Nearly Equal Distances in Metric Spaces

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Abstract

Let (X,d) be any finite metric space with n elements. We show that there are two pairs of distinct elements in X that determine two nearly equal distances in the sense that their ratio differs from 1 by at most $\frac{9\log n}{n^2}$. This bound (apart for the multiplicative constant) is best possible and we construct a metric space that attains this bound.

We discus related questions and consider in particular the Euclidean metric space.

1 Introduction

Let (X,d) be a finite metric space with $n = |X|$ elements. Consider all the $\binom{n}{2}$ distances between pairs of distinct elements in X. In principle all these distances can be equal as is the case if the metric d is the discrete metric. However, can all those distances be very much different from each other? In Section 2 we prove the following theorem which asserts that there always must be two distances that are nearly equal.

Theorem 1. *Any finite metric space (X,d) of $n \geq 3$ elements contains two pairs (not necessarily disjoint) of distinct elements $\{x,y\}$ and $\{a,b\}$ such that $|\frac{d(x,y)}{d(a,b)} - 1| \leq \frac{9\log n}{n^2}$.*

The bound in Theorem 1 is best possible for general metric spaces:

Theorem 2. *For every n there exists a metric space (X,d) on n elements such that for every two pairs (not necessarily disjoint) of distinct elements $\{x,y\}$ and $\{a,b\}$ we have $|\frac{d(x,y)}{d(a,b)} - 1| \geq \frac{\log n}{20n^2}$.*

The proof of Theorems 1 and 2 is given in Section 2, which contains also a generalization of Theorem 1 to the case of k nearly equal distances.

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Notice that there are two different kinds of pairs of distances in a metric space. That is, given two distinct pairs of elements $\{x, y\}$ and $\{a, b\}$, either they are disjoint, or they have one element in common. Theorem 1 guarantees two pairs of elements with nearly equal distances. However, it does not tell us anything on whether these two pairs are disjoint or not.

If we insist on finding two disjoint pairs of elements $\{x, y\}$ and $\{a, b\}$ such that $\frac{d(x,y)}{d(a,b)}$ is close to 1, then in very simple metric spaces we are doomed to fail. Consider for example the metric space $X = \{1, 3, 3^2, 3^3, \ldots, 3^{n-1}\}$, as a subset of $\mathbb{R}$ with the ordinary Euclidean metric. Given two disjoint pairs of elements of X, $\{3^x, 3^y\}$ and $\{3^a, 3^b\}$, assume without loss of generality that b is the maximum number among $\{x, y, a, b\}$. Then $d(3^a, 3^b) = 3^b - 3^a \geq 3^b - 3^{b-1} = 2\cdot 3^{b-1}$ while (assuming without loss of generality that $y > x$) $d(3^y, 3^x) = 3^y - 3^x \leq 3^y \leq 3^{b-1}$. Hence the ratio between $d(3^a, 3^b)$ and $d(3^y, 3^x)$ is at least 2, quite far from 1. It turns out however that one can always find two distinct pairs of elements in X that are not disjoint with nearly equal distances:

Theorem 3. *Let (X, d) be a metric space with n elements. There exist three distinct elements x, a, b in X such that $|\frac{d(x,a)}{d(x,b)} - 1| \leq \frac{3}{n}$.*

Here too we can show by construction that the bound in Theorem 3 is best possible:

Theorem 4. *For every n there exists a metric space (X, d) such that for every three distinct elements x, a, b in X we have $|\frac{d(x,a)}{d(x,b)} - 1| \geq \frac{1}{2n}$.*

The proofs of Theorem 3 and 4 are presented in Section 3 together with a generalization of Theorem 3 to the case of k nearly equal distances from the same point. Section 4 contains some discussion in the specific case of subsets of the Euclidean metric space $\mathbb{R}$, where there are some very interesting open problems. In Section 5 we bring some consequences of the above theorems to the Euclidean space as well as more open problems.

2 The existence of nearly equal distances in a finite metric space

We start with the proof of Theorem 1:

Let $x, y \in X$ be a pair of distinct elements in X such that $d(x, y)$ is minimum. Without loss of generality we may assume that $d(x, y) = 1$, for otherwise we replace the metric $d(\cdot, \cdot)$ by the metric $\frac{1}{d(x,y)} d(\cdot, \cdot)$.

