Crossing edges and faces of line arrangements in the plane

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Abstract

For any natural number n we define $f(n)$ to be the minimum number with the following property. Given any arrangement $\mathcal{A}(\mathcal{L})$ of n blue lines in the real projective plane one can find $f(n)$ red lines different from the blue lines such that any edge in the arrangement $\mathcal{A}(\mathcal{L})$ is crossed by a red line. We define $h(n)$ to be the minimum number with the following property. Given any arrangement $\mathcal{A}(\mathcal{L})$ of n blue lines in the real projective plane one can find $h(n)$ red lines different from the blue lines such that every face in the arrangement $\mathcal{A}(\mathcal{L})$ is crossed in its interior by a red line.

In this paper we show $f(n) = 2n - o(n)$ and $h(n) = n - o(n)$.

1 Introduction

Let $\mathcal{L}$ be a collection of n blue lines in the real projective plane. We consider the arrangement $\mathcal{A}(\mathcal{L})$ of all the lines in $\mathcal{L}$ and we wish to find another set of red lines $\mathcal{R}$, as small as possible, such that $\mathcal{L} \cap \mathcal{R} = \emptyset$ and every (blue) edge in the arrangement $\mathcal{A}(\mathcal{L})$ is crossed in its relative interior by a line in $\mathcal{R}$. We denote by $f(n)$ the maximum possible cardinality of $\mathcal{R}$ taken over all possible choices of the set of n blue lines $\mathcal{L}$. In other words $f(n)$ is the minimum number such that for *any* set $\mathcal{L}$ of n blue lines in the real projective plane one can find a set $\mathcal{R}$ of $f(n)$ red lines and every blue edge in the arrangement $\mathcal{A}(\mathcal{L})$ is crossed by a line from $\mathcal{R}$.

It is easy to see that $f(n) \geq n - 1$. Indeed, any arrangement of n blue lines in general position in the plane requires at least $n - 1$ red lines to cross all the blue edges. This is because every given blue line contains $n - 1$ blue edges and every red line can cross at most one such edge on this given blue line. Alternatively, the number of edges in an arrangement of n (blue) lines in general position is $n(n - 1)$ while every red line crosses at most (in fact precisely) n edges.

On the other hand it is also easy to see that $f(n) \leq 2n$. This is because given any family $\mathcal{L}$ of n blue lines one can take $\mathcal{R}$ to be the family of $2n$ red lines obtained from $\mathcal{L}$ by taking for each line $\ell \in \mathcal{L}$ two red lines parallel to it and very close to it one slightly above ℓ and the other slightly below ℓ (in any affine picture of the projective plane).

We remark that a "sister problem" of this one is studied in [4] where one seeks for the minimum possible cardinality of a set of $\mathcal{R}$ of red lines that cross all edges in *some* arrangement of n blue

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lines. The minimum is taken over all nontrivial (that is, non-concurrent) arrangements induced by a family $\mathcal{L}$ of n blue lines in the plane. This question is of completely different nature and is interesting when $\mathcal{L}$ is far from being in general position. It is shown in [4] that one always needs at least $\frac{n}{3.5}$ red lines to cross all edges in an arrangement of n blue lines not all concurrent. It is not hard to come up with an arrangement of n blue lines such that the number of red lines required to cross all the edges in the blue arrangement is at most n. In the current paper we are interested in arrangements of n blue lines such that the cardinality of the minimum set of red lines crossing all edges in the blue arrangement is as large as possible.

In this paper we will consider also the analogous problem about crossing all *faces* of a line arrangement: Consider an arrangement $\mathcal{A}(\mathcal{L})$ of n lines in the plane. We wish to find another set of red lines $\mathcal{R}$, as small as possible, such that every face in the arrangement $\mathcal{A}(\mathcal{L})$ is crossed in its interior by a line in $\mathcal{R}$. We denote by $h(n)$ the maximum possible cardinality of $\mathcal{R}$ taken over all possible choices of the set $\mathcal{L}$ of n blue lines. In other words $h(n)$ is the minimum number such that for *any* set $\mathcal{L}$ of n blue lines in the real projective plane one can find a set $\mathcal{R}$ of $h(n)$ red lines and every face in the arrangement $\mathcal{A}(\mathcal{L})$ is crossed in its interior by a line from $\mathcal{R}$.

It is easy to see that $h(n) \geq n/2$. Indeed, any arrangement of n blue lines in general position in the real projective plane requires at least $n-1$ red lines to cross all the blue edges. This is because every arrangement of n blue lines in general position in the plane determines precisely $n(n-1)/2+1$ faces, while every red line crosses the interior of at most n such faces.

As for an upper for $h(n)$, it is not hard to see that $h(n) \leq n$. This is because given any family $\mathcal{L}$ of n blue lines one can take R to be the family of n red lines obtained from $\mathcal{L}$ by taking for each line $\ell \in \mathcal{L}$ one red line parallel to ℓ that lies very close to ℓ and slightly below ℓ.

In this paper we almost completely close the gap between lower and upper bounds for both f and h by providing lower bounds (via constructions) that are very close to the easy upper bounds for both functions. Specifically, we show the following:

Theorem 1. *$h(n) = n - o(n)$.*

Theorem 2. *$f(n) = 2n - o(n)$.*

The main tool in our proofs is the density version of the Hales-Jewett theorem in [2].

