Points covered an odd number of times by translates

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Abstract

Let T be a fixed triangle and consider an odd number of translated copies of T in the plane. We show that the set of points in the plane that belong to an odd number of triangles has an area of at least half of the area of T. This result is best possible. We resolve also the more general case of a trapezoid and discuss related problems.

1 Introduction

While preparing puzzles for my 'puzzles course' that I gave in the Technion during 2009 and 2010, I found the following puzzle in the fall's contest of 'Tournament of Towns' for the year 2009 [2]:

On an infinite chessboard are placed 2009 $n \times n$ *cardboard pieces such that each of them covers exactly* n^2 *cells of the chessboard. Prove that the number of cells of the chessboard which are covered by odd numbers of cardboard pieces is at least* n^2.

As for the history of the problem as far as I could trace it back: A one dimensional version of this puzzle was suggested by Uri Rabinovich and communicated to Igor Pak, who added one more dimension and communicated it to Arseniy V. Akopyan, and then through some Russian network contacts to the organizers of the Tournament of Towns in Moscow.

Perhaps the shortest solution to this puzzle is an elegant use of the coloring technique: Color a grid square (a, b) by the color $((a \mod n), (b \mod n))$. We thus have n^2 different colors and the crucial observation is that each $n \times n$ cardboard contains precisely one grid square of each color. It follows that there must be at least one grid square of each color class that is covered an odd number of times (because the total number of grid-squares from each color is 2009 with multiple counting).

It is almost an immediate consequence of this puzzle that also the continuous version of this problem is true. That is, given an odd number of axes-parallel unit squares in the plane the total area of all points covered by an odd number of squares is at least 1 (see Figure 1 for an illustration). Equality is possible, for example if all squares coincide. One can also give a direct proof: Color each point (x, y) of the plane by the color $((x \mod 1), (y \mod 1))$. Then every axes-parallel unit square contains precisely one point of each color. The rest of the proof is identical to the discrete case, keeping in mind that any (measurable) set that contains at least one point of each color has an area of at least 1. We will use variants of this simple idea in the sequel.

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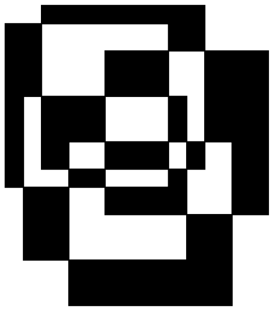


Figure 1: Five unit squares. The area in black is covered an odd number of times.

I then tried to consider the continuous version with respect to other planar shapes. The most natural that comes to one's mind is perhaps the unit disc, and also regular n-gons.

Problem. *Let $\mathcal{F}$ be a family of an odd number of unit discs in the plane. Is it true that the total area of all points in the plane that are contained in an odd number of discs in $\mathcal{F}$ is at least π?*

As it turns out Igor Pak preceded me in considering this question, as it appears as an (unsolved) Exercise 15.14 in [1].

Definition 1. Let T be any given measurable compact set in the plane with positive measure. We define $f_{odd}(T)$ to be the maximum number such that for any collection $\mathcal{F}$ of an odd number of translates of T the set of all points in the plane that belong to an odd number of members from $\mathcal{F}$ has area of at least $f_{odd}(T)$. We denote by $A(T)$ the area of the set T.

Clearly, $f_{odd}(T)/A(T) \in [0,1]$ for any T. As we observed earlier, if T is a square, then $f_{odd}(T) = A(T)$. In fact it can be shown that for any shape T that can tile the plane we have $f_{odd}(T) = A(T)$. At this point one could be tempted to conjecture that $f_{odd}(T) = A(T)$ for any T. This is perhaps the first guess after several failed attempts to construct T with $f_{odd}(T) < A(T)$. The truth however is different as shows the following theorem.

Theorem 1. *For any triangle T we have $f_{odd}(T) = \frac{1}{2}A(T)$.*

We will prove Theorem 1 as a special case of the following more general result about trapezoids:

Theorem 2. *Let T be a trapezoid that is not a parallelogram. Let h denote the height of T and let x_1 and x_2 denote the lengths of the two parallel sides of T, respectively. Then $f_{odd}(T) = \frac{1}{4}h|x_2 - x_1|$.*

Theorem 1 follows from Theorem 2 if we take $x_1 = 0$ in Theorem 2.

Notice that $f_{odd}(T)/A(T)$ can be arbitrary small when T is a trapezoid. This happens when the difference between the lengths of the two parallel sides of T is small compared to their sum. Somewhat surprisingly, this implies that the function $f_{odd}(T)/A(T)$ is far from being continuous. If T is a trapezoid that is very close to a square, then $f_{odd}(T)/A(T)$ is very close to 0 while if T is a square, then $f_{odd}(T)/A(T) = 1$.

