Extreme Intersection Points in Arrangements of Lines

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Abstract

We provide a strengthening of the classical theorem of Erdős [3] saying that n lines in the plane, no two of which are parallel and not all pass through the same point, determine at least n intersection points. We relate this theorem to another classical theorem about the maximum number of edges in a geometric graph with no pair of disjoint edges. We also present an independently interesting corollary of our result.

1 Introduction

A set P of points is said to determine a line ℓ if ℓ passes through (at least) two points of P. Dually (we refer to the point-line duality in the plane), a set $\mathcal{L}$ of lines is said to determine an intersection point x if x is incident to (at least) two lines in $\mathcal{L}$.

The celebrated Gallai-Sylvester theorem [5, 14] asserts that any finite set P of non-collinear points in the real projective plane determines an *ordinary* line, that is, a line through precisely two points of P. Dually, any finite collection of non-concurrent lines in the real projective plane determines an ordinary intersection point of precisely two lines.

It was Erdős [3] who first noticed that the famous Gallai-Sylvester theorem implies, by induction, that any set of n non-collinear points in the real projective plane determines at least n distinct lines. Dually, any collection of n non-concurrent lines in the real projective plane determines at least n distinct intersection points.

Later, de Bruijn and Erdős [1] showed that this result follows from a more general and purely combinatorial result. Given a set system $\ell_1, \ldots, \ell_m$ on the ground set P of cardinality n such that for every $1 \leq i < j \leq m$ $|\ell_i \cap \ell_j| \leq 1$ and for every pair $x, y \in P$ there is some $\ell_i \in \mathcal{L}$ that contains them both, then we must have $m \geq n$.

In this paper we provide another generalization of the original result of Erdős [3] as follows.

Given a finite set of lines $\mathcal{L}$ in the real affine plane, an intersection point x of lines in $\mathcal{L}$ is called *extreme* if on (at least) one of the lines $\ell \in \mathcal{L}$ through x the point x does not lie in the convex hull of the other intersection points on ℓ (see Figure 1).

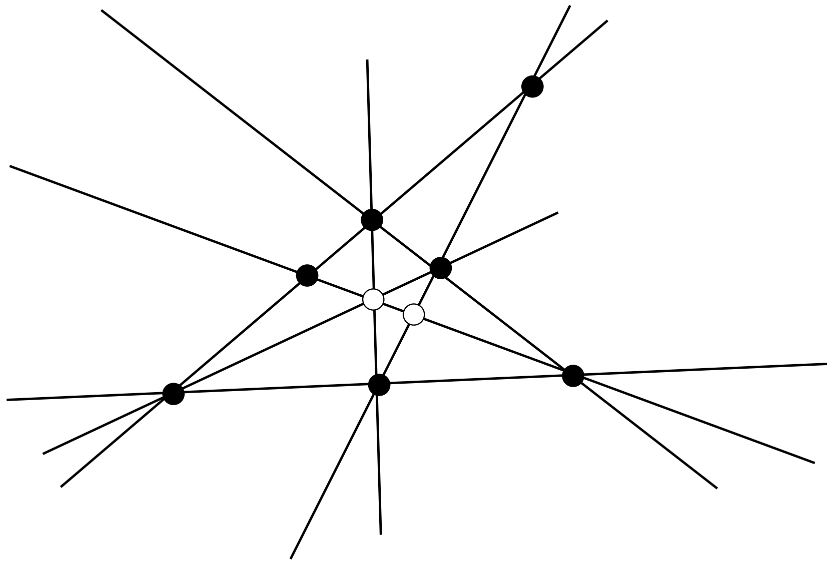

Figure 1: The extreme intersection points are colored black. The intersection points that are not extreme are colored white.