Let $1 = d_1 \leq d_2 \leq \ldots \leq d_{\binom{n}{2}}$ denote all the distances between pairs of distinct elements in X listed in an increasing order. In fact we may assume strict inequalities between these distances for otherwise we are done.

If we assume by contradiction that for every $1 \leq i < \binom{n}{2}$ we have $\frac{d_{i+1}}{d_i} \geq 1 + \frac{9 \log n}{n^2}$, then $d_{\binom{n}{2}} \geq (1 + c\frac{\log n}{n^2})^{\binom{n}{2}-1} \geq e^{\frac{9}{3} \log n} = n^3$. It follows that there is at least one element $z \in X$

such that $d(x,z) \geq \frac{1}{2}n^3$. Therefore, we have

$$|\frac{d(y,z)}{d(x,z)} - 1| = |\frac{d(y,z) - d(x,z)}{d(x,z)}| \leq \frac{d(x,y)}{d(x,z)} \leq \frac{2}{n^3}.$$

∎

In [4] Erdős and Turán show that there are sets of cn integers between 1 and n^2 with pairwise distinct differences, where $c > 0$ is an absolute constant that can be taken to be arbitrary close to $\frac{1}{\sqrt{2}}$ for large enough n. These sets are sometimes called *Sidon sets* after Simon Sidon who initiated their study.

Consider such a set X of integers as a finite metric space with the ordinary Euclidean metric induced from $\mathbb{R}$. Since any two distances among pairs of numbers from the set X differ by at least 1 (because they are distinct integers), we have for every two distinct pairs $\{x,y\}$ and $\{a,b\}$ from X:

$$|\frac{d(a,b)}{d(x,y)} - 1| = |\frac{d(a,b) - d(x,y)}{d(x,y)}| \geq \frac{1}{d(x,y)} \geq \frac{1}{n^2}.$$

This shows that apart from the $\log n$ term the bound in Theorem 1 is best possible already in subsets of $\mathbb{R}$ with the ordinary Euclidean metric.

Can we do better than Sidon sets in $\mathbb{R}$? This question is open and we state it as a problem:

Problem A. *Do there exists sets X of n real numbers such that for every two distinct pairs $\{x,y\}$ and $\{a,b\}$ from X we have $|\frac{d(a,b)}{d(x,y)} - 1| = \frac{\omega(n)}{n^2}$, where $\omega(n)$ is a function that goes to infinity with n.*

We shall now prove Theorem 2 and show by construction that there are metric spaces in which the answer to the analogous problem to Problem A is positive.

Proof of Theorem 2. We will assume that n is large enough ($n > 100$ should be enough). We define the following metric $d(\cdot,\cdot)$ on the elements $x_1, \ldots, x_n$: For $i = j$ we define $d(x_i, x_i) = 0$ and for $i < j$ we define $d(x_i, x_j) = d(x_j, x_i) = (1 + \frac{\log n}{10n^2})^{\binom{j+1}{2} - i}$. We first show that this is indeed a metric. We observe that for $1 \leq i < j$ we have $\binom{j}{2} < \binom{j+1}{2} - i < \binom{j+1}{2}$. Therefore, if $i < j$ and $a < b$ are all integers and $\binom{j+1}{2} - i = \binom{b+1}{2} - a$, then $j = b$ and $i = a$. Notice also that if $i < j$ and $a < b$ and $j > b$, then $d(x_i, x_j) \geq \binom{j}{2} \geq \binom{b+1}{2} \geq d(x_a, x_b)$.

Hence, in order to show that d satisfies the triangle inequality, then given $i < j < k$ it is enough to show that $d(x_i, x_k) \leq d(x_i, x_j) + d(x_j, x_k)$, as $d(x_i, x_k)$ is the largest among the three mutual distances between x_i, x_j, and x_k.

