Regular Matchstick Graphs

Sascha Kurz and Rom Pinchasi

Abstract

A *matchstick graph* is a plane geometric graph in which every edge has length 1 and no two edges cross each other. It was conjectured that no 5-regular matchstick graph exists. In this paper we prove this conjecture.

1 Introduction

A *matchstick graph* is a plane geometric graph in which every edge is a line segment of length 1 and no two edges cross. (See for example the Harborth graph in Figure 1.)

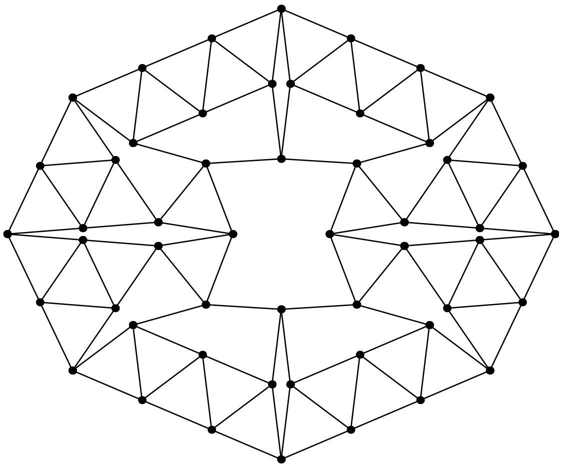

Figure 1: The Harborth graph.

We call a matchstick graph r-regular if every vertex has degree r. At an Oberwolfach meeting for discrete geometry in 1981 Heiko Harborth asked for the minimum vertex number $m(r)$ of an r-regular matchstick graph; see also [2]. Obviously we have $m(0) = 1$, $m(1) = 2$, and $m(2) = 3$, corresponding to a single vertex, a single edge, and a triangle, respectively.

The determination of $m(3) = 8$ is an entertaining amusement. For degree $r = 4$ the exact determination of $m(4)$ is unsettled so far. The smallest known example is the so-called Harborth graph, see, e.g., [3, p. 171], yielding $m(4) \leq 52$. While it is not too hard to show $m(4) \geq 20$ by hand, a customized exhaustive computer search [4] is needed to obtain the lower bound $m(4) \geq 34$. The hunt goes on to prove or disprove the optimality of the Harborth graph.

By the Eulerian polyhedron formula, every finite planar graph contains a vertex of degree at most five, so that we have $m(r) = \infty$ for $r \geq 6$. The question whether $m(5)$ is finite has a long history. As early as in 1982 Aart Blokhuis proposed a proof of the nonexistence of a 5-regular matchstick graph. It circulated for a while until a gap was discovered. In 2009 the old memorandum reappeared and turned out to be basically correct although containing several misprints; see [1] for a slightly edited version. During the years there was a strong belief, based on unpublished proofs, that no matchstick graph of degree 5 exists. In this paper we give a short proof of this conjecture.

2 5-regular matchstick graphs

Theorem 1. *Finite* 5*-regular matchstick graphs do not exist.*

Proof. Suppose to the contrary that there is such a graph M which we consider also as a planar *map*, that is, a crossing-free embedding of a planar graph in the plane. This drawing consists of vertices, edges, and faces. Without loss of generality we assume that this graph is connected and denote by V the number of its vertices, by E the number of its edges, and by F the number of faces in the planar map M. By Euler's formula we have $V - E + F = 2$. For every $k \geq 3$ we denote by f_k the number of faces in M with precisely k edges.

We observe that $2E = \sum kf_k = 5V$ and $F = \sum f_k$. Therefore

$$-6 = -3V + E + 2E - 3F = -3V + \frac{5}{2}V + \sum kf_k - 3\sum f_k = -\frac{1}{2}V + \sum (k-3)f_k. \tag{1}$$

The idea of the proof is to assign a charge to every vertex and every face in M so that the total charge is negative. Then we will redistribute the charges according to a simple local rule and reach a contradiction by showing that the charge of each vertex and each face becomes nonnegative.

We begin by giving a charge of $-\frac{1}{2}$ to each vertex and by giving a charge of $k - 3$ to each face in M with precisely k edges. By (1) the total charge of all the vertices and faces is negative.