Theorem 1.1. *Let $\mathcal{L}$ be a set of n non-concurrent lines in the real affine plane. Assume that no two lines in $\mathcal{L}$ are parallel. Then $\mathcal{L}$ determines at least n extreme intersection points.*

The proof of Theorem 1.1 is given in Section 2 while in Section 3 we bring a nontrivial consequence of this theorem: We show that if P is a set of n non-collinear blue points and R is a set of m red points such that $P \cap R = \emptyset$ and every line ℓ determined by P contains a point in R that is not extreme on ℓ with respect to the blue points on ℓ, then $m \geq \frac{n}{2}$.

2 Proof of Theorem 1.1

We will prove Theorem 1.1 by using a classical result in extremal theory of geometric graphs. A geometric graph is a graph drawn in the plane so that its vertices are drawn as points and its edges are drawn as straight line segments connecting corresponding points. It is assumed that the set of points representing the vertices of the geometric graph is in general position, namely no three of the points are collinear. This is in order to avoid a situation where an edge connecting two vertices passes through a third vertex. One of the first classical results about geometric graph is the following theorem of Hopf and Pannwitz [8] and independently Sutherland [13].

Theorem 2.1 ([8, 13])**.** *A geometric graph on n vertices with no two disjoint edges has at most n edges.*

We will need a slight generalization of Theorem 2.1 that is valid also in the case where the set of vertices of the graph is not in general position.

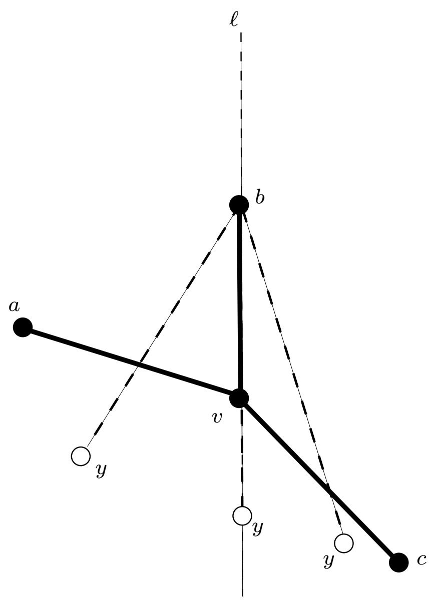

Figure 2: Illustration of the proof of Theorem 2.2.

Theorem 2.2. *Let P be a set of n points in the plane, possibly not in general position, and let G be a geometric graph whose vertices are the points of P. If no two edges are disjoint and no two are on the same line, then G has at most n edges.*

The requirement in Theorem 2.2 that no two edges in G are on the same line is important for otherwise G may consist of quadratically many edges in terms of n (consider, for example, the geometric graph with vertices $(1,0),\ldots,(n,0)$ and connect two vertices $(i,0)$ and $(j,0)$ if $i \leq n/2$ and $j > n/2$). This requirement can be weakened as it is enough that no two edges with a common vertex overlap in a segment. This will be seen in the proof and we leave the details to the reader. We will not need to use this stronger statement.

Proving Theorem 2.2 is not different than proving Theorem 2.1. For the sake of completeness, we will bring a short argument proving Theorem 2.2. We may assume that there is a vertex v in G whose degree is at least 3, for otherwise we are done. Let a, b, c be three of the neighbors of v in G. It must be that a line ℓ through v and one of a, b, c, say b, separates the two other vertices a and c. We claim that b does not have other neighbors but v. Indeed, suppose b has another neighbor y. If y is not on ℓ, then the edge (by) is disjoint from one of the edges (va) and (vc). If y lies on ℓ, then by must be disjoint from both edges (va) and (vc), unless (by) contains the vertex v, which is impossible because then the edges (by) and bv are contained in the same line (see Figure 2).