2 An auxiliary construction of a set $B = B(n_0, k, M)$

In this section we will introduce, by a construction, a planar set of points B which will depend on 3 parameters $B = B(n_0, k, M)$. We will also point out some properties of this set that will interest us in the sequel. The set B will later be used as a starting point in the proof of both Theorem 1 and Theorem 2.

For positive integers k and ℓ we define $B^k_\ell = \{0, 1, \ldots, \ell-1\}^k \subset \mathbb{R}^k$. The set B^k_ℓ contains ℓ^k points. Notice that no $\ell+1$ of the points of B^k_ℓ are collinear. Nevertheless, there are many ℓ-tuples of collinear points of B^k_ℓ. It is not very hard to count the number of collinear ℓ-tuples of points in B^k_ℓ. Indeed, let $P_1, \ldots, P_\ell$ be ℓ collinear points of B^k_ℓ. Without loss of generality assume that they are indexed according to the order in which they appear on the line through them. Then for every $1 \leq j \leq \ell$ either the j'th coordinate in each of $P_1, \ldots, P_\ell$ is the same, or the values of the j'th coordinate of the points $P_1, \ldots, P_\ell$ are pairwise different and hence contain all integer values in $\{0, 1, \ldots, \ell-1\}$. In the latter case either the values of the j'th coordinate of $P_1, \ldots, P_\ell$ are $0, 1, \ldots, \ell-1$, respectively, or they are $\ell-1, \ldots, 0$, respectively. It follows that the number of ℓ-tuples of collinear points in B_ℓ^k is equal to $\sum_{t=1}^{k} \binom{k}{t} \ell^{k-t} 2^t = (\ell+2)^k - \ell^k$. Notice that when k is large compared to ℓ then this number is significantly larger than $|B_\ell^k| = \ell^k$. In fact, by Lagrange theorem, $(\ell+2)^k - \ell^k \geq 2k\ell^{k-1} = \frac{2k}{\ell}\ell^k$. We recall the following result from [2].

Theorem 3 ([2])**.** *Let ℓ be a fixed positive integer and let k be any positive integer. If $A \subset B_\ell^k$ and A does not contain ℓ collinear points, then $|A| \leq g_\ell(k)|B_\ell^k|$, for some function g_ℓ such that $\lim_{k\to\infty} g_\ell(k) = 0$.*

The following lemma is corollary of Theorem 3.

Lemma 1. *Let $A \subset B_\ell^k$ be a subset of B_ℓ^k of cardinality at least $2g_\ell(k)|B_\ell^k|$. Then A contains at least $\frac{|A|^\ell}{2^\ell g_\ell(k)^{\ell-1}|B_\ell^k|^{\ell-1}}$ collinear ℓ-tuples of points.*

Proof. By Theorem 3, a subset A of B_ℓ^k must contain at least $|A| - g_\ell(k)|B_\ell^k|$ collinear ℓ-tuples of points. Denote by T the number of collinear ℓ-tuples in A. Pick every point in A with probability p, to be determined later. The expected number of collinear ℓ-tuples in the resulting subset of A is $p^\ell T$. On the other hand this expected number must be greater than or equal to $p|A| - g_\ell(k)|B_\ell^k|$. We conclude that $T \geq |A|/p^{\ell-1} - g_\ell(k)|B_\ell^k|/p^\ell$. Taking $p = 2g_\ell(k)|B_\ell^k|/|A|$ (observe that $0 < p \leq 1$) yields the bound in the lemma. ∎

We will now describe our construction of the set $B = B(n_0, k, M)$. Take n_0 to be any integer greater than or equal to 3. We consider the set $B_{n_0}^k$, for a positive integer k (in the proof of Theorem 1 we will take $n_0 = 3$ and k will just be a large number while in the proof of Theorem 2 k will depend on n_0 and n_0 will be taken to be large enough).

Let M be another positive integer (in the proofs of Theorem 1 and Theorem 2 M will be depend on k, and therefore also on n_0, and will be taken to be large enough).

We construct a sequence of sets $B_1, B_2, \ldots, B_M$ in $\mathbb{R}^2$. Those sets will hold the following properties:

- For every j the set B_j will consist of n_0^{jk} points.
- Every collinear n_0-tuple of points in B_j forms an arithmetic progression (in $\mathbb{R}^2$ and therefore in each of the two coordinates - this property will be of use only in the proof of Theorem 2).
- No more than n_0^k of the collinear n_0-tuples in B_j lie on parallel lines, or on lines that meet at a common point not in B_j.

B_1 will be the image of a generic linear transformation of the set $B_{n_0}^k$ to $\mathbb{R}^2$. In particular, no two points of B_1 have the same x-coordinate. We will also make sure that the points on B_1 as well as all the intersection points of nonparallel lines determined by B_1 lie extremely close to the origin. This can be done by multiplying B_1 by a very small constant. Notice also that no more than n_0^k of the collinear n_0-tuples in B_1 lie on parallel lines, or on lines that meet at a common point not in B_1, simply because the cardinality of B_1 is only n_0^k. We color each of the collinear n_0-tuples in B_1 by the color 1.

