than \mathcal{E}^W with W given by (3). Thus \mathcal{E} satisfies condition (i) of definition 1.

To verify condition (ii) we recall that an optimal ratio of $|W_i(X)|/(2\sqrt{3}r_i^2(X)) \gtrsim 1$ is obtained for hexagonal grids. Given $\mu \in \mathcal{B}(\Omega)$, we can approximate it (in the weak topology) by a sequentially constant density. Then we construct an hexagonal grid on every domain in Ω on which this density is a constant. For the details of the proof see [3].

3 Proof of Theorem 1.

For any two points X and Y in the plane we denote by $|XY|$ the Euclidean distance from X to Y , that is, $|XY| = |X - Y|$. Depending on the context (that will always be specified), XY will denote either the line through X and Y , or the straight line segment with endpoints X and Y .

For every disc D in \mathcal{F} we define a cell, that we denote by $cell(D)$, in the following way. Denote by O the center of D and let r be the radius of D . For every $D' \in \mathcal{F}$, different from D , let O'

be the center of D' and let r' denote the radius of D' . Let $\ell(D, D')$ be the line perpendicular to OO' such that the intersection point B of $\ell(D, D')$ and the segment OO' satisfies $\frac{|OB|}{|O'B|} = \frac{r}{r'}$ (see Figure 1). Notice that $\ell(D, D')$ separates D and D' because $|OO'| \geq \max(2r, 2r') \geq r + r'$. Let $H(D')$ denote the (open) half-plane determined by $\ell(D, D')$ that contains D . Finally, define $cell(D) = \cap_{D' \in \mathcal{F} \setminus \{D\}} H(D')$.

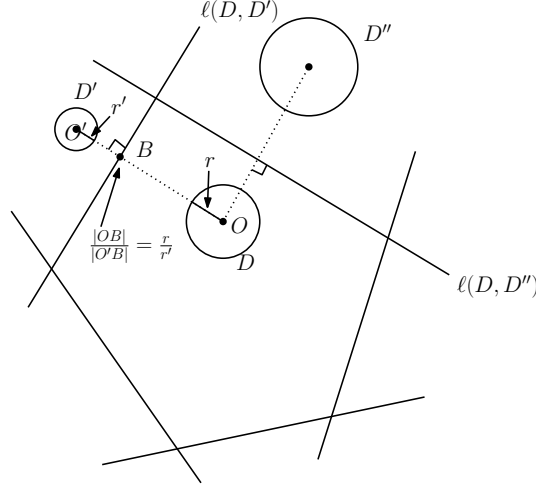


Figure 1: Definition of $\ell(D, D')$ and $cell(D)$

Observe that for every disc $D \in \mathcal{F}$ we have $D \subset cell(D)$ and for any two discs $D_1, D_2 \in \mathcal{F}$ we have $cell(D_1) \cap cell(D_2) = \emptyset$. If all the discs in \mathcal{F} are of the same radii, then the cells $\{cell(D) \mid D \in \mathcal{F}\}$ are just cells of the Voronoi diagram of the set of centers of the discs in \mathcal{F} . For arbitrary family of discs, however, it is possible that the collection of cells $cell(D)$ does not cover the plane.

In order to prove Theorem 1 we restrict our attention to a large ball \mathcal{B} around the origin. Fix c to be any number strictly greater than $\frac{\pi}{2\sqrt{3}}$. We would like to show that it is not possible to find larger and larger balls \mathcal{B} such that the area of \mathcal{B} contained in the union of all discs in \mathcal{F} is more than c times the area of \mathcal{B} .

Let \mathcal{B} be a large ball of radius n . Because \mathcal{F} has sub-linear radii growth we can discard from \mathcal{F} all discs that are not contained in \mathcal{B} . This is because the union of all discs in \mathcal{F} intersecting the boundary of \mathcal{B} is contained in an annulus of width $o(n)$ whose area is $o(n^2)$ and therefore negligible with respect to the area of \mathcal{B} . Because \mathcal{F} is locally finite, \mathcal{B} contains the centers of only finitely many discs in \mathcal{F} . We discard from \mathcal{F} all the discs that are not contained in \mathcal{B} .

We claim that it is enough to show that the portion of the area of any disc D in its cell $cell(D)$ is not greater than the portion of the area of a disc in its circumscribing hexagon, namely $\frac{\pi}{2\sqrt{3}}$. Indeed, let r denote the maximum radius of a disc in \mathcal{F} and recall that $r = o(n)$. Let \mathcal{B}' be the ball concentric with \mathcal{B} whose radius is equal to the radius of \mathcal{B} plus r . Add to \mathcal{F} many more artificial discs, each with extremely small radius, centered very densely at points on the boundary of \mathcal{B} . Notice that \mathcal{F} together with the additional artificial discs satisfies the conditions in Theorem 1. Observe that for every disc $D \in \mathcal{F}$ that is not artificial the new $cell(D)$ is a subset of the original $cell(D)$ before the artificial discs were added to \mathcal{F} . Notice moreover that the new $cell(D)$ is fully contained in \mathcal{B}' which is a ball concentric with \mathcal{B} whose radius is equal to the radius of \mathcal{B} plus $r + 1$. Hence, if we show that every disc D in \mathcal{F} cannot cover more than $\frac{\pi}{2\sqrt{3}}$ of the area of $cell(D)$, this will show that the union of all non-artificial discs in \mathcal{F} cannot cover more than $\frac{\pi}{2\sqrt{3}}$ of the area of

\mathcal{B}'' . Observe that the difference between the area of \mathcal{B} and the area of \mathcal{B}'' is $o(n^2)$ and is negligible compared to the area of \mathcal{B} when \mathcal{B} is a large ball (that is when n is large). This means that the union of all discs in the original family \mathcal{F} cannot cover from \mathcal{B} an area of at least c times the area of \mathcal{B} for fixed $c > \frac{\pi}{2\sqrt{3}}$ and a ball \mathcal{B} that is large enough.