We can now conclude the theorem by the induction hypothesis on the graph $G \setminus \{b\}$, which has one less vertex and only one less edge than G. ∎

We are now ready to prove Theorem 1.1. Given a set $\mathcal{L}$ of n non-concurrent lines in the real affine plane such that no two of the lines are parallel, we define a graph G on the set of extreme intersection points determined by $\mathcal{L}$. For every line $\ell \in \mathcal{L}$ we draw an edge

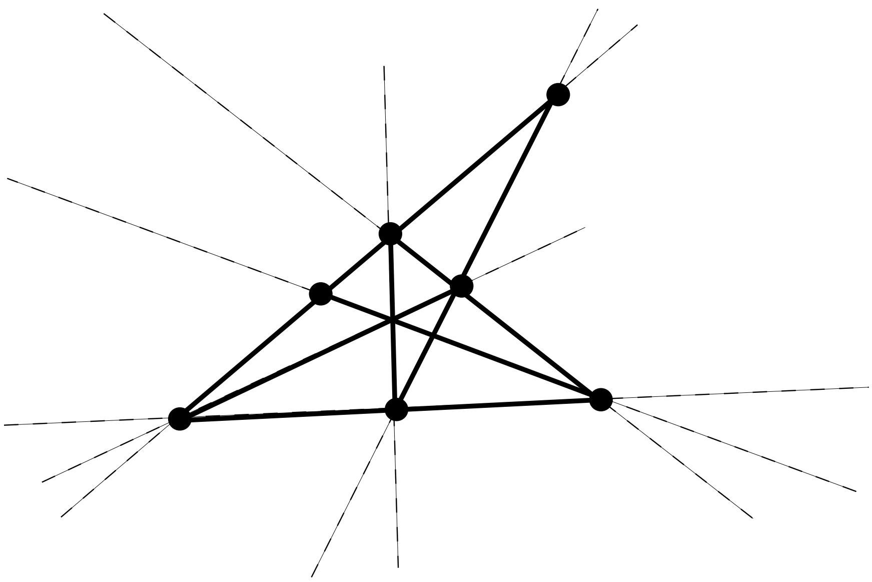

Figure 3: Illustration of the proof of Theorem 1.1

between the two extreme intersection points on ℓ (each of which is, of course, an extreme intersection point determined by $\mathcal{L}$). In particular the edge that corresponds to ℓ contains all the intersection points of ℓ with the other lines in $\mathcal{L}$. Notice that because the lines in $\mathcal{L}$ are not concurrent there are at least two intersection points on ℓ and hence two distinct extreme intersection points on ℓ. The crucial observation is that no two edges in G are disjoint. This is because for any two lines $\ell, \ell' \in \mathcal{L}$ the edges on ℓ and ℓ' both contain, by definition, the intersection point of the lines ℓ and ℓ'. Notice also that no two edges in G are contained in the same line (see Figure 3). We can therefore apply Theorem 2.2 and conclude that the number of edges in G, namely n, is smaller than or equal to the number of vertices of G, namely, the number of extreme intersection points determined by $\mathcal{L}$. ∎

3 A consequence of Theorem 1.1

A beautiful result of Motzkin [9], Rabin, and Chakerian [2] states that any set of non-collinear red and blue points in the plane determines a monochromatic line. Grünbaum and Motzkin [6] initiated the study of biased coloring, that is, coloring of the points such that no purely blue line is determined. The intuition behind this study is that if the number of blue points is much larger than the number of red points, then unless the set of blue points is collinear the set of blue and red points should determine a monochromatic blue line.

The same problem was independently considered by Erdős and Purdy [4] who stated it in a slightly different way.

Problem A. *Given a non-collinear set P of n points in the plane we wish to stab all the lines determined by this set by another set R of m red points such that $P \cap R = \emptyset$. Give a*