It is interesting to note however that the value of $\frac{1}{4}h|x_2 - x_1|$ in Theorem 2 is never actually attained:

Theorem 3. *Let T be a trapezoid that is not a parallelogram. Let h denote the height of T and let x_1 and x_2 denote the lengths of the two parallel sides of T, respectively. Let $\mathcal{F}$ be a finite collection* *of an odd number of translates of T. Then the area of all points in the plane that belong to an odd number of trapezoids in $\mathcal{F}$ is strictly greater than $\frac{1}{4}h|x_2 - x_1|$.*

The proof of Theorem 2 is presented in Section 2. We will provide the proof of Theorem 3 in Section 3 leaving some of the details to the reader.

2 Proof of Theorem 2

We assume without loss of generality that $x_2 > x_1$. Because T is not a parallelogram, it is enough to show the theorem for the trapezoid T' whose vertices are $A = (0,0)$, $B = (1,1)$, $C = (t,1)$ and $D = (t,0)$ where $t = \frac{x_2}{x_2 - x_1} \geq 1$ (the case $t = 1$ is the case where $x_1 = 0$ and T is a triangle). This is because T' is the image of T under a linear transformation that we denote by g (notice that when $x_1 > 0$, the ratio between the parallel sides of T and T' is the same as $\frac{t}{t-1} = \frac{x_2}{x_1}$. If $x_1 = 0$, then both T and T' are triangles and therefore linearly equivalent). Applying a linear transformation g changes the value of $f_{odd}(T)$ by a factor of $|\det g|$. Here we have $|\det g| = \frac{1}{h(x_2 - x_1)}$, because this is the ratio between the area of $T' = g(T)$ and the area of T. After applying the linear transformation g we have $h = 1$, $x_1 = t - 1$ and $x_2 = t$. Hence, if we show the theorem for T' namely, $f_{odd}(T') \geq \frac{1}{4}$, it will imply that $f_{odd}(T) = \frac{1}{|\det g|} f_{odd}(T') = \frac{1}{4}h(x_2 - x_1)$.

We color each point of the plane as follows: We color a point (a, b) by the color $((a \mod \frac{1}{2}), (b \mod \frac{1}{2}))$. Unlike with our coloring of the plane in the case of dealing with a square, it is not true in this case that every translated copy of T' contains precisely one point of each color. However, it is true that every translated copy of T' contains an odd number of points of each color. To see this observe that, up to sets of measure 0, T' is the disjoint union of five regions. $T' = D_1 \cup D_2 \cup Q \cup R_1 \cup R_2$ as shown in Figure 2:

