The Minimum Number of Edge-Directions of a Convex Polytope

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Abstract

We show that the minimum number of distinct edge-directions of a convex polytope with n vertices in R^d is $\theta(dn^{1/(d-1)})$.

1 Introduction

Let P be a convex polytope with n vertices in R^d. As usual, the *edges* of P are its 1-dimensional faces. An *edge-direction* of P is the class of all lines parallel to some edge of P. Edge-directions play a central role in the algorithmic solution and the complexity analysis of the *convex combinatorial optimization problem*, see [2] and references therein for details.

In this note, we are interested in the minimum number of distinct edge-directions of polytopes with n vertices in R^d. For example, if $d = 2$, then P is a convex polygon and clearly the minimum number of directions of the edges of P is $\lceil \frac{n}{2} \rceil$. Indeed, this lower bound is obtained in a regular n-gon when n is even, and in a regular $(n-1)$-gon with one corner chopped off when n is odd.

For higher dimensions we prove the following theorem.

Theorem 1.1. *The number m of distinct edge-directions of any d-dimensional convex polytope with n vertices satisfies $m \geq \frac{d-1}{2e} n^{1/(d-1)}$.*

The lower bound in Theorem 1.1 is asymptotically tight: it is attained, up to a multiplicative constant factor, by the following *zonotope* Z, which is equivalent to the member $\mathcal{M}_d^2(m)$ of the class of *momentopes* introduced

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in [1] in connection with certain partitioning problems. We outline the construction of Z. For any j, let $e^j := (1, j, \ldots, j^{d-1})$ be the corresponding point on the moment curve in R^d. Let Z be the zonotope generated by the e^j,

$$Z \quad := \quad \sum_{j=1}^{m} [0,1] e^j \quad = \quad \mathrm{conv} \left\{ \sum_{j=1}^{m} \mu_j e^j \;:\; \mu = (\mu_1, \ldots, \mu_m) \in \{0,1\}^m \right\} .$$

It can be verified that Z is equivalent to the special momentope $\mathcal{M}_d^2(m)$ of [1]. Then a tuple $\mu = (\mu_1, \ldots, \mu_m) \in \{0,1\}^m$ gives a vertex $\sum_{j=1}^m \mu_j e^j$ of Z if and only if there are at most $d-1$ alterations between segments of 0 and segments of 1 in μ: indeed, for such (and only such) μ there is a univariate polynomial $p(z) = \sum_{i=1}^d p_i z^{i-1}$ satisfying $p(j) > 0$ if and only if $\mu_j = 1$, implying that the linear functional $\sum_{i=1}^d p_i x_i$ on R^d is uniquely maximized over Z at $\sum_{j=1}^m \mu_j e^j$. Since, as is easy to see, the number of such μ is $n := 2 \sum_{k=0}^{d-1} \binom{m-1}{k}$, it follows that Z is a d-dimensional polytope with m edge-directions and n vertices, attaining the claimed lower bound.

We note, however, that if P is not a *convex* polytope, then the minimum number of distinct directions of the edges of P may be bounded by a constant independent of n. This can clearly be seen, for example, by taking P to be any union of axes-parallel cubes in R^d.

Moreover, if we don't restrict attention to the edges of P, but seek the minimum number of distinct directions among all segments connecting pairs of vertices of P, then the lower bound is always linear in n and in higher dimensions is conjectured to be roughly $(d-1)n$. In dimension 3 it follows from [3] that the minimum number of these directions is at least $2n-3$. When $d = 2$, it is easy to see (and follows also from a more general theorem by Ungar [4]) that the minimum number of distinct directions determined by the vertices of a convex n-gon is at least n.

As a nice corollary of Theorem 1.1, we prove the following theorem, about the minimum number of distinct directions of the edges of a convex polytope with n vertices, regardless of its dimension.

Theorem 1.2. *The number of distinct edge-directions of any convex polytope with n vertices is $\Omega(\log n)$.*

Note that the d-dimensional cube, which has $n := 2^d$ vertices and d edge-directions, demonstrates that the number of distinct edge-directions of a convex polytope on n vertices can be as small as $\log_2 n = \log_2 e \log n$. Therefore, Theorem 1.2 is asymptotically tight as well.

2 Proofs of Theorem 1.1 and Theorem 1.2

In the proofs below it is convenient to consider *oriented edge-directions* rather than just edge-directions. An oriented edge-direction which corresponds to the pair (e, v) where e is an edge of a polytope with a vertex v on it, is the class of oriented lines l parallel to the edge e where the positive direction of l is indicated by the ray whose apex is v and includes e. Thus, every edge-direction gives rise to two oriented edge-directions.