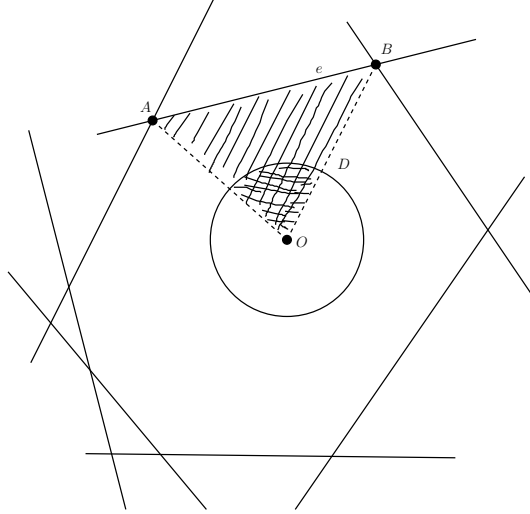


Figure 2: cell(D)

Therefore, we will concentrate on showing that the portion of the area of any disc D in its cell $cell(D)$ (we may assume that $cell(D)$ is bounded) is not greater than the portion of the area of a disc in its circumscribing hexagon, namely, $\frac{\pi}{2\sqrt{3}}$. (We note that $\frac{\pi}{2\sqrt{3}} \leq 0.906$.)

To this end we will show something stronger. Fix a disc $D \in \mathcal{F}$, denote its center by O , and assume without loss of generality that it is a unit disc. Notice that $cell(D)$ is a convex polygon. We will show that for every edge e of $cell(D)$ the portion of the area of D inside the triangle determined by O and e is at most $\frac{\pi}{2\sqrt{3}}$ (see Figure 2). We further strengthen our statement as follows: Let C be the point on the line ℓ through e such that OC is perpendicular to ℓ . We will show that if A is a vertex of the edge e such that AC overlaps with e , then the portion of the area of D inside the triangle ΔOAC is at most $\frac{\pi}{2\sqrt{3}}$ (see Figure 3).

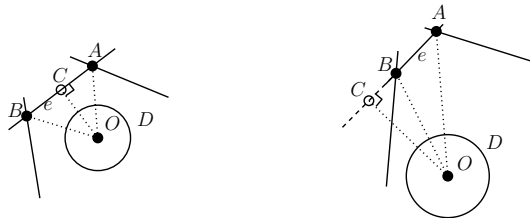


Figure 3: Some simple reduction steps

To see that this is indeed a stronger statement, let A and B be the two vertices of the edge e . We split into two possible cases. If C is a point in the segment AB (as in the left part of Figure 3), then both AC and BC overlap with e . Notice that $\frac{area(D \cap \Delta OAB)}{area(\Delta OAB)} \leq \max(\frac{area(D \cap \Delta OAC)}{area(\Delta OAC)}, \frac{area(D \cap \Delta OBC)}{area(\Delta OBC)})$. If C does not belong to the segment AB (as in the right part of Figure 3), then assume without loss of generality that B is a point in the segment AC . We claim that $\frac{area(D \cap \Delta OAB)}{area(\Delta OAB)} \leq \frac{area(D \cap \Delta OAC)}{area(\Delta OAC)}$.

The reason is that the expression $\frac{\text{area}(D \cap \Delta OAC)}{\text{area}(\Delta OAC)}$ is monotone decreasing in the distance of A from C , or equivalently in the angle $\angle AOC$ (this is because $\frac{\text{area}(D \cap \Delta OAC)}{\text{area}(\Delta OAC)} = \frac{\angle AOC}{\tan \angle AOC}$). Therefore, $\frac{\text{area}(D \cap \Delta OAC)}{\text{area}(\Delta OAC)} \leq \frac{\text{area}(D \cap \Delta OBC)}{\text{area}(\Delta OBC)}$. This implies

$$\frac{\text{area}(D \cap \Delta OAB)}{\text{area}(\Delta OAB)} = \frac{\text{area}(D \cap \Delta OAC) - \text{area}(D \cap \Delta OBC)}{\text{area}(\Delta OAC) - \text{area}(\Delta OAB)} \leq \frac{\text{area}(D \cap \Delta OAC)}{\text{area}(\Delta OAC)}.$$

We leave the verification of the last inequality to the reader.

Fix an edge e of $\text{cell}(D)$ and let D_1 be the disc in \mathcal{F} that gives rise to the edge e , that is, $\ell(D, D_1)$ contains e . Denote by A_1 the point of intersection of $\ell(D, D_1)$ and the line through OO_1 . Let A_2 be one vertex of e .

We have $\angle OA_1A_2 = \frac{\pi}{2}$. Denote by O_1 the center of D_1 and denote by r_1 the radius of D_1 . Recall, because of the definition of $\text{cell}(D)$ and the fact that the radius of D is equal to 1, that we have $\frac{|OA_1|}{|O_1A_1|} = \frac{1}{r_1}$. Let D_2 be the disc in \mathcal{F} that gives rise to the edge e' of $\text{cell}(D)$ that is adjacent to A_2 but different from e . Denote by O_2 the center of D_2 and let r_2 denote the radius of D_2 (see Figure 4).

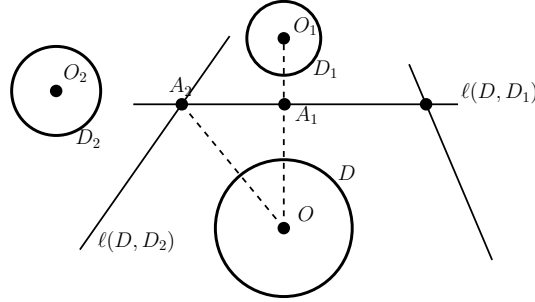


Figure 4: Definition of A_1 , A_2 , D_1 , and D_2 .

For three points A, B , and C in \mathbb{R}^2 we denote by $f(A, B, C)$ the ratio between the area of $D \cap \Delta ABC$ and the area of the triangle ΔABC . We need to show that $f(O, A_1, A_2) \leq \frac{\pi}{2\sqrt{3}}$. Assume to the contrary that $f(O, A_1, A_2) > \frac{\pi}{2\sqrt{3}}$.