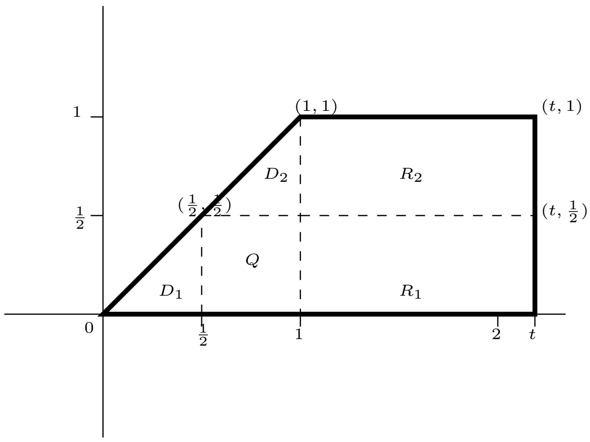


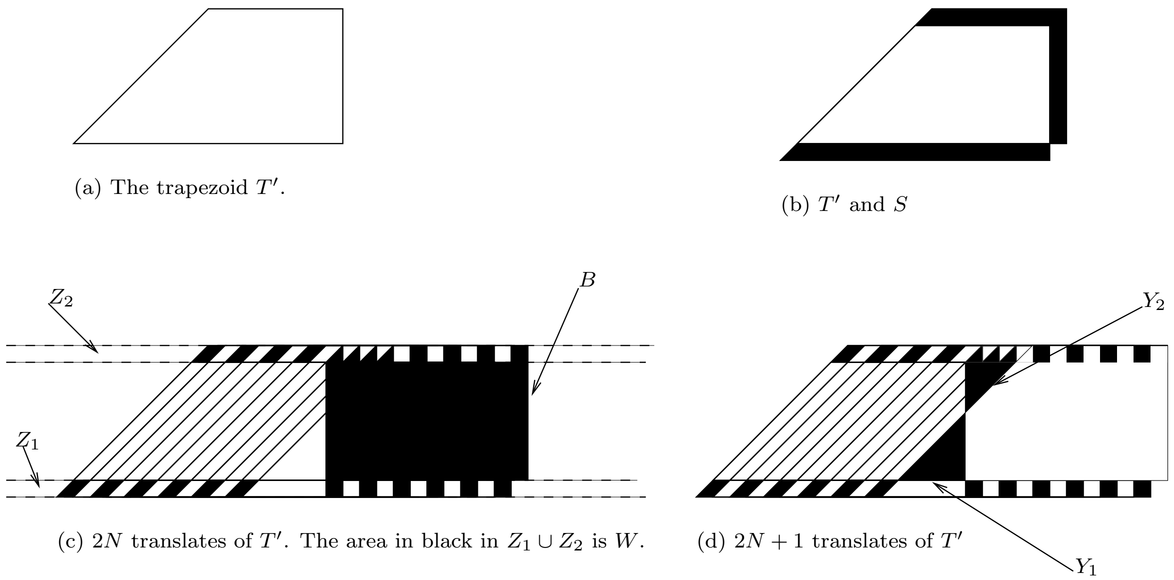
Figure 2: T' is the disjoint union of five regions.

D_1 is the triangle with vertices $(0,0)$, $(\frac{1}{2},\frac{1}{2})$, and $(\frac{1}{2},0)$. D_2 is the triangle with vertices $(\frac{1}{2},\frac{1}{2})$, $(1,1)$, and $(1,\frac{1}{2})$. Q is the square with vertices $(\frac{1}{2},0)$, $(\frac{1}{2},\frac{1}{2})$, $(1,\frac{1}{2})$, and $(1,0)$. R_1 is the rectangle with vertices $(1,0), (1,\frac{1}{2}), (t,\frac{1}{2})$, and $(t,0)$. R_2 is the rectangle with vertices $(1,1), (t,1), (t,\frac{1}{2})$, and $(1,\frac{1}{2})$.

Consider a translated copy of T', that is, $T' + (x, y)$ for some vector (x, y). As we have already seen, $Q + (x, y)$ contains exactly one point of each color. The two rectangles $R_1 + (x, y)$ and $R_2 + (x, y)$ contain the same multi-set of colors and are in fact identical from the coloring point of view. This is because the color of a point (a, b) in the plane is the same as the color of $(a, b + \frac{1}{2})$. Therefore, together they contain an even number of points of each color. The same is true for the two triangles D_1 and D_2. We conclude that every translated copy of T' contains an odd number of points from each color.

As in the case of the square, it follows that for each color there must be a point of this color that belongs to an odd number of translated copies of T'. Because the area of the basic square of colors is $\frac{1}{4}$ this shows that $f_{odd}(T') \geq \frac{1}{4}$.

To complete the proof of Theorem 2 we will show, by construction, that $f_{odd}(T')$ is not greater than $\frac{1}{4}$. Let $S = T' - (\epsilon, \epsilon)$ for some small $\epsilon > 0$. The set of points that belong to just one of T' and S consists of the rectangle whose vertices are $(t, 1-\epsilon), (t-\epsilon, 1-\epsilon), (t, 0)$, and $(t-\epsilon, 0)$ together with another set of points contained in the two stripes $Z_1 = \{-\epsilon \leq y \leq 0\}$ and $Z_2 = \{1 - \epsilon \leq y \leq 1\}$ (see Figure 3(b)). Take $N = \frac{t-1/2}{\epsilon}$ and assume ϵ is chosen so that N is an integer. For every $i = 1, \ldots, N$ let $T'_i = T' + (i\epsilon, 0)$ and $S_i = S + (i\epsilon, 0)$.



(a) The trapezoid T'.

(b) T' and S

(c) $2N$ translates of T'. The area in black in $Z_1 \cup Z_2$ is W.

(d) $2N + 1$ translates of T'

Figure 3: Illustrating the construction in the proof of Theorem 2.

Consider now the family $\mathcal{F}$ of $2N$ translated copies of T' which consists of $T'_1, \ldots, T'_N$ and of $S_1, \ldots, S_N$. The set of points that belong to an even odd number of the sets in $\mathcal{F}$ consists of an axes parallel rectangle B and additional set of points that we denote by W, contained in the stripes Z_1 and Z_2 (see Figure 3(c)). B is the rectangle whose vertices are $(t, 0), (t, 1 - \epsilon), (2t - \frac{1}{2}, 1 - \epsilon)$, and $(2t - \frac{1}{2}, 0)$. Hence B is a rectangle of dimensions $(t - 1/2) \times (1 - \epsilon)$.