Having defined already $B_1, \ldots, B_j$ we construct B_{j+1} as follows. We let B'_j be the image of B_j under a generic linear transformation, so that in particular no two points in B'_j will have the same x coordinate. We let C_j be the image of a generic projective transformation of $B^k_{n_0}$ to $\mathbb{R}^2$. We will also make sure that the points in C_j as well as all the intersection points of nonparallel lines determined by C_j lie extremely close to the origin. This can be done by multiplying C_j by a very small constant.

For every $x \in \mathbb{R}$ we define a linear map $T_x : \mathbb{R}^2 \to \mathbb{R}^2$ by $T_x(a, b) = (a, ax + b)$. We define B_{j+1} as

$$B_{j+1} = \cup_{(x,y)\in B'_j}((x, y) + T_x(C_j)).$$

We notice two kinds of collinear n_0-tuples of points in B_{j+1} as follows. If $p_1 = (x_1, y_1), \ldots, p_{n_0} = (x_{n_0}, y_{n_0})$ is a collinear n_0-tuple of points in B'_j (and therefore is an arithmetic progression in $\mathbb{R}^2$) and $q = (a, b)$ is a point in C_j, then the points $p'_1 = p_1 + T_{x_1}(q), \ldots, p'_{n_0} = p_{n_0} + T_{x_{n_0}}(q)$ is a collinear n_0-tuple in B_{j+1}. To see this notice that we have $p_i + T_{x_i}(q) = (x_i + a, y_i + ax_i + b)$. We see from here that the points $p_i + T_{x_i}(q)$ for $i = 1, \ldots, n_0$ are still collinear (and form an arithmetic progression in $\mathbb{R}^2$). We say that n_0-tuple $(p'_1, \ldots, p'_{n_0})$ has an anchor point $q = (a, b)$ in C_j. Notice that if the equation of the line through $p_1, \ldots, p_{n_0}$ is $y = \alpha x + \beta$, then the equation of the line through $p'_1, \ldots, p'_{n_0}$ is $y = (\alpha + a)x + \beta + b - (\alpha + a)a$. We color this n_0-tuple (namely, the n_0-tuple $(p'_1, \ldots, p'_{n_0})$) in B_{j+1} by the color of the n_0-tuple $(p_1, \ldots, p_{n_0})$ in B'_j (this will be one of the colors in $\{1, \ldots, j\}$). Notice that two such tuples sit on parallel lines in B_{j+1} only if they arise from two parallel n_0-tuples in B'_j and the same point q in C (because the points in C have pairwise distinct x-coordinates).

The second (and last) kind of collinear n_0-tuple of points in B_{j+1} that we will care about arises from collinear n_0-tuples of points in C_j: Suppose $q_1 = (x_1, y_1), \ldots, q_{n_0} = (x_{n_0}, y_{n_0})$ is a collinear n_0-tuple of points in C_j (and therefore also an arithmetic progression in $\mathbb{R}^2$) and $p = (a, b)$ is a point in B'_j, then the points $q'_1 = p+T_a(q_1), \ldots, q'_{n_0} = p+T_a(q_{n_0})$ form a collinear n_0-tuple of points in B_{j+1}. To see this notice that $p+T_a(q_i) = (a+x_i, b+x_ia+y_i)$. We see from here that the points $q'_i = p + T_a(q_i)$ for $i = 1, \ldots, n_0$ are still collinear (and form an arithmetic progression in $\mathbb{R}^2$). We say that the n_0-tuple $(q'_1, \ldots, q'_{n_0})$ has an anchor point $p = (a, b)$ in B'_j. Notice that if the equation of the line through $q_1, \ldots, q_{n_0}$ is $y = \alpha x + \beta$, then the equation of the line through $q'_1, \ldots, q'_{n_0}$ is $y = (\alpha + a)x + \beta + b - (\alpha + a)a$. We color this n_0-tuple (namely, the n_0-tuple $q'_1, \ldots, q'_{n_0}$) in B_{j+1} by the color $j + 1$. Notice that two such tuples are parallel in B_{j+1} only if they arise from two parallel n_0-tuples in C_j and the same point p in B'_j (because no two points in B'_j have the same x-coordinate). Therefore, the number of pairwise parallel n_0-tuples in B_{j+1} of color $j + 1$ is at most the number of pairwise parallel n_0-tuples in C_j which is clearly not larger than n_0^k (in fact it is precisely equal to n_0^{k-1}), namely the cardinality of C_j.

It remains to explain why no more than n_0^k n_0-tuples in B_{j+1} lie on lines that meet at a common point that is not in B_{j+1}. The short answer to this is that this is the typical generic situation (recall that at step $j+1$ C_j is a generic linear copy of $B^k_{n_0}$ and that B'_j is a generic linear transformation of B_j). However, because this is an important point (and in fact this is the reason why our construction of the set B is more complicated than just taking a generic projection of B_3^{kM} for some large M) we explain it here in detail.

We prove this by induction of j. As we have seen, this is true for $j = 1$. Let W be a collection of n_0-tuple of collinear points in B_{j+1} whose containing lines meet at a common point $Z \notin B_{j+1}$.