Claim 1. $\angle A_2OO_1 \leq \frac{\pi}{6}$.

Proof. Notice that $f(O, A_1, A_2) \leq \frac{\angle A_2OO_1}{\tan \angle A_2OO_1}$. It follows that we must have $\angle A_2OO_1 \leq \frac{\pi}{6}$ for otherwise $f(O, A_1, A_2) \leq \frac{\pi}{2\sqrt{3}}$. ■

The following observation follows directly from our definitions:

Observation 1. Suppose S_1 and S_2 are two discs in \mathcal{F} of radii r_1 and r_2 , respectively, and let d be the distance between the centers of S_1 and S_2 . Then the distance t from the center of S_1 to $\ell(S_1, S_2)$ is equal to $\frac{r_1 d}{r_1 + r_2}$.

Proof. Indeed, this is because we have $\frac{d-t}{r_2} = \frac{t}{r_1}$. ■

Lemma 1. The angle $\angle O_2OO_1$ is greater than $\frac{\pi}{6}$.

Proof.

We will need the following observation:

Claim 2. *Let D' be a disc in \mathcal{F} with center O' and radius r' . Suppose that the line $\ell(D, D')$ is at distance $1 + x$ from the center O of D . Then r' satisfies $\frac{1-x}{1+x} \leq r' \leq \frac{1+x}{1-x}$. Moreover, the distance between the O and O' satisfies: $2 \leq |OO'| \leq 2\frac{1+x}{1-x}$.*

Proof. Recall that the radius of D is equal to 1. Denote by t the distance from O' to $\ell(D, D')$. We have $\frac{t}{r'} = \frac{1+x}{1} = 1 + x$. Because $2r' \leq OO' = t + 1 + x = r'(1 + x) + (1 + x)$, we get

$$r' \leq \frac{1+x}{1-x}.$$

On the other hand we also have $2 \leq OO' = t + 1 + x = r'(1 + x) + (1 + x)$ implying

$$r' \geq \frac{1-x}{1+x}.$$

To see the second part of the claim about the distance from O to O' , notice that, by our construction of $\ell(D, D')$, we have $\frac{|OO'| - (1+x)}{r'} = \frac{1+x}{1}$. Therefore, $|OO'| = (1+x)(1+r')$ and hence, as a consequence of the first part of the claim, $2 \leq |OO'| \leq 2\frac{1+x}{1-x}$. ■

We claim that

$$|OA_2| \leq \frac{2}{\sqrt{3}} \leq 1.155. \quad (4)$$

To see this, let $\alpha = \angle A_1OA_2$. By Claim 1, $\alpha \leq \frac{\pi}{6}$. We have $\frac{\pi}{2\sqrt{3}} \leq f(O, A_1, A_2) = \frac{\alpha}{|OA_2|^2 \sin \alpha \cos \alpha}$. Notice that $\frac{\alpha}{\sin \alpha \cos \alpha}$ is monotone increasing function of α and hence (recall $\alpha \leq \frac{\pi}{6}$) $\frac{\pi}{2\sqrt{3}} \leq \frac{\pi/6}{|OA_2|^2 (1/2)(\sqrt{3}/2)}$, implying (4).

As a consequence of (4), the distance from O to $\ell(D, D_2)$ is at most 1.155. Moreover, the distance from O to A_1 is smaller than the distance from O to A_2 and therefore we also deduce that the distance from O to $\ell(D, D_1)$ is at most 1.155. By Claim 2, both distances from O to O_1 and from O to O_2 are at least 2 and at most $2\frac{1+0.155}{1-0.155} < 2.74$.

By Claim 2 and the fact that the distance from O to both $\ell(D, D_1)$ and $\ell(D, D_2)$ is at most 1.155, we have that both r_1 and r_2 are at least $\frac{1-0.155}{1+0.155} \geq 0.73$. Hence $|O_1O_2| \geq r_1 + r_2 \geq 1.46$. Because $2 \leq |OO_1|, |OO_2| \leq 2.74$ there are two extreme options. In one $\angle O_1OO_2$ is at least as large as the angle at P in a triangle ΔPQR such that $|PQ| = 2.74$, $|QR| = 1.46$, and $|PR| = 2$. The cosine of this angle a satisfies $\cos a = \frac{2.74^2 + 2^2 - 1.46^2}{2 \cdot 2 \cdot 2.74} \leq 0.856 < \cos \frac{\pi}{6}$.

The other extreme case is where $\angle O_1OO_2$ is at least as large as the angle at P in a triangle ΔPQR such that $|PQ| = |PR| = 2.74$ and $|QR| = 1.46$. The cosine of this angle a satisfies $\cos a = \frac{2.74^2 + 2.74^2 - 1.46^2}{2 \cdot 2.74 \cdot 2.74} \leq 0.859 < \cos \frac{\pi}{6}$. ■

Because $\angle O_2OO_1$ is the angle generated between $\ell(D, D_2)$ and (the right ray of) $\ell(D, D_1)$ we have the following

Corollary 1. *The angle generated between $\ell(D, D_2)$ and (the right ray of) $\ell(D, D_1)$ is greater than $\frac{\pi}{6}$.*

The next lemma will turn to be quite useful.

Lemma 2. *Let ℓ be a line parallel to $\ell(D, D_1)$ that separates $\ell(D, D_1)$ and D . In particular, ℓ is closer to O than $\ell(D, D_1)$ is. Let A_1^* be the intersection point of ℓ with the line through O and O_1 and let A_2^* be the intersection point of ℓ with the line $\ell(D, D_2)$. Then $f(O, A_1, A_2) \leq f(O, A_1^*, A_2^*)$.*

Proof. Denote by B the intersection point of $\ell(D_2, D)$ and the line through O and O_1 . Let α denote the measure of the angle between $\ell(D_2, D)$ and the (positive part of the) x -axis, that is, $\alpha = \angle BA_2A_1$. Denote by x the angle $\angle A_2OA_1$ (see Figure 5).