Set $\alpha = 1 + \frac{\log n}{10n^2}$. Now, $d(x_i, x_k) \leq d(x_i, x_j) + d(x_j, x_k)$ is equivalent to $\alpha^{\binom{k+1}{2} - i} \leq \alpha^{\binom{j+1}{2} - i} + \alpha^{\binom{k+1}{2} - j}$. After some easy manipulation we obtain the following equivalent inequality:

$$\alpha^{\binom{k+1}{2}} \leq \frac{\alpha^{\binom{j+1}{2} + j - i}}{\alpha^{j-i} - 1}. \tag{1}$$

The left hand side of (1) is maximum when $k = n$. Notice that $\alpha^{\binom{k+1}{2}} \leq \alpha^{\binom{n+1}{2}} \leq (1 + \frac{\log n}{10n^2})^{n^2} \leq n^{1/5}$.

The right hand side of (1) is greater or equal to $\frac{1}{\alpha^n - 1}$. Therefore, it is enough to show that $n^{1/5} \leq \frac{1}{\alpha^n - 1}$. This last inequality is true because $\alpha^n - 1 = (1 + \frac{\log n}{10n^2})^n - 1 \leq \frac{\log n}{5n}$ for large enough n.

It is left to show that the ratio between every two distances in our metric space is not too close to 1. This is quite easy to show: Assume $i < j$ and $a < b$ and $\{i, j\} \neq \{a, b\}$. We have $d(x_i, x_j) = \alpha^{\binom{j+1}{2} - i}$ and $d(x_a, x_b) = \alpha^{\binom{b+1}{2} - a}$. As we observed earlier, unless $\{i, j\} = \{a, b\}$ (and indeed we assume they are distinct pairs), it must be that $\binom{j+1}{2} - i \neq \binom{b+1}{2} - a$. Therefore, $\frac{d(x_i, x_j)}{d(x_a, x_b)}$ is either at least α or at most $1/\alpha$. In either case, $|\frac{d(x_i, x_j)}{d(x_a, x_b)} - 1| > \frac{\log n}{20n^2}$. ∎

We end this section with a generalization of Theorem 1. We show that in every metric space one can find not only two distances that are nearly equal but any fixed number of them:

Theorem 5. *Let (X, d) be a metric space with n elements and let $k < \frac{n}{100}$ be a positive integer. Then there exist k distinct pairs of elements from X $(a_1, b_1), \ldots, (a_k, b_k)$ such that for every $1 \leq i < j \leq k$ we have $|\frac{d(a_i, b_i)}{d(a_j, b_j)} - 1| \leq \frac{20k \log n}{n^2}$.*

Proof. The idea of the proof is similar to the one in proof of Theorem 1 and we will put more emphasis on the differences between the two.

For every $r > 0$ and $x \in X$ we denote by $B_r(x)$ the set of all elements in X whose distance from x is at most r. Let r be the minimum number such that there exists $x \in X$ with $|B_r(x)| \geq k + 1$. Let $x_1, \ldots, x_k$ be k elements different from x at distance at most r from x.

We claim that there are at most kn distances that are smaller than or equal to r. Indeed, this is because the number of distances smaller than or equal to r is exactly $\frac{1}{2} \sum_{y \in X} (|B_r(y)| - 1)$. If this sum is greater than kn, then there must exist y with $|B_r(y)| - 1 > 2k$. Notice that y is at distance r from at most $k - 1$ elements of X or else there are k equal distances in X and we are done. This implies that for small enough $\epsilon > 0$ we have $|B_{r-\epsilon}(y)| \geq k + 2$ contradicting the minimality of r.

Because $k \leq \frac{n}{100}$, we conclude that there are at least $\frac{n^2}{3}$ pairs in X whose distance from each other is greater than r.

The rest of the proof is very similar to the proof of Theorem 1. We may assume that the maximum distance between two elements of X is at most $2rn^3$. This is because otherwise there exists $z \in X$ with $d(x, z) > rn^3$. We claim that in this case the ratio between any two of the k distances $d(x_1, z), \ldots, d(x_k, z)$ differs from 1 by at most $\frac{1}{n^3}$. Indeed, notice that $d(x_i, z) > rn^3 - r$ for every $1 \leq i \leq k$ and therefore for every $1 \leq i, j \leq n$:

$$|\frac{d(x_i, z)}{d(x_j, z)} - 1| = \frac{|d(x_i, z) - d(x_j, z)|}{d(x_j, z)} \leq \frac{2r}{rn^3 - r} \leq \frac{1}{n^3}.$$

