

Points covered an odd number of times by translates

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Abstract

Let T be a fixed triangle and consider an odd number of translated copies of T in the plane. We show that the set of points in the plane that belongs to an odd number of triangles has area of at least half of the area of T . This result is best possible. We discuss related problems as well.

1 Introduction

While preparing puzzles for my 'puzzles course' that I gave in the Technion during 2009 and 2010, I found the following puzzle in the fall's contest of 'the tournament of towns' for the year 2009:

On an infinite grid block 2009 squares are drawn, each composed of N grid squares. Show that at least N grid squares are covered an odd number of times by the original 2009 squares.

Perhaps the shortest solution to this puzzle is an elegant use of the coloring technique: Color the a grid square (a, b) by the color $((a \bmod k), (b \bmod k))$, where $k^2 = N$. We thus have $k^2 = N$ different colors and the crucial observation is that each big square of size N contains precisely one grid square of each color. It follows that there must be at least one grid square of each color class that is covered an odd number of times (because the total number of gridsquares from each color is 2009 with multiple counting).

It is an immediate consequence of this puzzle that also the continuous version of this problem is true. That is, given an odd number of axis-parallel unit squares in the plane the total area of all points covered by an odd number of squares is at least 1. Equality is possible of course, for example if all squares coincide.

It does not take more than a few moments of thinking to see that this easily generalizes to any figure whose translated copies can tile the plane - regular hexagons for instance. Moreover, higher dimensional generalizations are clearly valid.

I then tried to consider the continuous version with respect to other planar shapes. The most natural that comes to one's mind is perhaps the unit disc, and also regular n -gons.

Definition 1. Let T be any given measurable compact set in the plane with positive measure. We define $f_{\text{odd}}(T)$ to be the maximum number such that for any collection \mathcal{F} of an odd number of translates of T the set of all points in the plane that belong to an odd number of members from \mathcal{F} has area of at least $f_{\text{odd}}(T)$. We denote by $A(T)$ the area of the set T .

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Clearly, $f_{\text{odd}}(T)/A(T) \in [0, 1]$ for any T . For any set T whose translated copies can tile the plane we have $f_{\text{odd}}(T) = A(T)$.

Theorem 1. *For any triangle T we have $f_{\text{odd}}(T) = \frac{1}{2}A(T)$.*

It is interesting to note however that the value of $\frac{1}{2}$ in Theorem 1 is never actually attained:

Theorem 2. *Let \mathcal{F} be any finite collection of an odd number of translates of a fixed triangle T . Then the area of all point in the plane that belong to an odd number of members of \mathcal{F} is strictly greater than $\frac{1}{2}A(T)$.*

2 Proof of Theorem 1

We may assume without loss of generality that T is the triangle whose vertices are $(0, 0)$, $(0, 1)$ and $(1, 0)$. This is because we can always apply an affine transformation to the plane taking T to any other given triangle, and observe that affine transformations change the area by a constant factor that is equal to the Jacobian of the transformation (absolute value of its determinant in this case).

For convenience we denote by B the square whose opposite vertices are $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$. We denote by T' and T'' the two smaller triangles composing $T \setminus B$.

We color each point of the plane as follows: We color a point (a, b) by the color $((a \bmod \frac{1}{2}), (b \bmod \frac{1}{2}))$. It is not true anymore that every translated copy of T contains precisely one point of each color. However, it is true that any translated copy of T contains an odd number of points of each color. As a matter of fact either 3 or 1 points of each color. Indeed, let the translated copy of T be $T + (x, y)$. The square $B + (x, y)$ contains precisely one point of each color. The two triangles $T' + (x, y)$ and $T'' + (x, y)$ contain the same set of colors and thus together they contain either 0 or 2 points of each color. It follows that for each color there must be a point of this color that belongs to an odd number of translated copies of T . Because the area of B is half of the area of T , this shows that $f_{\text{odd}}(T) \geq \frac{1}{2}A(T)$. It is left to show that this result cannot be improved. We will show by construction that $f_{\text{odd}}(T)$ is not greater than $\frac{1}{2}A(T)$.

Let $\epsilon = \frac{1}{4N} > 0$ be given, where N is a large integer. We consider the following $2N + 1$ translated copies of T :

$$\{T + (k\epsilon, (k + 1)\epsilon) \mid 0 \leq k < N\} \cup \{T + ((k + 1)\epsilon, k\epsilon) \mid 0 \leq k < N\} \cup \{T\}$$

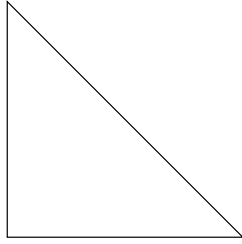
It is left to the reader to verify that the area covered an odd number of times is a subset of the set:

$R \cup T_1 \cup T_2 \cup T_3$, where

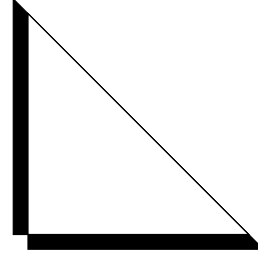
$R = \cup_{k=0}^{N-1} H + (k\epsilon, k\epsilon)$ and H is the square with vertices $(0, 0)$, $(\epsilon, 0)$, $(0, \epsilon)$, (ϵ, ϵ) . Consequently the area of R is $N\epsilon^2 = \frac{1}{4}\epsilon$.