Suppose first that there are two n_0-tuples $w_1, w_2 \in W$ with color smaller than $j+1$ that have the same anchor point q in C_j. In this case every n_0-tuple in W of color smaller than $j+1$ must have q as an anchor point. Indeed, assume $w_3 \in W$ has a different anchor point $q' \in C_j$. Observe that Z lies on the intersection of the two lines through the points of w_1 and of w_2. Because C_j is a generic linear transformation of $B^k_{n_0}$, we can move q' without moving q. This will not change w_1 and w_2 and hence also not change the point Z, but the line through w_3 will no longer pass through Z. This shows that generically this situation is not possible. We also claim that there cannot be any n_0-tuple in W that has color $j+1$. Indeed, assume to the contrary that $w \in W$ is such an n_0-tuple. Recall that w is an n_0-tuple of points in some copy of C_j. w passes through n_0 points of (a copy) of C_j. Let q' be one such point that is different from q. By taking a generic copy of C_j where q remains fixed and q' moves we see that w will no longer pass through Z. This means again that generically this situation is not possible.

We conclude that W contains only n_0-tuples of colors $1, \ldots, j$ with q as an anchor point. This means that all the n_0-tuples in W belong to the same copy of B'_j in B_{j+1} (viewed as a Cartesian product of B'_j and C_j). By the induction hypothesis W contains at most n_0^k n_0-tuples.

We assume therefore that every n_0-tuple in W that has color smaller than $j+1$ has a distinct anchor point in C_j. We claim that the number of the n_0-tuples in W with color smaller than $j+1$ is at most n_0. To see this observe that no n_0+1 points in C_j may be collinear. Assume to the contrary that W contains n_0+1 n_0-tuples of color smaller than $j+1$. Let $q_1, \ldots, q_{n_0+1}$ denote their pairwise distinct anchor points in C_j. One can choose three of these n_0+1 points that are not collinear. Without loss of generality assume these are q_1, q_2, q_3. Because C_j is a generic linear copy of $B^k_{n_0}$ we can leave q_1 and q_2 fixed (hence fixing also Z as an intersection point of two lines that remain fixed) and change the position of q_3 so that the line through the n_0-tuple whose anchor point is q_3 will no longer pass through Z. This shows that generically, this situation should not happen.

We conclude that the number of n_0-tuples in W that have color smaller than $j+1$ is at most n_0. Suppose next that there are two n_0-tuples $w_1, w_2 \in W$ of color $j+1$ with the same anchor point q in B'_j. We claim that every n_0-tuple in W of color $j+1$ must have q as its anchor point. Indeed, assume to the contrary that w_3 is an n_0-tuple in W of color $j+1$ with a distinct anchor point $q' \in B'_j$. Because B'_j is a generic linear image of B_j we can leave q fixed and therefore also w_1 and w_2 and Z and change the position of q'. The line through the points of w_3 will no longer pass through Z. This shows that generically, this situation is not possible. We conclude that all n_0-tuple in W of color $j+1$ has q as their anchor point. This means that all the n_0-tuples in W of color $j+1$ belong to the same copy of $C_j$in B_{j+1}. Because they are pairwise disjoint their number is at most $|C_j|/n_0 = n_0^{k-1}$. We conclude that the cardinality of W is at most $n_0 + n_0^{k-1} < n_0^k$, as desired.

It is left to consider the case where the n_0-tuples in W with color $j+1$ have pairwise distinct anchor points in B'_j. We claim that in this case the number of such n_0-tuples in W is at most n_0. Indeed, among any n_0+1 (anchor) points in B'_j there are 3 that are not collinear. Therefore, by taking a generic linear image of B_j as B'_j we can fix two of them (and therefore also Z) and move a third one thus causing the line through its n_0-tuple not to pass through Z anymore. This shows that generically this situation should not happen. We conclude that W contains at most $2n_0 < n_0^k$ tuples as desired.

We therefore completed the induction step and showed that for every j the set B_j does not contain more than n_0^k collinear n_0-tuples whose containing lines meet at a common point.

After M steps we obtain (for $j = M$) the set B_M that we denote by $B = B(n_0, k, M)$. This set consists of $n = n_0^{kM}$ points.

Claim 1. *Let A be a subset of B such that $|A| > 4g_{n_0}(k)n$. Then A contains at least $M\frac{|A|^{n_0}}{4^{n_0}g_{n_0}(k)^{n_0-1}n^{n_0-1}}$ colored (in one of the colors $\{1, \ldots, M\}$) collinear n_0-tuples of points in B.*

Proof. We count separately how many collinear n_0-tuples of each color A contains. Fix a color j. B can be viewed as a disjoint union of $n/|B_1| = n_0^{k(M-1)}$ (affine) copies of B_1, where in each such copy all collinear n_0-tuples have the color j. We denote $m = n/|B_1| = n_0^{k(M-1)}$. We call each such copy of B_1 in B a j*-special* copy of B_1 in B. Denote by $x_1, \ldots, x_m$ the number of points of A in each of these m j-special copies of B_1 in B. By Lemma 1, for every $i = 1, \ldots, m$, if $x_i > 2g_{n_0}(k)n_0^k$, then there are at least $\frac{x_i^{n_0}}{2^{n_0}g_{n_0}(k)^{n_0-1}(n_0^k)^{n_0-1}}$ collinear n_0-tuples of color j in the i'th j-special copy of B_1 in B. Discard from A all the points that belong to a j-special copy of B_1 that contains less than $2g_{n_0}(k)n_0^k$ points of A. The remaining subset of A, that we denote by A', contains at least

$$|A| - 2g_{n_0}(k)n_0^k m = |A| - 2g_{n_0}(k)n \geq |A| - \frac{1}{2}|A| = \frac{1}{2}|A|$$

points of B.