Let $d_1 \leq d_2 \leq \ldots \leq d_{n^2/3}$ be $n_2/3$ distances greater than r between distinct pairs of elements in X. If there is an index $1 \leq i \leq n^2/3 - k + 1$ such that $\frac{d_{i+k-1}}{d_i} \leq 1 + \frac{20k \log n}{n^2}$, then the ratio between any two of the distances $d_i, \ldots, d_{i+k-1}$ differs from 1 by at most $\frac{20k \log n}{n^2}$. Hence we may assume that for every $1 \leq i \leq n^2/3 - k + 1$ we have $\frac{d_{i+k-1}}{d_i} > 1 + \frac{20k \log n}{n^2}$. This implies that $\frac{d_{n^2/3}}{d_1} > (1 + \frac{20k \log n}{n^2})^{\frac{n^2}{3k}} > 2n^3$ and consequently $d_{n^2/3} > 2rn^3$ contradicting our assumption. ∎

3 Nearly equal distances from a point

Theorem 1 guarantees that in any metric space with n elements there exist two distances that are nearly equal and quantitatively "nearly equal" means here that their ratio differs from 1 by at most $\frac{c \log n}{n^2}$ for some absolute constant c. The idea of the proof is that either there is an element $z \in X$ that is so much distanced from $x, y \in X$ so that $d(z, x)$ and $d(z, y)$ are almost the same, or if this is not the case, then the maximum distance cannot be so much larger than the minimum distance and then because there are n^2 distances we can find two that are very close.

As we already observed in Section 1, if one seeks for two nearly equal distances, then one cannot avoid considering two distances from the same element (z above) in X. Theorem 3, that we shall now prove, does indeed show that there must exist two nearly equal distances from the same element in X. However, if we only seek for such two nearly equal distances, then we can no longer guarantee as good bound as in Theorem 1. To get the bound in Theorem 1 one needs to consider also distances between two disjoint pairs of element in X, as we shall see that the bound in Theorem 3 is best possible (Theorem 4 below).

Before proving Theorem 3, we note that using a similar approach to the one in Theorem 1 one can immediately show that there are two distinct distances from the same point whose ratio differs from 1 by at most $\frac{c \log n}{n}$ for some absolute constant $c > 0$. However, with some extra effort we can remove the $\log n$ factor to obtain the tight result in Theorem 3:

Proof of Theorem 3. Assume the contrary, that is, for every three distinct element $x, a, b \in X$ we have $|\frac{d(x,a)}{d(x,b)} - 1| > \frac{3}{n}$. Let $M = d(x, y)$ be the maximum distance in X, between two elements x and y. Let $z \in X$ be such that there are at least $\frac{n}{3}$ elements $p \in X$ with $d(x, p) \geq d(x, z)$ and there are at least $\frac{n}{3}$ elements $q \in X$ with $d(y, q) \geq d(y, z)$. We can easily find such z by taking z that is not among the $n/3$ farthest elements from x and that is not among the $n/3$ farthest elements from y.

We claim that $d(x, z) < M/2$. Indeed, let $p_1, \ldots, p_{n/3}$ be such that $d(x, z) \leq d(x, p_1) \leq \ldots \leq d(x, p_{n/3})$. Because we have $\frac{d(x,p_{i+1})}{d(x,p_i)} > 1 + \frac{3}{n}$, then $d(x, p_{n/3}) \geq (1 + \frac{3}{n})^{n/3} d(x, z) > 2d(x, z)$. Therefore $d(x, z) < \frac{1}{2} d(x, p_{n/3}) \leq \frac{1}{2} M$.

In a symmetric way we show that $d(y, z) < M/2$. This leads to a contradiction as $M = d(x, y) \leq d(x, z) + d(y, z) < \frac{1}{2} M + \frac{1}{2} M$. ∎

We will now show, by construction, that the result in Theorem 3 is best possible (up to the constant factor 3) in general metric spaces:

Proof of Theorem 4. Let $X = \{x_1, \dots, x_n\}$ and define the following metric d on X. For every $i \neq j$ we define $d(x_i, x_j) = n + (i + j \mod n)$ and of course we define $d(x_i, x_i) = 0$ for every $1 \leq i \leq n$. It is almost immediate to see that d is indeed a metric. The triangle inequality is almost trivially satisfied as $n \leq d(x_i, x_j) < 2n$ for every $i \neq j$.