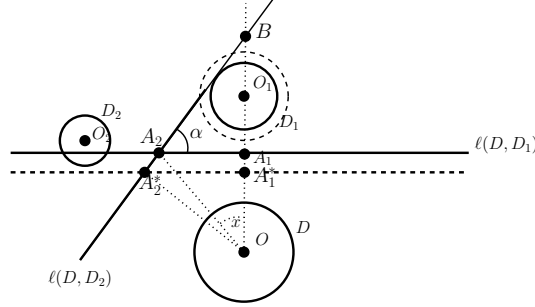


Figure 5: As we increase r_2 we increase x .

It is not hard to express $f(O, A_1, A_2)$ as a function of x : We have $f(O, A_1, A_2) = \frac{x/2}{\text{area of } \triangle OA_1A_2}$.

Notice that from the theorem of sines $|A_2O| = \frac{|OB|\sin(\pi/2-\alpha)}{\sin(\pi/2-x+\alpha)} = \frac{|OB|\cos\alpha}{\cos(x-\alpha)}$. Therefore, the area of $\triangle OA_1A_2$ is equal to

$$\frac{1}{2}|OA_1||A_1A_2| = \frac{1}{2}\sin x \cos x |OA_2|^2 = \frac{1}{2}\sin x \cos x \frac{|OB|^2 \cos^2 \alpha}{\cos^2(x-\alpha)} = \frac{1}{2}\sin x \cos x \frac{|OB|^2 \cos^2 \alpha}{(1 + \cos(2x - 2\alpha))/2}.$$

Since $f(O, A_1, A_2) = \frac{x/2}{\text{area of } \triangle OA_1A_2}$, then up to positive constant multipliers that depend only on B , O , and α , this function is equal to $g(x) = \frac{x(1+\cos(2x-2\alpha))}{\sin 2x}$.

Let x' denote the angle $\angle A_2^*OA_1^*$. Notice that $x' > x$. Therefore, in order to show that $f(O, A_1, A_2) \leq f(O, A_1^*, A_2^*)$ it is enough to show that the function $g(x)$ is an increasing function of x , or equivalently that $g'(x) \geq 0$.

A direct attempt to prove $g'(x) \geq 0$ leads to the equivalent inequality

$$\sin 2x(1 + \cos(2x - 2\alpha) - x2 \sin(2x - 2\alpha)) \geq x(1 + \cos(2x - 2\alpha))2 \cos 2x.$$

This reduces to

$$(1 + \cos(2x - 2\alpha)) \sin 2x \geq x(2 \cos 2x + 2 \cos 2\alpha)$$

and then to

$$\cos(x - \alpha) \sin 2x \geq 2x \cos(x + \alpha).$$

Using the fact that $\frac{\sin 2x}{2x} \geq \cos 2x$, it will be enough to show that

$$\cos(x - \alpha) \cos 2x \geq \cos(x + \alpha).$$

This is equivalent to

$$\frac{1}{2}(\cos(3x - \alpha) + \cos(x + \alpha)) \geq \cos(x + \alpha).$$

This finally reduces to

$$\cos(3x - \alpha) \geq \cos(x + \alpha)$$

which is equivalent to $x \leq \alpha$. This last inequality holds because we have $x \leq \frac{\pi}{6} \leq \alpha$ (the first inequality is Claim 1 and the second inequality is by Corollary 1). ■

Lemma 3. $\angle O_2 O O_1 \leq \frac{\pi}{3}$.

Proof. We will show that if $\angle O_2 O O_1 > \frac{\pi}{3}$, then $f(O, A_1, A_2) \leq \frac{\pi}{2\sqrt{3}}$. By Lemma 2, it is enough to consider the case where $\ell(D, D_1)$ is tangent to D . In this case notice that if $\angle O_2 O O_1 = \frac{\pi}{3}$, then $f(O, A_1, A_2) = \frac{\pi}{2\sqrt{3}}$. As we further increase $\angle O_2 O O_1$, the value of $f(O, A_1, A_2)$ decreases. ■

Reducing to the critical case

We say that D is *critical* if its radius, namely 1, is equal to $\frac{1}{2} \min(|OO_1|, |OO_2|)$. Intuitively speaking, we imagine D is inflated around its center as much as we can so that the conditions in Theorem 1 are still satisfied when restricting our attention only to the three discs D , D_1 , and D_2 .

In a similar way we define the notion of critical for D_1 and D_2 . That is, D_1 is critical if r_1 is equal to $\frac{1}{2} \min(|O_1 O|, |O_1 O_2|)$. We say that D_2 is critical if $r_2 = \frac{1}{2} \min(|O_2 O|, |O_2 O_1|)$.

In this subsection we will show that one can assume, without loss of generality, that all three discs D , D_1 , and D_2 are critical. This reduction will simplify a lot the presentation of the rest of the proof.

Without loss of generality we will assume that O is the origin, O_1 lies strictly above O on the y -axis, and O_2 lies in the half-plane $\{x < 0\}$.

It is easiest to see that we may assume that D_2 is critical. Indeed, by increasing the value of r_2 we push the line $\ell(D_2, D)$ towards O , thus shifting the point A_2 to the right. This increases the value of $f(O, A_1, A_2)$ (see Figure 6). Formally, denote by x the angle $\angle A_2 O A_1$. We have $f(O, A_1, A_2) = \frac{x/2}{\frac{1}{2} \tan x |OA_1|}$. This is a decreasing function of x . Hence, as A_2 moves to the right x decreases and consequently $f(O, A_1, A_2)$ increases.

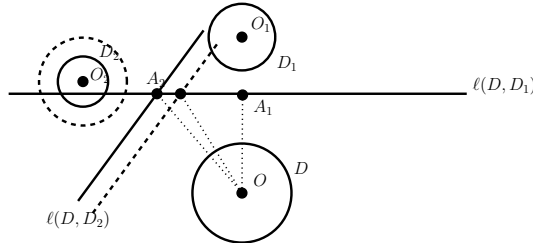


Figure 6: We may assume D_2 is critical.