We redistribute the charge in the following very simple way. Consider a face T of M and a vertex x of T. Let α denote the measure of the internal angle of T at x. If $\alpha > \frac{\pi}{3}$ we take a charge of $\min\left(\frac{1}{2}, \frac{3}{2\pi}\alpha - \frac{1}{2}\right)$ from T and move it to x (see Figure 2).

We now show that after the redistribution of charges every vertex and every face has a nonnegative charge. Consider a vertex x. Let ℓ denote the number of internal angles at x that are greater than $\frac{\pi}{3}$. As the degree of x is equal to 5 we must have $\ell > 0$. If due to one of these ℓ angles we transfered a charge of $\frac{1}{2}$ to x, then the charge at x is nonnegative. Otherwise, denote by $\alpha_1, \ldots, \alpha_\ell$ the measures of these ℓ internal angles at x that are greater than $\frac{\pi}{3}$. For $i = 1, \ldots, \ell$ we transfer a charge of $\frac{3}{2\pi}\alpha_i - \frac{1}{2}$ to x due to α_i. Hence the total charge transfered to x is $\frac{3}{2\pi}\sum_{i=1}^{\ell}\alpha_i - \frac{\ell}{2}$. The angle count at x gives $2\pi \leq \sum_{i=1}^{\ell}\alpha_i + (5-\ell)\frac{\pi}{3}$. This implies $\sum_{i=1}^{\ell}\alpha_i \geq 2\pi - (5-\ell)\frac{\pi}{3} = \frac{\pi}{3}(\ell+1)$. We conclude that the total charge transfered to x is at least $\frac{3}{2\pi}\frac{\pi}{3}(\ell+1) - \frac{\ell}{2} = \frac{1}{2}$. This leaves x with a nonnegative charge.

Consider now a face T in M with $k \geq 3$ edges. Assume first that T is a bounded face. The initial charge of T is $k - 3 \geq 0$. Therefore, if the charge at T becomes negative this implies that one of the internal angles of T is greater than $\frac{\pi}{3}$. If $k = 3$, that is, T is a triangle, then it must be an equilateral triangle, hence having all internal angles equal to $\frac{\pi}{3}$. Consequently, the charge of T is unaffected and remains $k - 3 = 0$.

If $k = 4$, then T is a rhombus. If only two internal angles of T are greater than $\frac{\pi}{3}$, then at most a total charge of 1 was deducted from the initial charge of T, leaving its charge nonnegative.

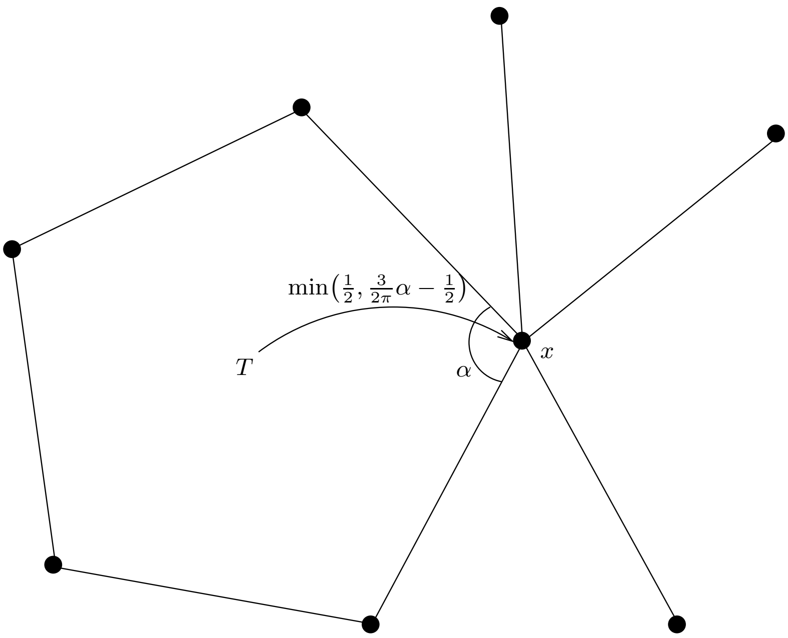

Figure 2: The distribution of a charge from a face T to its vertex x.