To see that (X, d) realizes the bound in Theorem 3, notice that for every three distinct indices $1 \leq i, j, k \leq n$ we have

$$|\frac{d(x_i, x_j)}{d(x_i, x_k)} - 1| = \frac{|d(x_i, x_j) - d(x_i, x_k)|}{d(x_i, x_k)} \leq \frac{1}{2n}.$$

The last inequality is because $|d(x_i, x_j) - d(x_i, x_k)| \geq 1$, as the two distances $d(x_i, x_j)$ and $d(x_i, x_k)$ are distinct, and because $d(x_i, x_k) < 2n$. ∎

Just as in the case of Theorem 1, we can generalize Theorem 3 and find k nearly equal distances from the same point:

Theorem 6. *Let $k < \frac{n}{100}$ be an integers and let (X, d) be a metric space with n elements. Then there exist distinct elements $x, a_1, \dots, a_k \in X$ such that for every two distinct indices i and j between 1 and n we have $|\frac{d(x,a_i)}{d(x,a_j)} - 1| \leq \frac{3k}{n}$.*

Here the proof is a straight forward generalization of the proof of Theorem 3, and we leave the details to the reader.

4 What about finite subsets of $\mathbb{R}$?

The bounds in Theorems 1 and 3 are true for general metric spaces and as we have seen they cannot be improved in this generality. It is of course very well possible that for specific metric spaces the bounds are very different and much better. In the extreme case of a discrete metric space one can always find two (or any fixed number of) equal distances.

Of the most interesting metric spaces are subsets of the Euclidean metric spaces. Because the situation in one dimensional Euclidean space (namely $\mathbb{R}$) is already interesting and (probably) difficult enough, we shall concentrate our discussion on this case.

As we have already noted before the proof of Theorem 1, Sidon sets in $\mathbb{R}$ give rise to metric spaces A of n real numbers such that for every $a, b, x, y \in A$ where $\{a, b\} \neq \{x, y\}$ we have $|\frac{|a-b|}{|x-y|} - 1| \geq \frac{1}{n^2}$. This almost matches the upper bound in Theorem 1 and the big open problem here is whether the $\log n$ factor in Theorem 1 is indeed required in the case of subsets of $\mathbb{R}$. We leave this question open without even daring to conjecture anything here (see also Problem A in Section 2).

We can ask an analogous question about Theorem 3 and it turns out that this is not less interesting:

Problem B. Given a $A \subset \mathbb{R}$ with $|A| = n$ can we always find $a, b, c \in A$ such that $|\frac{|a-b|}{|a-c|} - 1| = o(\frac{1}{n})$?

If the answer to Problem B is positive this will mean that a construction analogous to the one in Theorem 4 cannot be realized in the one dimensional Euclidean metric space. It is possible however to find subsets of $\mathbb{R}$ that come close to the bound in Theorem 3. Perhaps not very surprisingly this has to do with subsets of integers with no 3-term arithmetic progression.

Suppose $A \subset \{1, \ldots, N\}$ consists of n integers and contains no 3-term arithmetic progression. Then for every distinct $a, b, c \in A$ $|a-b| \neq |a-c|$ and this implies $||a-b| - |a-c|| \geq 1$. Therefore,

$$|\frac{|a-b|}{|a-c|} - 1| \geq \frac{||a-b| - |a-c||}{|a-c|} \geq \frac{1}{|a-c|} \geq \frac{1}{N}.$$

Hence if we can find such A where N is not to big compared to n, then no two distances from the same point can be too close together, in the sense that their ratio will not be too close to 1. This is equivalent to finding as large as possible subset of $\{1, \ldots, N\}$ with no 3-term arithmetic progression. It is a well known result of Semerédi [6] that the cardinality of a subset of $\{1, \ldots, N\}$ with no 3-term arithmetic progression must have cardinality $o(N)$. Therefore, no such construction can match the bound in Theorem 3 as does the construction in Theorem 4.