T_1 is the triangle whose vertices are $(\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{3}{4})$, and $(\frac{3}{4}, \frac{1}{4})$ and whose area is therefore equal to $\frac{1}{8}$.

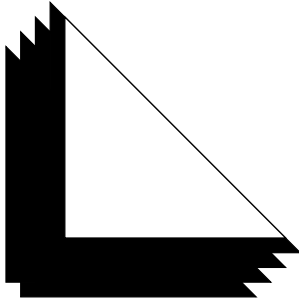
T_2 is the right isosceles triangle whose hypotenuse is the segment between $(\frac{1}{4}, \frac{3}{4})$ and $(\frac{1}{4}, \frac{5}{4} + \epsilon)$. The area of T_2 is $\frac{1}{4}(\frac{1}{2} + \epsilon)^2$.



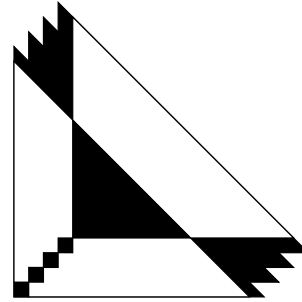
The triangle T



Two copies of T



$2N$ copies of T



$2N + 1$ copies of T

Figure 1: A construction showing that $f_{\text{odd}}(\text{Triangle}) \leq \frac{1}{2}A(\text{Triangle})$.

T_3 is a triangle with the same area as T_2 . T_3 is too a right isosceles triangle. The hypotenuse of T_3 is the interval between $(\frac{3}{4}, \frac{1}{4})$ and $(\frac{5}{4} + \epsilon, \frac{1}{4})$.

Therefore, the area of all points covered by an odd number of translated copies of T is at most $\frac{1}{4} + \frac{\epsilon}{4} + \frac{\epsilon + \epsilon^2}{2}$. This value can be as close as desired to $\frac{1}{4}$ which is half of the area of T . ■

3 Proof of Theorem 2

In this section we prove theorem 2 and show that the kind of construction we had in the proof of Theorem 1 is best possible in the sense that no construction of odd number of copies of T is such that the set of points that belong to an odd number of copies has area precisely half of the area of T , but it is always strictly greater than half of the area of T .

Let \mathcal{F} be the collection of $2n + 1$ translated copies of T . We may assume that no two triangles in \mathcal{F} coincide or else we may ignore them in pairs. Let ℓ be the line whose slope is -1 supporting a hypotenuse of at least one triangle from \mathcal{F} such that no other hypotenuse of a triangle in \mathcal{F} lies above ℓ . Denote by T_1, \dots, T_k all the triangles in \mathcal{F} whose hypotenuse is contained in ℓ .

Let $\epsilon > 0$ be a small constant to be determined later and let ℓ_ϵ be the line obtained from ℓ by translating it downwards a distance of ϵ . We assume that ϵ is small enough so that ℓ_ϵ does not meet any triangle in \mathcal{F} but T_1, \dots, T_k . Replace each triangle T_1, \dots, T_k in \mathcal{F} by its intersection T'_i ($i = 1, 2, \dots, k$) with the half-plane below ℓ_ϵ . We denote the new family of triangles by \mathcal{F}' .

Repeating the proof of Theorem 1, for all triangles in \mathcal{F}' but T'_1, \dots, T'_k it is still true that each of them contains an odd number of points from each color. In fact it is true also for T'_1, \dots, T'_k except for a small portion of the colors (points) whose measure is at most $\epsilon^2/2$ in each of the k triangles. Therefore, the area of the set of all points in the plane that belong to an odd number of triangles from \mathcal{F}' is at least $\frac{1}{2} - k\epsilon^2/2$. Now assume without loss of generality that T_1, \dots, T_k are indexed according to the height of their top vertex, so that the top vertex of T_1 is the highest. Let d denote the distance between the top vertex A of T_1 and the top vertex B of T_2 (note that both A and B lie on ℓ). Here we assume that $k > 1$, however if this is not the case then just consider B to be the other vertex of T_1 on ℓ . We observe that the entire parallelogram between ℓ and ℓ_{ϵ} below the segment AB belongs only to T_1 , from all triangles in \mathcal{F} , and therefore its area should be counted with the points in the plane that belong to an odd number of triangles from \mathcal{F} . It follows now that the area of all point in the plane that belong to an odd number of triangles from \mathcal{F} is at least $\frac{1}{2} - k\epsilon^2/2 + \frac{d}{\sqrt{2}}\epsilon$. Taking $\epsilon > 0$ small enough we conclude that this area is strictly greater than $\frac{1}{2}$. ■

4 The case of trapezoids

In this section we completely resolve the case of a trapezoid.