If all internal angles of T are greater than $\frac{\pi}{3}$, then the total charge deducted from T is at most $\frac{3}{2\pi} \cdot 2\pi - \frac{4}{2} = 1$, leaving the charge at T nonnegative.

If $k = 5$, then we split into two subcases. If at most four internal angles of T are greater than $\frac{\pi}{3}$, then there is a deduction of at most 2 from the initial charge of T, leaving the charge at T nonnegative. If all five internal angles of T are greater than $\frac{\pi}{3}$, then as the sum of the internal angles of T is equal to 3π, the total charge deducted from T amounts to $\frac{3}{2\pi} \cdot 3\pi - \frac{5}{2} = 2$, again leaving the charge at T nonnegative.

Finally, if $k \geq 6$, then the charge deducted from T is at most $\frac{k}{2}$ leaving T with a charge of at least $k - 3 - \frac{k}{2} \geq 0$.

It is left to consider the unbounded face S of M. If the number of edges of S is at least 6, we are done as in the case of a bounded face. The cases where the unbounded face consists of at most 5 edges can easily be excluded. Another way to settle this issue is to observe that if S consists of at most 5 edges, then the total charge deducted from S is at most $\frac{5}{2}$, leaving the charge of S at least $-\frac{5}{2}$ (and in fact at least $-\frac{3}{2}$). We still obtain a contradiction as the sum of all charges should be equal to -6, while only the unbounded face may remain with a negative charge, and its charge is not smaller than $-\frac{5}{2}$. ∎

3 Concluding remarks

It is interesting to note that Theorem 1 is not true if we consider it on the sphere. A matchstick graph drawn on a sphere is a drawing of the vertices as points on the sphere and edges as great arcs connecting corresponding points, with the property that the lengths of all connecting arcs are equal and no two arcs cross. The example of the icosahedron shows that 5-regular matchstick graphs may exist on a sphere (see Figure 3). In this example the charges of all triangles are negative after the redistribution, since the angle sum of a spherical triangle is strictly larger than π.

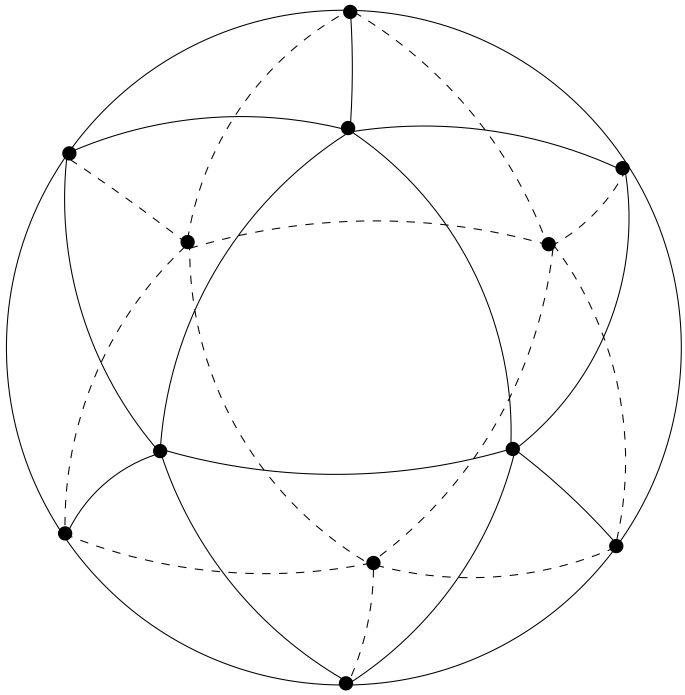

Figure 3: Icosahedron on a sphere.

References

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Dep. of Mathematics, University of Bayreuth, Universitätsstr. 30, 95447 Bayreuth, Germany.
sascha.kurz@uni-bayreuth.de

Department of Mathematics, Technion—Israel Institute of Technology, Haifa 32000, Israel.
room@math.technion.ac.il