Finding large subsets of $\{1, \ldots, N\}$ with no 3-term arithmetic progression is a well studied area of mathematics that started with Erdős and Turán [3] who also constructed sets of $\Omega(N^{\log_3 2})$ integers from $\{1, \ldots, N\}$ with no 3-term arithmetic progression. Their construction was improved in [5] and then by Behernd [1] who gave the first near linear construction. The best known construction is due to Elkin [2] who slightly improved on Behernd construction and found a subset of $\{1, \ldots, N\}$ of cardinality $\Omega(\frac{N \log^{1/4} N}{2^{2\sqrt{2}\sqrt{\log_2 N}}})$ with no 3-term arithmetic progression.

Behernd's and Elkin's constructions imply that one can find n integers from the set $\{1, \ldots, n2^{c\sqrt{\log n}}\}$ with no 3-term arithmetic progression for some absolute constant c. As we have seen above, this directly implies a construction of n real numbers such that the ratio between any two distances from the same point differs from 1 by at least $\frac{1}{n2^{c\sqrt{\log n}}}$. This almost matches the bound of $\frac{1}{n}$ in Theorem 3 but not quite.

This raises the following interesting question closely relates to sets with no arithmetic progressions. For a set A of real numbers we say that $a, b, c \in A$ form a $\frac{1}{n}$-3-*term* arithmetic progression if $a < b < c$ and $|\frac{b-a}{c-b} - 1| \leq \frac{1}{n}$.

Problem C. *What is the largest cardinality of a set $A \subset \mathbb{R}$ with no $\frac{1}{n}$-3-term arithmetic progression?*

Theorem 3 implies that the answer to Problem C must be $O(n)$. Can it be $\Omega(n)$, or must it be $o(n)$? We leave this question open as well.

It is interesting to note that the bound in Theorem 6 is best possible for fixed $k \geq 3$ even for subsets of $\mathbb{R}$. Indeed, consider the set $A = \{1, 2, \ldots, n\}$. Now for every distinct

$x, a_1, a_2, a_3 \in A$, two of the distances $|x - a_1|, |x - a_2|$, and $|x - a_3|$ must be different and thus differ by at least 1. It follows that the ratio of these two distances differs from 1 by at least $\frac{1}{n}$.

5 An example of an application in $\mathbb{R}^2$

In this section we raise some questions in the Euclidean space in the spirit of Theorem 1. We use our results in previous sections to provide nontrivial answer to one of these problems.

Consider n points in the plane and the $\binom{n}{3}$ triangles determined by these n points. That is, the $\binom{n}{3}$ triangles whose vertices are from a given set of n points. Can we always find two triangles with nearly equal area? nearly equal perimeter? We should define of course what "nearly equal" means here, but we can always look for the best estimate we can give in terms of n.

Analogous questions can be asked in higher dimensions and in various similar ways. In order to illustrate possible applications of Theorem 1 and 3, we prove the following theorem about triangles with nearly equal areas in the plane.

Theorem 7. *Let P be a set of n points in general position in the plane. There exist two triangles determined by P, Δ_{xyz} and Δ_{abc} such that:*

$$|\frac{Area(\Delta_{abc})}{Area(\Delta_{xyz})} - 1| \leq \frac{60 \log^{1/3} n}{n^{2/3}}.$$

Proof. Set $k = \sqrt[3]{n \log n}$. By Theorem 6, there exist $x, y_1, \ldots, y_k$ such that for every distinct i and j we have

$$|\frac{||x - y_i||}{||x - y_j||} - 1| \leq \frac{3k}{n} = \frac{3 \log^{1/3} n}{n^{2/3}}.$$