For every $1 \leq i \leq m$ denote by x'_i the number of points of A' in the i'th j-special copy of B_1 in B. Then for every i either $x'_i = 0$ or $x'_i \geq 2g_{n_0}(k)n$. In the latter case it follows from Lemma 1 that the set of points of A' in the i'th j-special copy of B_1 contains at least $\frac{x_i^{n_0}}{2^{n_0}g_{n_0}(k)^{n_0-1}(n_0^k)^{n_0-1}}$ collinear n_0-tuples, all of which are colored with the color j. Notice that

$$\sum_{x'_i \geq 2g_{n_0}(k)n} x'_i = \sum_{i=1}^{m} x'_i = |A'| \geq \frac{1}{2}|A| \geq 2g_{n_0}(k)n.$$

Consequently, from convexity, we have that the total number of collinear n_0-tuples of color j is at least

$$\begin{aligned}\sum_{x_i > 2g_{n_0}(k)n_0^k} \frac{x_i^{n_0}}{2^{n_0}g_{n_0}(k)^{n_0-1}(n_0^k)^{n_0-1}} &\geq m\frac{(|A'|/m)^{n_0}}{2^{n_0}g_{n_0}(k)^{n_0-1}(n_0^k)^{n_0-1}} \geq m\frac{(|A|/(2m))^{n_0}}{2^{n_0}g_{n_0}(k)^{n_0-1}(n_0^k)^{n_0-1}} = \\ &\frac{|A|^{n_0}}{4^{n_0}g_{n_0}(k)^{n_0-1}(n_0^k)^{n_0-1}m^{n_0-1}} = \frac{|A|^{n_0}}{4^{n_0}g_{n_0}(k)^{n_0-1}n^{n_0-1}}.\end{aligned}$$

Therefore, the total number of special n_0-tuples in A, of every possible color, is at least $M\frac{|A|^{n_0}}{4^{n_0}g_{n_0}(k)^{n_0-1}n^{n_0-1}}$. ∎

3 The lower bound for $h(n)$ - Proof of Theorem 1

In this section we prove Theorem 1 and show that $h(n) \geq n - o(n)$. We do this by constructing a set $\mathcal{L}$ of n blue lines such that any set $\mathcal{R}$ of red lines that together cross all the faces in $\mathcal{A}(\mathcal{L})$ must consist of at least $n - o(n)$ red lines.

For the proof of Theorem 1 we take $n_0 = 3$. and consider the set $B = B(3, k, M)$ where k is a large integer and M is another large integer that will depend on k and will be determined later.

Consider now the standard point-line duality that takes every point (a, b) in the plane to the line $\{y + ax + b = 0\}$. We apply this duality transformation to the set B and obtain a set of lines $\mathcal{L}$. $\mathcal{L}$ consists of $n = n_0^{kM} = 3^{kM}$ lines. No 4 of the lines in $\mathcal{L}$ are concurrent. It follows from Claim 1 that every subset A of $\mathcal{L}$ of cardinality greater than $4g_3(k)n$ must contain at least $M\frac{|A|^3}{4^3 g_3(k)^2 n^2}$ 3-tuples (each 3-tuple is colored by one of the colors $1, 2, \ldots, M$) of concurrent lines. We call a point incident to 3 such concurrent lines in A *special*. Because of the property of the set B which is that no more than n_0^k lines containing n_0-tuples of B meet at a common point that does not belong to B, the following is true about $\mathcal{L}$: There is no line not in $\mathcal{L}$ that is incident to more than $n_0^k = 3^k$ special points.

We slightly perturb every line in L in such a way that the resulting set of lines $\mathcal{L}'$ is in general position, that is, no three lines in $\mathcal{L}'$ pass through the same point. Observe that every 3 lines in $\mathcal{L}$ that pass through the same (special) point create, after perturbed, a small bounded triangular face in the arrangement $\mathcal{A}(\mathcal{L}')$. We call this triangular face *special*. Therefore, by Claim 1, every subset A of $\mathcal{L}'$ of cardinality greater than $4g_3(k)n$ must contains at least $M\frac{|A|^3}{4^3 g_3(k)^2 n^2}$ triples of lines each of which supports all edges of some special (triangular) face. Moreover, if there is a line that intersects more than 3^k special faces, then those special faces must be supported by the same line $\ell \in \mathcal{L}'$. We can assume that the perturbation of the lines in $\mathcal{L}$ is so small that a line that crosses two special faces supported by the same line $\ell \in \mathcal{L}$ cannot cross any special face not supported by ℓ.