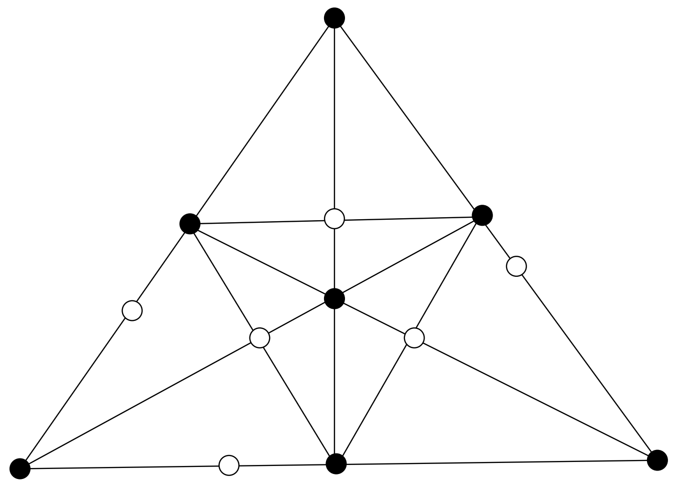

Figure 4: 7 blue points (drawn black) and only 6 red points (drawn white), showing that it may happen that $m < n$ in Theorem 3.1.

lower bound for m in terms of n.

In [11] it is shown that in Problem A we must have $m \geq \frac{n-1}{3}$. This is the best known answer to this problem. As for an upper bound, constructions found by Grünbaum show that m can be as small as $n-4$ in Problem A and there are sporadic constructions (that is, for small values of n) in which $|R|$ is equal to $|P|-6$ (see [7]).

An important special case of Problem A was solved in [15] and extended in [10]: In 1970 Scott [12] conjectured that any set of n non-collinear points in the plane determines at least $2\lfloor \frac{n}{2} \rfloor$ lines with distinct directions. Scott's conjecture was proved by Ungar [15]. Notice that this statement is equivalent to saying that given a set of n blue points in the plane and a set of m red points on the line at infinity such that any line determined by the blue points is stabbed by a red point, then $m \geq 2\lfloor \frac{n}{2} \rfloor$. This bound is best possible.

In [10] this result is extended as follows: Suppose that P is a set of n non-collinear blue points in the plane and R is a set of m red points such that $P \cap R = \emptyset$ and every line determined by P contains a red point *that is extreme* on that line (with respect to its incident blue points), then $m \geq 2\lfloor \frac{n}{2} \rfloor$. This bound is best possible.

Using Theorem 1.1, we can show the following result about the other special case of Problem A:

Theorem 3.1. *Let P be a set of n non-collinear blue points in the plane and let R be a set of m red points such that $P \cap R = \emptyset$ and every line determined by P contains a red point* that is NOT extreme *on that line with respect to its incident blue points, then $m \geq \frac{n}{2}$.*

We are not aware of any construction of a set P of n non-collinear blue points such that the set R in Theorem 3.1 contains fewer than n red points. The only exception we are aware of is the set of 7 blue points and only 6 red points depicted in Figure 4.

It will be more convenient for us to prove the dual statement to Theorem 3.1:

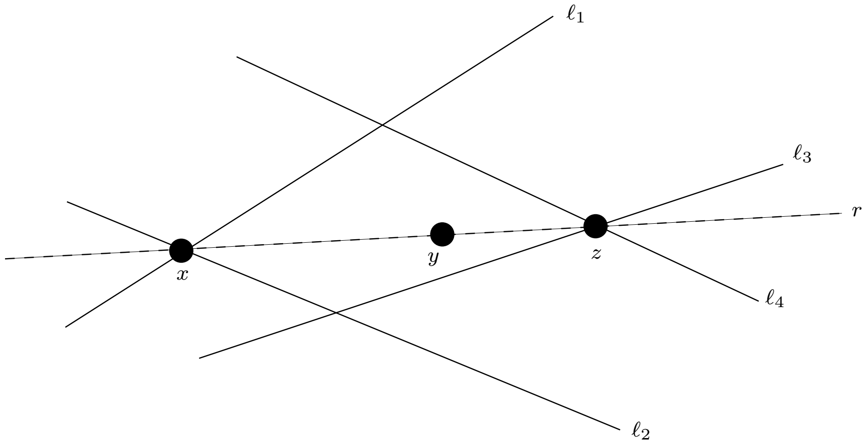

Figure 5: Illustration of the proof of Theorem 3.2.