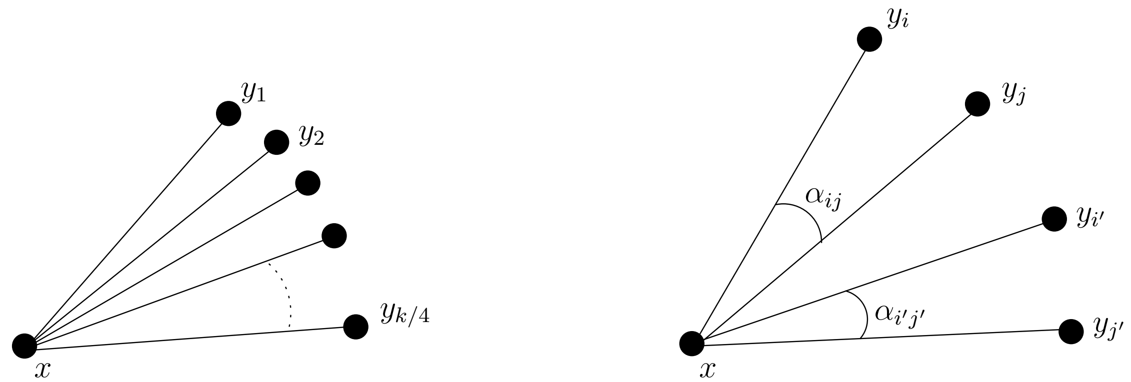

Figure 1: Illustration of the proof of Theorem 7.

Without loss of generality assume that $y_1 - x, \ldots, y_{k/4} - x$ lie in the first quadrant (see Figure 1). For every $1 \leq i, j \leq k/2$ we denote by α_{ij} the angle $\measuredangle y_i x y_j$. We define the following metric d on $y_1, \ldots, y_{k/2}$: $d(y_i, y_j) = \sin \alpha_{ij}$. To see the d is indeed a metric notice that for every $0 \leq \alpha, \alpha' \leq \frac{\pi}{2}$ we have $\sin(\alpha + \alpha') \leq \sin(\alpha) + \sin(\alpha')$.

By Theorem 1, there exist two distinct pairs of indices $\{i, j\}$ and $\{i', j'\}$ such that

$$|\frac{d(y_i, y_j)}{d(y_{i'}, y_{j'})} - 1| \leq \frac{9 \log(k/4)}{(k/4)^2} < \frac{40 \log^{1/3} n}{n^{2/3}}.$$

Notice that we have:

$$\begin{aligned} Area(\Delta_{xy_iy_j}) &= \frac{1}{2} ||x - y_i|| ||x - y_j|| d(y_i, y_j), \\ Area(\Delta_{xy_{i'}y_{j'}}) &= \frac{1}{2} ||x - y_{i'}|| ||x - y_{j'}|| d(y_{i'}, y_{j'}). \end{aligned}$$

Without loss of generality assume that $Area(\Delta_{xy_iy_j}) \geq Area(\Delta_{xy_{i'}y_{j'}})$.

Therefore,

$$\begin{aligned} 1 &\leq \frac{Area(\Delta_{xy_iy_j})}{Area(\Delta_{xy_{i'}y_{j'}})} = \frac{||x - y_i||}{||x - y_{i'}||} \frac{||x - y_j||}{||x - y_{j'}||} \frac{d(y_i, y_j)}{d(y_{i'}, y_{j'})} \\ &\leq (1 + \frac{3 \log^{1/3} n}{n^{2/3}})(1 + \frac{3 \log^{1/3} n}{n^{2/3}})(1 + \frac{40 \log^{1/3} n}{n^{2/3}}) < 1 + \frac{60 \log^{1/3} n}{n^{2/3}}. \qquad (2) \end{aligned}$$

∎

We note that the bound in Theorem 7 is not likely to be best possible, and the best possible bound is unknown to us.

References

[1] F. A. Behrend, On sets of integers which contain no three terms in arithmetic progression, *Proc. Nat. Acad. Sci. USA* **32** (1946) 331–332.

[2] M. Elkin, An improved construction of progression-free sets, *Israel J. Math.* **184** (2011), 93–128.

[3] P. Erdős and P. Turán, On some sequences of integers, *J. London Math. Society*, **11** 261–264, 1936.

[4] P. Erdős, P. Turán, On a problem of Sidon in additive number theory, and on some related problems. *J. London Math. Soc.* **16**, (1941) 212–215.

[5] R. Salem and D. Spencer, On sets of integers which contain no three in arithmetic progression, *Proc. Nat. Acad. Sci. (USA)*, **28** 561–563, 1942.

[6] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, *Acta Arithm.*, **27** 299–345, 1975.