Next, we claim that we may assume without loss of generality that D_1 is critical. To see this notice that as we increase r_1 , we push the line $\ell(D, D_1)$ towards O (this operation has an effect both

on A_1 and on A_2). By Lemma 2, as we push the line $\ell(D, D_1)$ towards O , the value of $f(O, A_1, A_2)$ does not decrease.

Finally, we claim that we may assume without loss of generality that D is critical. To see this, we will now show that the effect of increasing the radius of D is equivalent to keeping D a unit disc and pushing the lines $\ell(D, D_1)$ and $\ell(D, D_2)$ closer to O . Once we show this then the claim follows from Lemma 2 because it is shown there that pushing $\ell(D, D_1)$ closer to O (keeping $\ell(D, D_2)$ fixed) increases $f(O, A_1, A_2)$. If in addition we also push $\ell(D, D_2)$ closer to O , then $f(O, A_1, A_2)$ can only further increase.

To see the effect of increasing the radius of D , let D' be any other disc in \mathcal{F} and let O' and r' be its center and radius, respectively. The distance d from O to $\ell(D, D')$ satisfies $\frac{1}{d} = \frac{r'}{|OO'| - d}$, namely, $d = \frac{|OO'|}{1+r'}$. If we increase the radius of D to be $r > 1$, then the new distance d' from O to $\ell(D, D')$ satisfies $\frac{r}{d'} = \frac{r'}{|OO'| - d'}$, namely, $d' = \frac{|OO'|}{1+r'/r}$. Scaling back the picture so that D is again a unit disc, this distance reduces to $\frac{1}{r} \frac{|OO'|}{1+r'/r} = \frac{|OO'|}{r+r'}$. Because $r > 1$ we have $\frac{|OO'|}{r+r'} < \frac{|OO'|}{1+r'} = d$.

Concluding the proof

We henceforth assume that all three discs D, D_1 , and D_2 are critical. We split into three cases according to which is the closest pair of centers among O, O_1 , and O_2 .

Case 1. $|OO_1| \leq |OO_2|, |O_1O_2|$. In this case, because D and D_1 are both critical, the radii of both D and D_1 are the same and are equal to $\frac{|OO_1|}{2}$. As we assume that D is a unit disc, the radii of both D and D_1 are equal to 1 and hence $|OO_1| = 2$. The discs D and D_1 touch each other at A_1 and $\ell(D, D_1)$ is their common tangent at A_1 . Let A'_2 be the point on $\ell(D, D_1)$ to the left of A_1 such that $f(O, A_1, A'_2) = \frac{\pi}{2\sqrt{3}}$. In order for $f(O, A_1, A_2)$ to be greater than $\frac{\pi}{2\sqrt{3}}$ the line $\ell(D, D_2)$ must cross $\ell(D, D_1)$ at a point A_2 to the right of A'_2 . Let D'_1 be the disc centered at O_1 whose radius is 2 (double the radius of D_1). The point O_2 , the center of D_2 , must lie outside D'_1 because of the assumptions in Theorem 1.

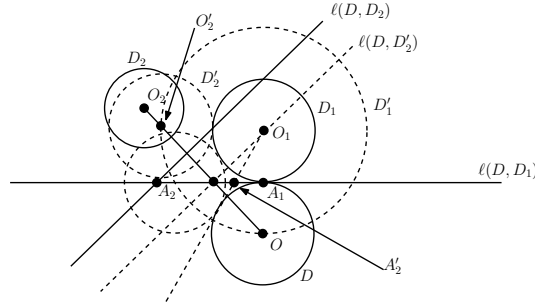


Figure 7: Illustrating Case 1.

We will now show that we may assume without loss of generality that O_2 lies on the boundary of D'_1 . Let O'_2 denote the intersection point of the line through O and O_2 with the boundary of D'_1 . We will replace D_2 with D'_2 , the disc of radius 1 centered at O'_2 . By Lemma 3, $\angle O_2OO_1 < \frac{\pi}{3}$. This implies that both points O_2 and O'_2 are closer to O_1 than to O . We will show that O is closer to the line $\ell(D, D'_2)$ than to the line $\ell(D, D_2)$ (see Figure 7). This will imply that by replacing D_2 with D'_2 we push A_2 further to the right (on $\ell(D, D_1)$) and therefore can only increase $f(O, A_1, A_2)$.

Recall that as D_2 is critical then r_2 , the radius of D_2 , is equal to $\frac{1}{2}|O_2O_1|$ (this is because

$|O_2O_1| \leq |O_2O|$ and $r_2 = \frac{1}{2} \min(|O_2O|, |O_2O_1|)$.

The distance d from O to $\ell(D, D_2)$ satisfies $\frac{1}{d} = \frac{r_2}{|OO_2| - d}$. Therefore, $d = \frac{|OO_2|}{r_2 + 1} = \frac{|OO_2|}{\frac{1}{2}|O_2O_1| + 1}$.

The distance d' from O to $\ell(D, D'_2)$ satisfies $d' = \frac{1}{2}|OO'_2|$ (this is because both D and D'_2 are unit discs and therefore $\ell(D, D'_2)$ is the perpendicular bisector of OO'_2).