We now add to $\mathcal{F}$ the trapezoid $T + (t - 1/2, 0)$. Now $\mathcal{F}$ is a family of an odd number of translated copies of T. The set of those points in the plane that belong to an odd number of sets from $\mathcal{F}$ is the union of the two triangles Y_1 and Y_2 together with points that belong to $Z_1 \cup Z_2$, as illustrated in Figure 3(d). Y_1 is the triangle whose vertices are $(t - \frac{1}{2}, 0), (t, 0)$, and $(t, \frac{1}{2})$. Y_2 is the triangle whose vertices are $(t, \frac{1}{2}), (t, 1 - \epsilon)$, and $(t + \frac{1}{2} - \epsilon, 1 - \epsilon)$. The area of $Y_1 \cup Y_2$ is equal to $\frac{1}{8} + \frac{1}{2}(\frac{1}{2} - \epsilon)^2 = \frac{1}{4} - \frac{\epsilon}{2} + \frac{\epsilon^2}{2}$. The area of those points that belong to $Z_1 \cup Z_2$ and to one of the sets in $\mathcal{F}$ is bounded from above by $4t\epsilon$. It follows that $f_{odd}(T) \leq \frac{1}{4} + 4t\epsilon$. Since this is true for every $\epsilon > 0$ we conclude that $f_{odd}(T) \leq \frac{1}{4}$. ∎

3 Proof of Theorem 3

By applying a suitable linear transformation in the plane, as in the proof of Theorem 2, we may assume that the vertices of T are $A = (0,0)$, $B = (1,1)$, $C = (t,1)$ and $D = (t,0)$ for $t = \frac{x_2}{x_2 - x_1} > 1$ (here we assume $x_1 > 0$, that is, T is a proper trapezoid). Then $h = 1$ and $|x_2 - x_1| = 1$ and we need to show that given a collection $\mathcal{F}$ of odd number of translated copies of T, the area of those points in the plane that belong to odd number of trapezoids from $\mathcal{F}$ is strictly greater than $\frac{1}{4}$.

Let $\mathcal{F}$ be such a collection of an odd number of translated copies of T. We may assume that no two trapezoids in $\mathcal{F}$ coincide or else we can remove such a pair from $\mathcal{F}$, as this will not change the set of points in the plane that is covered an odd number of times by trapezoids in $\mathcal{F}$. Let ℓ be the vertical line supporting an edge of at least one trapezoid from $\mathcal{F}$ such that all trapezoids in $\mathcal{F}$ lie in the closed half-plane bounded to the left of ℓ.

Let $\epsilon > 0$ be a small constant to be determined later. Let ℓ_ϵ be the vertical line that lies to the left of ℓ at distance ϵ from ℓ. We assume that ϵ is small enough so that ℓ_ϵ does not meet any trapezoid not supported by ℓ.

We color the points of the plane in the same way as we did in the proof of Theorem 2, namely, we color the point (x, y) by the color $((x \mod \frac{1}{2}), (y \mod \frac{1}{2}))$. We know, as we argued in the proof of Theorem 2, that every trapezoid in $\mathcal{F}$ contains and odd number of points of each color. The crucial observation is that if we cut off the picture the region H bounded between ℓ and ℓ_ϵ, then it is still true that every trapezoid contains an odd number points of each color. This is because every trapezoid is either disjoint from H or intersects H in a rectangle of dimensions $\epsilon \times 1$. In the latter case notice that the color of a point (x, y) in the plane is the same as the color of $(x, y + \frac{1}{2})$ and therefore a rectangle of dimensions $\epsilon \times 1$ contains either zero or two points of each color.

It follows now that the total area of all points not in H that are covered an odd number of times is at least $\frac{1}{4}$, as in the proof of Theorem 2. The crucial point is that the region H contains points that belong to exactly one trapezoid (and 1 is an odd number), namely, the area that belongs to the highest trapezoid supported by ℓ, but not to the second highest trapezoid supported by ℓ. ∎

We note that the assumption in the proof of Theorem 3 that T is a proper trapezoid is not crucial for the proof and with small modifications the same idea works also if T is a triangle.

References

[1] I. Pak, *Lectures on Discrete Geometry and Convex Polyhedra*, Cambridge U. Press (to appear). Also at http://www.math.ucla.edu/ pak/geompol8.pdf.

[2] The International Mathematics Tournament of the Towns 2009, *http://www.math.toronto.edu/oz/turgor/archives/TT2009F_JAproblems.pdf*