Assume that $\mathcal{R}$ is a set of red lines that together cross the interior of each of the faces in the arrangement $\mathcal{A}(\mathcal{L}')$. In particular, every special face in $\mathcal{A}(\mathcal{L}')$ is crossed by a line in $\mathcal{R}$. Call a line $\ell \in \mathcal{L}'$ *good* if there is a red line $r \in \mathcal{R}$ that crosses at least two special faces supported by ℓ. Notice that in this case r cannot cross a special face not supported by ℓ. Notice also that if $\ell_1, \ell_2 \in \mathcal{L}'$ are two good lines, then the corresponding red lines $r_1, r_2 \in \mathcal{R}$ are distinct, as no two special faces can be supported by the same two given red lines in $\mathcal{L}'$. If a line $\ell \in \mathcal{L}'$ is not good we call it *bad*. If the number of bad lines in $\mathcal{L}'$ is smaller than $4g_3(k)n$, then the number of red lines in $\mathcal{R}$ is at least $|\mathcal{L}'| - 4g_3(k)n = n - o(n)$. Let A denote the set of bad lines in $\mathcal{L}'$ and suppose that its cardinality is at least $4g_3(k)n$. By the property of our construction (that follows from Claim 1), A must contain at least

$$M\frac{|A|^3}{4^3 g_3(k)^2 n^2} \geq Mng_3(k)$$

triples of lines, where the lines in each triple support all the three edges of some special triangular face. Notice that no red line can cross more than 3^k of those special faces. The reason for this is that if a line crosses more than 3^k special faces, then all of these special faces are supported by the same blue line ℓ in A. But because ℓ is bad there cannot be a red line that crosses more than one of these special faces.

We can now conclude that the number of lines in $\mathcal{R}$ must be at least $\frac{Mng_3(k)}{3^k}$. If we choose M large enough so that $M \geq 3^k/g_3(k)$, then the number of red lines must be greater than n. ∎

4 The lower bound for $f(n)$ - Proof of theorem 2

In this section we will show that $f(n) \geq 2n - o(n)$. We do this by constructing a set $\mathcal{L}$ of n blue lines such that any set $\mathcal{R}$ of red lines that together cross all the (blue) edges in $\mathcal{A}(\mathcal{L})$ must consist of at least $2n - o(n)$ red lines.

We start with the following easy observation.

Lemma 2. *Let $\mathcal{L}$ be a set of $n > 3$ blue lines in general position in the real* affine *plane. Assume that the arrangement $\mathcal{A}(\mathcal{L})$, determined by $\mathcal{L}$, contains t bounded triangular faces. Then any set of red lines that together cross every bounded blue edge in $A(\mathcal{L})$ must consist of at least $n - 2 + t/n$ lines.*

Proof. $\mathcal{A}(\mathcal{L})$ contains $n(n-2)$ bounded edges. Every triangular face must have at least one blue edge that is crossed by two red lines. It follows that the number of times that a red line crosses a bounded blue edge of $\mathcal{A}(\mathcal{L}))$ is at least $n(n-2)+t$. The result now follows by observing that every red line can cross at most n bounded edges of $\mathcal{A}(\mathcal{L})$. ∎

We consider the set $B = B(n_0, k, M)$ that was constructed in Section 2 where M is a large integer that will depend on k and k is a large integer that will depend on n_0, and n_0 is large integer to be determined later.

We recall that in Section 2 we constructed a sequence of sets $B_1, \ldots, B_M$ and we set $B = B_M$. Recall that B_1 is a generically projected copy of $B^k_{n_0}$ to the plane. Observe that from the purely combinatorial point of view the set B_M is in a natural one to one correspondence with the Cartesian product of M copies of B_1. We therefore assign to every point b in $B = B_M$ an M-tuple $(p_1(b), \ldots, p_M(b))$ where each $p_i(b)$ is a point in $B^k_{n_0}$. Observe that if $b_1, \ldots, b_{n_0}$ is a collinear n_0-tuple of points in B that has color equal to j, then for every $1 \leq i \leq M$ we have $p_i(b_1) = \ldots = p_i(b_{n_0})$, unless $i = j$. In the latter case (where $i = j$) the points $p_j(b_1), \ldots, p_j(b_{n_0})$ form a collinear n_0-tuple in $B^k_{n_0}$.

We consider the set B^s_3, for some s to be determined later. We will use a specific projection to the two dimensional plane that takes a point $X = (x_1, \ldots, x_s) \in \mathbb{R}^s$ to the point

$$T(X) = (x_1 + 3x_2 + 3^2x_3 + \ldots + 3^{s-1}x_s, a_1x_1 + \ldots + a_sx_s),$$

where $a = (a_1, \ldots, a_s)$ is a generic vector in $\mathbb{R}^s$ with very small but nonzero coordinates.

The image of the set B^s_3 under the projection T is a set, that we denote by S, of $n_0 = 3^s$ (here we finally define n_0) points that lie very close to the x-axis and the set of their x-coordinates is the set of all the integers between 0 and $3^s - 1$. By our observation in Section 3, S contains $(3+2)^s - 3^s$ collinear triples of points. For $i = 0, \ldots, s-1$ we denote by c_i the y-coordinate of the point in $T(B^s_3)$ whose x-coordinate is equal to i.

Next we perturb the set B in the following way. For $i = 1, \ldots, M$ we let z_i be a very small positive number. We make sure that z_{i+1} is negligible compared to z_i. Let $q = (q_x, q_y) \in B$ and let $(p_1(q), \ldots, p_M(q))$ be the M-tuple of points in $B^k_{n_0}$ assigned to q. Therefore, for every $1 \leq i \leq M$ we have $p_i(q) \in B^k_{n_0}$. We write each $p_i(q)$ in $B^k_{n_0}$ as $p_i(q) = (p_{i,1}(q), \ldots, p_{i,k}(q))$, where each $p_{i,j}(q)$ is in $\{0, \ldots, n_0 - 1\}$.