We claim that $d' \leq d$, or equivalently,

$$\frac{1}{2}|OO'_2| \leq \frac{|OO_2|}{\frac{1}{2}|O_2O_1| + 1}.$$

After dividing by 2 we get

$$\frac{|OO'_2|}{4} \leq \frac{|OO_2|}{|O_2O_1| + 2}.$$

Keeping in mind that $2 = |OO_1|$ and $4 = |OO_1| + |O'_2O_1|$, we need to show that

$$\frac{|OO'_2|}{|OO_1| + |O'_2O_1|} \leq \frac{|OO_2|}{|O_2O_1| + |OO_1|}. \quad (5)$$

Notice that

$$\frac{|OO_2|}{|O_2O_1| + |OO_1|} = \frac{|OO'_2| + |O'_2O_2|}{|O_2O_1| + |OO_1|} \geq \frac{|OO'_2| + |O'_2O_2|}{|O'_2O_1| + |O'_2O_2| + |OO_1|}.$$

Hence, in order to show (5) it is enough to show

$$\frac{|OO'_2|}{|OO_1| + |O'_2O_1|} \leq \frac{|OO'_2| + |O'_2O_2|}{|O'_2O_1| + |O'_2O_2| + |OO_1|}.$$

This last inequality reduces, after elementary manipulations, to the triangle inequality $|OO'_2| \leq |OO_1| + |O'_2O_1|$.

Therefore, we assume that the center O_2 of D_2 is on the boundary of D'_1 and that the radius of D_2 is equal to 1 (as D_2 can be assumed to be critical). Now it is easy to see that $\ell(D_2, D)$ passes through O_1 and therefore it intersects with $\ell(D, D_1)$ (at the point A_2) to the left of A'_2 and not as required. Hence $f(O, A_1, A_2) \leq \frac{\pi}{2\sqrt{3}}$.

Case 2. $|OO_2| \leq |OO_1|, |O_1, O_2|$. In this case the radii of both D and D_2 are equal, and therefore are equal to 1, which, in turn, is half of the distance from O to O_2 . Moreover, the discs D and D_2 touch each other and $\ell(D, D_2)$ is their common tangent at the point where they touch.

By Lemma 2, $\angle O_2OO_1 \leq \frac{\pi}{3}$. Similar to the argument in Case 1, we let D'_2 denote the disc of radius 2 centered at O_2 . Observe that O_1 must be outside D'_2 . This, together with the fact that $\angle O_2OO_1 \leq \frac{\pi}{3}$, implies that $|O_1O_2| \leq |O_1O|$. This is equivalent to saying that O_1 and O_2 lie in the same half-plane bounded by $\ell(D, D_2)$.

Let O'_1 be the intersection point of the line through O and O_1 with the boundary of D'_2 . Let D'_1 be the unit disc centered at O'_1 .

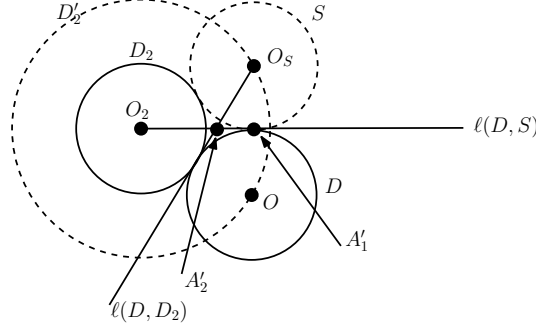


Figure 8: Illustrating the argument in Case 2.

Claim 3. *The distance from O to $\ell(D, D_1)$ is greater than or equal to the distance from O to $\ell(D, D'_1)$.*

Proof. Let x denote $\angle OO_1O_2$ and let $\alpha = \angle O_1OO_2$. We will now express the distance from O to $\ell(D, D_1)$ as a function of x . By Observation 1, the distance from O to $\ell(D, D_1)$ is equal to $\frac{|OO_1|}{r_1+1} = \frac{|OO_1|}{|O_1O_2|/2+1}$ (here $r_1 = |O_1O_2|/2$ because D_1 is critical and $|O_1O_2| \leq |O_1O|$). From the theorem of sines with respect to triangle ΔOO_1O_2 , $|O_1O_2| = \frac{2\sin\alpha}{\sin x}$ and $|OO_1| = \frac{2\sin(\alpha+x)}{\sin x}$.

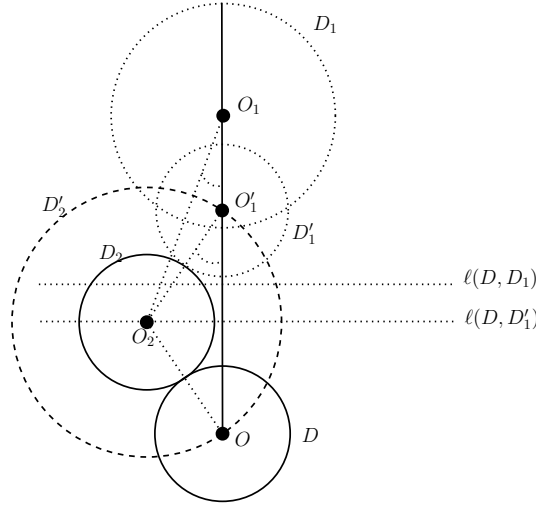


Figure 9: Illustrating the proof of Claim 3.

Therefore, the distance from O to $\ell(D, D_1)$ is equal to $\frac{2\sin(\alpha+x)}{\sin\alpha+\sin x}$. By checking the derivative of this function with respect to x one can see that this function is monotone decreasing in x . Because $\angle OO_1O_2 \leq \angle OO'_1O_2$, this shows that the distance from O to $\ell(D, D'_1)$ is smaller than the distance from O to $\ell(D, D_1)$. ■

Hence by taking $D_1 = D'_1$ we push $\ell(D, D_1)$ closer to O and therefore, by Lemma 2, we increase the value of $f(O, A_1, A_2)$.

Finally, observe that when $D_1 = D'_1$ the line $\ell(D, D_1)$ passes through O_2 . Let $x = \angle A_1OA_2$ and notice that x is monotone decreasing in $\angle O_1O_2O$ while $|OA_2|$ is monotone increasing in $\angle O_1O_2O$ (see Figure 10).