Theorem 3.2. *Let $\mathcal{L}$ be a set of n lines in the plane. We assume that no two of the lines are parallel and not all are concurrent. We also assume that no line is vertical. Let $\mathcal{R}$ be a set of m red non-vertical lines such that $\mathcal{L} \cap \mathcal{R} = \emptyset$. If for every intersection point x, determined by $\mathcal{L}$, there is a line $r_x \in \mathcal{R}$ incident to x such that there is a line in $\mathcal{L}$ through x with larger slope than r_x and there is a line in $\mathcal{L}$ through x with smaller slope than r_x, then $m \geq \frac{n}{2}$.*

Proof. Let P be the set of extreme intersection points determined by $\mathcal{L}$. By Theorem 1.1, $|P| \geq n$. We claim that no three points $x, y, z \in P$ satisfy $r_x = r_y = r_z$. Indeed, assume the contrary and denote by r the line $r_x = r_y = r_z$. Then r passes through all three points x, y, and z. Without loss of generality assume that y is between x and z on r. The crucial observation is that y cannot be an extreme intersection point determined by $\mathcal{L}$. This is because there is a line $\ell_1 \in \mathcal{L}$ through x with larger slope than r and there is a line $\ell_2 \in \mathcal{L}$ through x with smaller slope than r. Similarly, there is a line $\ell_3 \in \mathcal{L}$ through z with larger slope than r and there is a line $\ell_4 \in \mathcal{L}$ through z with smaller slope than r. The four line $\ell_1, \ell_2, \ell_3, \ell_4$ bound a quadrilateral containing y in its interior (see Figure 5). In particular, y cannot be an extreme intersection point determined by $\mathcal{L}$. This contradiction proves that at least $|P|/2 \geq \frac{n}{2}$ red lines must belong to the set R. ∎

References

[1] N.G. de Bruijn and P. Erdős, On a combinatorial problem. *Nederl. Akad. Wetensch.*, Proc. **51**, (1948) 1277–1279.

[2] G. D. Chakerian, Sylvester's problem on collinear points and a relative, *Amer. Math. Monthly* **77** (1970), 164–167.

[3] P. Erdős, Problem 4065, *Amer. Math. Monthly* **50** (1943), p. 65.

[4] P. Erdős and G. Purdy, Some combinatorial problems in the plane, *J. Combinatorial Theory, Ser. A* **25** (1978), 205–210.

[5] T. Gallai, Solution of problem 4065, *Amer. Math. Monthly* **51** (1944), 169–171.

[6] B. Grünbaum, Arrangements of colored lines, Abstract 720-50-5, *Notices Amer. Math. Soc.* **22** (1975), A-200.

[7] B. Grünbaum, Monochromatic intersection points in families of colored lines, *Geombinatorics* **9** (1999), 3–9.

[8] H. Hopf and E. Pannwitz, Aufgabe Nr. 167, *Jahresbericht Deutsch. Math.-Verein.* **43** (1934) p.114.

[9] T. S. Motzkin, Nonmixed connecting lines, Abstract 67T 605, *Notices Amer. Math. Soc.* **14** (1967), 837.

[10] J. Pach, R. Pinchasi, and M. Sharir, On the number of directions determined by a three-dimensional point set, *J. Combinat. Theory, Ser. A*, **108** (2004), 1–16. Also in *Proc. 19th Annu. ACM Sympos. Comput. Geom.* (2003), 106–113.

[11] R. Pinchasi, A solution to a problem of Grünbaum and Motzkin and of Erdős and Purdy about bichromatic configurations of points in the plane. Israeli Journal of Math., accepted.

[12] P. R. Scott, On the sets of directions determined by n points, *Amer. Math. Monthly* **77** (1970), 502–505.

[13] J.W. Sutherland, Lösung der Aufgabe 167, *Jahresbericht Deutsch. Math.-Verein.* **45** (1935) 33–35.

[14] J. J. Sylvester, Mathematical question 11851, *Educational Times* **59** (1893), 98–99.

[15] P. Ungar, $2N$ noncollinear points determine at least $2N$ directions, *J. Combin. Theory, Ser. A* **33** (1982), 343–347.