Theorem 3. *Let T be a trapezoid that is not a parallelogram. Let h denote the height of T and let x_1 and x_2 denote the lengths of the two parallel sides of T , respectively. Then $f_{\text{odd}}(T) = \frac{1}{4}h|x_1 - x_2|$.*

Notice that the value $\frac{1}{4}h|x_1 - x_2|$ is nothing else but half of the area of the triangle whose removal from T creates a parallelogram. This of course generalizes the result in Theorem 1 and the proof is indeed very similar and so is the construction.

We note also that $f_{\text{odd}}(T)/A(T)$ can be arbitrary small in terms of the area of T when T is a trapezoid. This shows that f_{odd}/A attains all values between 0 (still not inclusive) and $\frac{1}{4}$.

Proof of Theorem 3. Let A, B, C, D be the vertices of T in a clockwise order on its boundary. By applying a suitable linear transformation of the plane (just as in the proof of Theorem 1) we may assume that $A = (0, 0)$, $B = (1, 1)$, $C = (t, 1)$ and $D = (t, 0)$ for some $t \geq 1$ (note that the case $t = 1$ is the case of a triangle). Here $h = 1$, $x_1 = t$ and $x_2 = t - 1$. We need to show that $f_{\text{odd}}(T) = \frac{1}{4}$.

We color each point of the plane as follows: We color a point (a, b) by the color $((a \bmod \frac{1}{2}), (b \bmod \frac{1}{2}))$. As in the case of a triangle it is not true that every translated copy of T contains precisely one point of each color. However, it is true that any translated copy of T contains an odd number of points of each color. To see this observe that T is the disjoint union of three shapes. $T = T' \cup R_1 \cup R_2$ where T' is the triangle with vertices $(0, 0)$, $(1, 1)$, and $(1, 0)$, R_1 is the rectangle with vertices $(1, 1)$, $(t, 1)$, $(t, \frac{1}{2})$, and $(1, \frac{1}{2})$, and R_2 is the rectangle with vertices $(1, \frac{1}{2})$, $(t, \frac{1}{2})$, $(1, 0)$, and $(t, 0)$. Consider a translated copy of T $T + (x, y)$. As we have already seen, $T' + (x, y)$ contains odd number of points of each color. The two rectangles $R_1 + (x, y)$ and $R_2 + (x, y)$ contain the same multi-set of colors and are in fact identical from the coloring point of view. Therefore, together they contain an even number of points of each color. We conclude that every translated copy of T contains an odd number of points from each color.

As in the proof of Theorem 1, it follows that for each color there must be a point of this color that belongs to an odd number of translated copies of T . Because the area of the basic square of colors is $\frac{1}{4}$ this shows that $f_{\text{odd}}(T) \geq \frac{1}{4}$.

It is left to show that this result cannot be improved. We will show by construction that $f_{\text{odd}}(T)$ is not greater than $\frac{1}{4}$. Let $S = T + (\epsilon, \epsilon)$ for some small $\epsilon > 0$. The set of points that belong to just one of T and S consists of the rectangle whose vertices are $(t, 1)$, $(t + \epsilon, 1)$, (t, ϵ) , and $(t + \epsilon, \epsilon)$ as well as another set of points contained in the two stripes $0 \leq y \leq \epsilon$ and $1 \leq y \leq 1 + \epsilon$. Now let $T_i = T + (i\epsilon, 0)$ and $S_i = S + (i\epsilon, 0)$ and set $N = \lceil \frac{t-1/2}{\epsilon} \rceil$.

The set of points that belong to an even odd number of the sets in $\cup_{i=0}^N (T_i \cup S_i)$ consists of an axes parallel rectangle B of dimensions $(1 - \epsilon) \times t - 1/2$ and additional points whose total measure is $O(\epsilon)$. Taking now the additional one translated copy T that is $T + (t - 1/2, \epsilon)$, we observe a collection of an odd number of translated copies of T such that the set of points contained in an odd number of those copies has an area $\frac{1}{4} + O(\epsilon)$.

As in the case of triangles here too there is no construction of an odd number of copies of T such that the area of all points belonging to an odd number of these copies equals $1/4$. We leave the details to the reader. ■