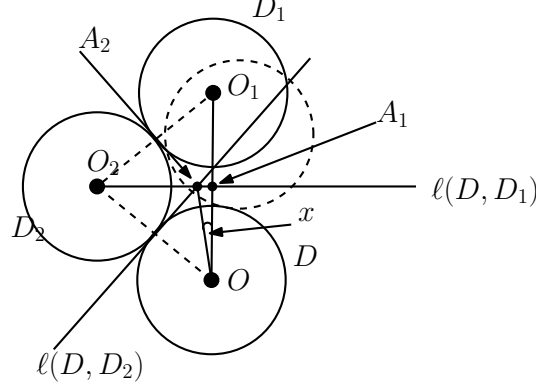


Figure 10: Illustrating the argument in Case 2.

We have

$$f(O, A_1, A_2) = \frac{x/2}{\frac{1}{2}|OA_2|^2 \sin x \cos x} = \frac{x}{\frac{1}{2}|OA_2|^2 \sin 2x}.$$

Therefore, as $\angle O_1 O_2 O$ increases x decreases and so $\frac{x}{\sin 2x}$ decreases (as can be easily verified this is an increasing function of x). On the top of this $|OA_2|$ increases and hence $f(O, A_1, A_2) = \frac{x}{\sin 2x |OA_2|^2}$ decreases.

Notice that $\angle O_1 O_2 O$ is minimum when D, D_2 and D_1 (which is now equal to D'_1) are three pairwise touching unit discs. In the latter case we have $f(O, A_1, A_2) = \frac{\pi}{2\sqrt{3}}$, showing that indeed $f(O, A_1, A_2) \leq \frac{\pi}{2\sqrt{3}}$ in general.

Case 3. $|O_1 O_2| \leq |OO_1|, |OO_2|$. In this case the radii of both D_1 and D_2 are equal and D_1 and D_2 touch each other. Denote by r the radii of D_1 and D_2 . Notice that $r \leq 1$. This is because D is critical and therefore either $|OO_1| = 2$, or $|OO_2| = 2$ and in either case we have $2r \leq 2$. Because of the assumption in Theorem 1 both $|OO_1|$ and $|OO_2|$ are greater than or equal to twice the radius of D , namely 2. We split into two sub-cases according to which of $|OO_1|$ and $|OO_2|$ is equal to 2.

Subcase a. $|OO_1| = 2$ and $|OO_2| \geq 2$. We claim that we may assume that $|OO_2| = 2$. To see this we rotate the disc D_2 around the center O_1 of D_1 in the clockwise direction until $|OO_2| = 2$ and we keep track of $f(O, A_1, A_2)$. Notice that by rotating the disc D_2 around O_1 we only change the position of A_2 while O and A_1 remain fixed.

Let α denote the angle $\angle O_2 O O_1$ and notice that as we rotate D_2 clockwise around O_1 until $|O_2 O| = 2$ α increases.

We have $|O_1 O_2| = 2r$, $|OO_1| = 2$ and it is not hard to see that $|OO_2| = 2(\cos \alpha + \sqrt{r^2 - \sin^2 \alpha})$. Notice that $|OO_2|$ is a monotone decreasing function of α .

By Observation 1, the distance from O to $\ell(D, D_1)$ is equal to $\frac{2}{r+1}$. The distance from O to $\ell(D, D_2)$ is equal to $\frac{|OO_2|}{r+1}$. Let x denote the angle $\angle A_2 O A_1$. We have $\cos x = \frac{|OA_1|}{|OA_2|}$. Let B denote the intersection point of $\ell(D, D_2)$ and the line OO_2 (see Figure 11).

Recall that $|OB|$, the distance from O to $\ell(D, D_2)$ is equal to $\frac{|OO_2|}{r+1}$. We have $\cos(\alpha - x) = \frac{|OB|}{|OA_2|}$.

Hence

$$\frac{\cos x}{|OA_1|} = \frac{\cos(\alpha - x)}{|OB|} = \frac{\cos \alpha \cos x + \sin \alpha \sin x}{|OB|}.$$

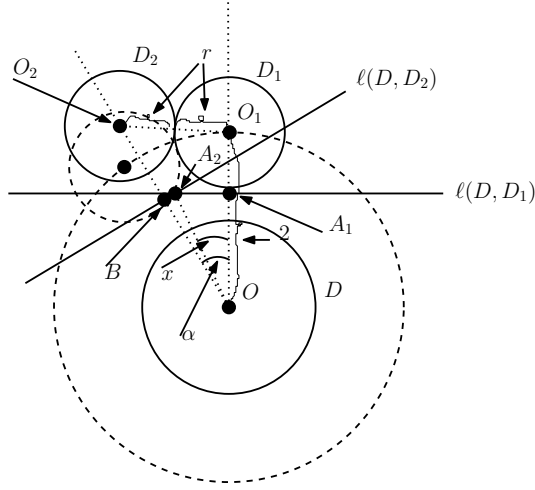


Figure 11: Illustrating Case 3(a).

From here we conclude that

$$|A_2A_1| = |OA_1| \tan x = \frac{|OB|}{\sin \alpha} - \frac{|OA_1| \cos \alpha}{\sin \alpha}. \quad (6)$$

It will therefore be enough to show that the right hand side of (6) decreases as we increase α . Keeping in mind that $|OA_1| = \frac{2}{r+1}$ and

$$|OB| = \frac{|OO_2|}{r+1} = \frac{2(\cos \alpha + \sqrt{r^2 - \sin^2 \alpha})}{r+1},$$

The right hand side of (6) becomes

$$\frac{2}{r+1} \sqrt{\frac{r^2}{\sin^2 \alpha} - 1},$$

which is evidently a decreasing function of α .

We conclude that we may assume in Subcase a of Case 3 that $|OO_1| = |OO_2| = 2$. Let α denote the angle $\angle A_1OA_2$. Notice that α is a monotone increasing function of r the radii of both D_1 and D_2 . We will show that $f(O, A_1, A_2)$ is an increasing function of α . From this it will follow that one can assume that r is maximum possible, namely $r = 1$, but in this case $f(O, A_1, A_2) = \frac{\pi}{2\sqrt{3}}$, as can be easily seen.

Notice that $r = 2 \sin \alpha$ and therefore $|OA_1| = \frac{2}{r+1} = \frac{2}{1+2\sin \alpha}$.