We take q to the point $(q_x, q_y + r(q))$ where $r(q) > 0$ is a small number defined by

$$r(q) = \sum_{i=1}^{M} \sum_{j=1}^{k} z_j c_{p_{i,j}(q)}.$$

We claim that this way every collinear n_0-tuple of points in B is perturbed into a (slightly perturbed) affine copy of the set S. To see this, consider such a collinear n_0-tuple $q_1, \ldots, q_{n_0}$ and suppose that this n_0-tuple is colored by the color j. For every $i = 1, \ldots, n_0$ let $(p_1(q_i), \ldots, p_M(q_i))$

be the M-tuple assigned to q_i and recall that every $p_\ell(q_i)$ is formally a point in $B_{n_0}^k$. For every $\ell \neq j$ we have $p_\ell(q_1) = \ldots = p_\ell(q_{n_0})$. For $\ell = j$ the points $p_j(q_1), \ldots, p_j(q_{n_0})$ in $B_{n_0}^k$ form a collinear n_0-tuple.

Recall that a collinear n_0-tuple in $B = B_M$ forms an arithmetic progression (in $\mathbb{R}^2$). For every $i = 1, \ldots, n_0$ write $q_i = (x_i, y_i)$. The point q_i is perturbed to the point $(x_i, y_i + r(q_i))$. We recall the definition of $r(q_i)$. Write $p_\ell(q_i) = (p_{\ell,1}(q_i), \ldots, p_{\ell,k}(q_i)) \in B_{n_0}^k$ and recall that each $p_{\ell,t}(q_i)$ is a number in $\{0, \ldots, n_0 - 1\}$. We have

$$r(q_i) = \sum_{m=1}^{M} \sum_{t=1}^{k} z_t c_{p_{m,t}(q_i)}.$$

Let $A_i = \sum_{m \neq j} \sum_{t=1}^{k} z_t c_{p_{m,t}(q_i)}$. Then for every $i = 1, \ldots, n_0$ we have $r(q_i) = A_i + \sum_{t=1}^{k} z_t c_{p_{j,t}(q_i)}$.

Because the color of the collinear n_0-tuple $q_1, \ldots, q_{n_0}$ is j, we have $A_1 = \ldots = A_{n_0}$ and we denote this number by A.

Therefore, for every $i = 1, \ldots, n_0$

$$r(q_i) = A + \sum_{t=1}^{k} z_t c_{p_{j,t}(q_i)}.$$

The points $p_j(q_1), \ldots, p_j(q_{n_0})$ form a collinear n_0-tuple in $B_{n_0}^k$. Let $1 \leq t \leq k$ be the smallest coordinate whose value is not constant in the points $p_j(q_1), \ldots, p_j(q_{n_0}) \in B_{n_0}^k$. This means that for every $i = 1, \ldots, n_0$ $r(q_i)$ is equal to a constant plus $z_t c_{p_{j,t}(q_i)}$ plus additional negligible terms (multiplied by $z_{t'}$ for $t' > t$). Because $q_1, \ldots q_{n_0}$ is an arithmetic progression, there exist a, b, u, w such that $q_i = (x_i, y_i) = (a, b) + (ui, wi)$. Therefore, up to negligible terms, the perturbed copy of q_i is equal either to $(a, b + Constant) + (ui, wi + z_t c_i)$, or to $(a, b + Constant) + (ui, wi + z_t c_{n_0 - i})$. In either way this is an affine copy of the set S (or a flipped affine copy of the set S).

The next thing we do is we take the dual arrangement $\mathcal{L}$ of lines of the perturbed copy B. We claim that this is the desired set of $n = n_0^{kM}$ blue lines such that any set of red lines crossing all edges in this arrangement must consist of at least $2n - o(n)$ red lines.

Call an n_0-tuple of lines in $\mathcal{L}$ *virtually concurrent* if they are dual to perturbed n_0-collinear points in B. We can assume that all the crossings between virtually concurrent n_0-tuple of lines in $\ell_1, \ldots, \ell_{n_0} \in \mathcal{L}$ are contained in a very small disc that does not meet any other line in $\mathcal{L}$. We call this disc the *virtual point of intersection* of the n_0-tuple of lines $\ell_1, \ldots, \ell_{n_0}$.

We say that a line $\ell' \notin \mathcal{L}$ *follows* a line $\ell \in \mathcal{L}$ if ℓ' meets at least two virtual points of intersection met also by ℓ. If all of our perturbations are very small, we may assume that if ℓ' follows ℓ, then every virtual point of intersection that ℓ' meets is met also by ℓ. In particular it follows that no line ℓ' can follow two distinct lines in $\mathcal{L}$, because two lines in $\mathcal{L}$ can meet at most one common virtual point of intersection (where they cross).

Every virtually concurrent n_0-tuple of lines from $\mathcal{L}$ is dual to a set of points that is arbitrarily close to an affine copy of S. Recall that S contains 3^s points and $5^s - 3^s$ collinear triples of points. Therefore, assuming s is large enough, the number of bounded triangular faces contained in each small disc that serves as a virtual intersection point of $n_0 = 3^s$ lines from $\mathcal{L}$ is at least $3 \cdot 3^s$. By Lemma 2, any set of red lines crossing together all the edges bounded in one disc that is a virtual intersection point must consist of at least $3^s - 2 + (3 \cdot 3^s)/3^s = 3^s + 1 = n_0 + 1$ red lines.