We have

$$f(O, A_1, A_2) = \frac{\alpha}{|OA_1||OA_2|} = \frac{\alpha}{\tan \alpha |OA_1|^2} = \frac{\alpha(1+2\sin \alpha)^2}{4 \tan \alpha}.$$

It remains to show that this is an increasing function of α . Considering the derivative of this function, it is equivalent to showing that

$$\frac{\alpha}{\sin \alpha} \leq \cos \alpha \frac{1+2\sin \alpha}{1+2\sin \alpha - 4\sin \alpha \cos^2 \alpha}.$$

As $\frac{\alpha}{\sin \alpha} \leq \frac{1}{\cos \alpha}$ for every $0 \leq \alpha < \frac{\pi}{2}$, it is enough to show that

$$\frac{1}{\cos \alpha} \leq \cos \alpha \frac{1 + 2 \sin \alpha}{1 + 2 \sin \alpha - 4 \sin^2 \alpha}. \quad (7)$$

This reduces, after elementary manipulations, to

$$\cos^2 \alpha + \sin 2\alpha \cos \alpha + 2 \sin \alpha \cos 2\alpha \geq 1 \quad (8)$$

This clearly holds for every $\alpha \leq \frac{\pi}{6}$ (which we assume) because for those α we have $\cos 2\alpha \geq \sin \alpha$ and therefore the left hand side of (8) is at least $\cos^2 \alpha + \sin^2 \alpha$, that is, at least 1.

Subcase b. $|OO_2| = 2$ and $|OO_1| \geq 2$. We claim that we may assume in this case that $r_1 = r_2 = 1$. This will imply $|O_1O_2| = |OO_2| \leq |OO_1|$ and we may thus reduce to Case 2. To see that we may assume $r_1 = r_2 = 1$, we will increase the value of r keeping the angle $\angle OO_2O_1$, that we denote by β , fixed. Through this increment we will keep D_1 and D_2 touching each other. At every moment denote by α the angle $\angle O_2OO_1$ and notice that α is a monotone increasing function of r . We will show that as α increases the value of $f(O, A_1, A_2)$ increases.

Let B denote the intersection point of $\ell(D, D_2)$ with the line OO_2 . As $|OO_2| = 2$, it follows from Observation 1 that $|OB| = \frac{2}{r+1}$. By considering the triangle ΔOO_2O_1 and using the theorem of sines, we see that $|OO_1| = \frac{2 \sin \beta}{\sin(\alpha+\beta)}$. Therefore, again by Observation 1, we have $|OA_1| = \frac{|OO_1|}{r+1} = \frac{2 \sin \beta}{(r+1) \sin(\alpha+\beta)}$. Recall that A_2 is the intersection point of $\ell(D, D_2)$ and $\ell(D, D_1)$. Denote by x the angle $\angle A_2OA_1$ and notice that $f(O, A_1, A_2) = \frac{x}{\tan x |OA_1|^2}$ (see Figure 12).

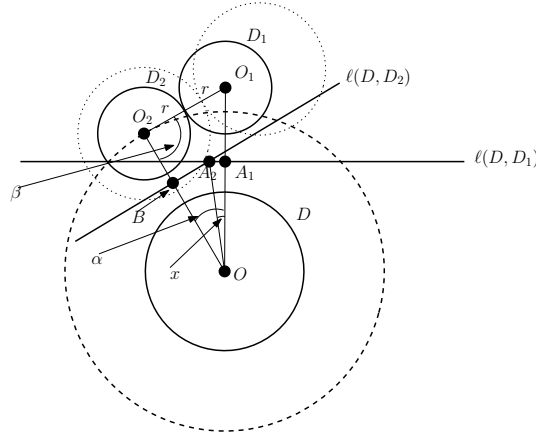


Figure 12: Illustrating Case 3(b).

By considering triangle ΔOA_2A_1 , we see that

$$\frac{1}{|OA_2|} = \frac{\cos x}{|OA_1|} = \frac{\cos x (r+1) \sin(\alpha + \beta)}{2 \sin \beta}. \quad (9)$$

By considering the triangle ΔOA_2B , we see that

$$\frac{1}{|OA_2|} = \frac{\cos(\alpha - x)}{|OB|} = \frac{(r+1) \cos(\alpha - x)}{2}. \quad (10)$$

From (9) and (10) it follows that

$$\frac{\cos x \sin(\alpha + \beta)}{\sin \beta} = \cos(\alpha - x) = \cos \alpha \cos x + \sin \alpha \sin x.$$

This implies

$$\frac{\sin x}{\cos x} = \frac{\cos \beta}{\sin \beta}.$$

This means that the angle x remains fixed through the increment of the value of r and therefore, in order to show that the value of $f(O, A_1, A_2)$ increases it is enough to show that $|OA_1|$ decreases, because $f(O, A_1, A_2) = \frac{x}{\tan x |OA_1|^2}$. To see that the value of $|OA_1|$ decreases as we increase α , we recall that $|OA_1| = \frac{2 \sin \beta}{(r+1) \sin(\alpha+\beta)}$ and therefore it is enough to show that $(r+1) \sin(\alpha+\beta)$ increases as we increase α . To this end consider triangle $\Delta O_1 O O_2$ and use the theorem of sines to see that $\frac{2r}{\sin \alpha} = \frac{2}{\sin(\alpha+\beta)}$. This implies $r = \frac{\sin \alpha}{\sin(\alpha+\beta)}$. Using this, we see that

$$(r+1) \sin(\alpha+\beta) = \sin \alpha + \sin(\alpha+\beta) = 2 \sin\left(\alpha + \frac{1}{2}\beta\right) \cos\left(\frac{\beta}{2}\right).$$

Now, it is enough to observe that $0 \leq \alpha + \frac{1}{2}\beta \leq \pi/2$. This is because $\alpha + \beta + \alpha \leq \alpha + \beta + \angle O_2 O_1 O = \pi$.

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