Because of Claim 1 we know that any subset A of $\mathcal{L}$ such that $|A| > 4g_{n_0}(k)n$ must contain at least $M\frac{|A|^{n_0}}{4^{n_0}g_{n_0}(k)^{n_0-1}n^{n_0-1}}$ virtually concurrent n_0-tuples of lines in $\mathcal{L}$.

Suppose now that $\mathcal{R}$ is a set of red lines that together cross all the bounded edges in the arrangement generated by the lines in $\mathcal{L}$. Call a line ℓ in $\mathcal{L}$ *good* if there are at least two red lines in $\mathcal{R}$ that follow ℓ. Otherwise ℓ is called *bad*.

If the number of bad lines in $\mathcal{L}$ is at most $4g_{n_0}(k)n$, then the number of good lines in $\mathcal{L}$ is at least $n - 4g_{n_0}(k)n$ and consequently there are at least twice this number of red lines in $\mathcal{R}$. This is because no red line can follow two different lines in $\mathcal{L}$ and every good blue line is followed by at least two red lines. We conclude that in this case there are at least $2n - 8g_{n_0}(k)n$ red lines in $\mathcal{R}$ and we are done because as n goes to infinity also k goes to infinity and $g_{n_0}(k)$ goes to 0.

Assume therefore that the number of bad blue lines in $\mathcal{L}$ is at least $4g_{n_0}(k)n$ and denote by A the set of all of these bad lines in $\mathcal{L}$. Because $|A| \geq 4g_{n_0}(k)n$ there are at least $M\frac{|A|^{n_0}}{4^{n_0}g_{n_0}(k)^{n_0-1}n^{n_0-1}}$ virtually concurrent n_0-tuples of lines in $\mathcal{L}$ that are also in A. Each of these virtually concurrent n_0-tuple of lines determine a collection of bounded edges that are contained in a disc which is a virtual intersection point. The number of red lines needed in order to cross all the edges bounded in one virtual intersection point is strictly greater than n_0. This implies that through every virtual intersection point there is a red line that does not follow any blue line. This is because each of the n_0 blue lines through a virtual intersection point of blue lines in A is followed by at most one red line.

This means that through each of the (at least) $M\frac{|A|^{n_0}}{4^{n_0}g_{n_0}(k)^{n_0-1}n^{n_0-1}}$ virtual intersection points associated with the virtually concurrent n_0-tuples of lines in A there must be a red line that does not follow any of the lines in $\mathcal{L}$.

As we have seen already, a line that does not follow a line in $\mathcal{L}$ can meet at most n_0^k virtual intersection points. Therefore, the number of lines in $\mathcal{R}$ (even only those who do not follow a line in $\mathcal{L}$) must be at least

$$\frac{1}{n_0^k} \cdot M\frac{|A|^{n_0}}{4^{n_0}g_{n_0}(k)^{n_0-1}n^{n_0-1}} \geq \frac{1}{n_0^k} \cdot M\frac{(4g_{n_0}(k))^{n_0}}{4^{n_0}g_{n_0}(k)^{n_0-1}n^{n_0-1}} \geq \frac{Mg_{n_0}(k)}{n_0^k}n.$$

Taking M large enough (as a function of k) such that $\frac{Mg_{n_0}(k)}{n_0^k} \geq 2$, we conclude that the number of red lines in $\mathcal{R}$ is at least $2n$ which completes the proof of Theorem 2. ∎

5 Concluding remarks

In both Theorems 1 and 2 it would be nice to settle the true order of magnitude of the $o(n)$ term. In particular it is interesting to find whether $h(n)$ and $f(n)$ are bounded away from their "trivial" upper bounds (n and $2n$, respectively) by more than an absolute additive constant.

One simple observation is that the trivial upper bound $f(n) \leq 2n$ that is mentioned in the introduction can be improved to $2n - 2$, at least in the projective plane. To see this notice that given any collection of n blue lines in general position in the real projective plane, one can discard one of them and for every other blue line ℓ take two red lines parallel and very close to it, one slightly above ℓ and one slightly below ℓ (in an affine picture of the projective plane). The resulting set of $2n-2$ red lines still crosses all blue edges in the arrangement of all blue lines. This is because every blue edge is delimited by two intersection points with two blue lines and at most one of them is the discarded blue line.

Another important remark is that one could think that in ANY arrangement of n blue lines in general position in the plane one needs MANY red lines to cross all faces, respectively edges, of the blue arrangement, where MANY is very close to $h(n)$, respectively $f(n)$, red lines. This is however not the case. Let $\mathcal{L}$ be the set of n lines that support the n edges of a regular n-gon Q. It is not hard to see that the set of all intersection points of lines in $\mathcal{L}$ sit on n red lines through the origin. Taking a slight clockwise rotation of the red lines about the center of Q results in a collection of n (which is the "trivial" lower bound for $f(n)$) red lines that together intersect all the blue edges in the arrangement generated by $\mathcal{L}$.

The same construction of n blue lines supporting the edges of a regular n-gon Q also shows that some times one needs only $n/2$ red lines to cross all faces in an arrangement of n blue lines in general position. It is easiest to see when n is even: Take the $n/2$ red lines through the origin that bisect edges of Q and then slightly rotate them clockwise about the center of Q. We leave it to the reader to verify the details.

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