Proof of Theorem 1.1: To each vertex v of P we assign a sequence of $d-1$ oriented edge-directions $(t^v_{d-1}, \ldots, t^v_1)$ and a corresponding decreasing chain of faces of P $(F^v_{d-1}, \ldots, F^v_1, F^v_0)$ (F^v_i has dimension i) in the following way.

Let F^v_{d-1} be a facet of P which has v as a vertex. We take F^v_{d-2} to be a $(d-2)$-dimensional face of F^v_{d-1} which has v as a vertex. Then we take an edge e_{d-1} of P, adjacent to v, which is in F^v_{d-1} but not in F^v_{d-2}, and define t^v_{d-1} to be the oriented edge-direction which corresponds to (e_{d-1}, p). Then we take F^v_{d-3} to be a $(d-3)$-dimensional face of F^v_{d-2}, having v as a vertex and let t^v_{d-2} be the oriented edge-direction which corresponds to (e_{d-1}, v), where e_{d-2} is an edge of P, adjacent to v, which is in F^v_{d-2} but not in F^v_{d-3}. We continue in this manner $d-1$ steps. At the $i'th$ step we take F^v_{d-i-1} to be a $(d-i-1)$-dimensional face of F^v_{d-i} having v as a vertex. Then we define t^v_{d-i} to be the oriented edge-direction which corresponds to (e_{d-i}, v) , where e_{d-i} is an edge of P, adjacent to v, which is in F^v_{d-i} but not in F^v_{d-i-1}.

Claim 2.1. *If x, y are two distinct vertices of P such that $F^x_{d-1} = F^y_{d-1}$, then x, y can not have the same sequence of $d-1$ oriented edge-directions.*

Proof: By induction on the dimension d of P. Clearly, the claim is true if $d = 1$ (for then P is a segment). For arbitrary dimension, let x, y be two distinct vertices of P such that $F^x_{d-1} = F^y_{d-1}$. Denote that facet by F. If $F^x_{d-2} = F^y_{d-2}$, then x, y belong to the same $(d-2)$-dimensional face of F and we conclude by the induction hypothesis for the $(d-1)$-dimensional polytope F. Therefore, we may assume that $F^x_{d-2} \neq F^y_{d-2}$. However, if we restrict our attention to the $(d-1)$-dimensional flat which contains F, then within it the hyper-plane which contains F^x_{d-2} is determined by the directions $t^x_{d-2}, \ldots, t^x_1$ and similarly for F^y_{d-2}. It follows that F^x_{d-2} and F^y_{d-2} are two parallel facets of F. But then it is not possible that $t^x_{d-1} = t^y_{d-1}$. This proves the claim. ■

Claim 2.2. *No three vertices of P are assigned the same $(d-1)$-sequence of oriented edge-directions.*

Proof: Assume to the contrary that x, y, z are three vertices that are assigned the same $(d-1)$-sequence of oriented edge-directions. By Claim 2.1, $F^x_{d-1}, F^y_{d-1}, F^z_{d-1}$ are three different facets of P. Since the sequences of $d-1$ oriented edge-directions assigned to x, y, and z determine the directions of the hyper-planes containing the corresponding facets $F^x_{d-1}, F^y_{d-1}, F^z_{d-1}$, then clearly these three facets must lie in parallel hyper-planes in R^d, which is a contradiction. ■

For each vertex v of P we now create all possible $(d-1)$-sequences of oriented edge-directions as described above. Observe that for every vertex v there are at least $d!$ different such sequences. Indeed, given a vertex v, every k-face of P having v as a vertex, has at least k $(k-1)$-faces which also have v as a vertex. Therefore, there are at least $d!$ decreasing chains of faces of P which contain v. For every such chain $F_{d-1}, F_{d-2}, \ldots, F_0$, there is at least one sequence of oriented edge-directions $t_{d-1}, t_{d-2}, \ldots, t_1$ which corresponds to it, for example, simply take t_{d-i} be the oriented edge-direction which corresponds to the pair (e_{d-i}, v), where e_{d-i} is an edge (adjacent to v) which is in F_{d-i} but not in F_{d-i-1}. Since the sequence $t_{d-1}, \ldots, t_1$ determines the chain $F_{d-1}, \ldots, F_0$, we obtain at least $d!$ $(d-1)$-sequences of oriented edge-directions for each v.

Let m denote the number of distinct edge-directions of edges of P. The maximum number of $(d-1)$-sequences of oriented edge directions which can be formed from these m edge-directions is $2^{d-1}\binom{m}{d-1}(d-1)!$. By Claim 2.2 and our observation above, this number should be greater than or equal to $d!n/2$. We thus obtain

$$\binom{m}{d-1} \geq \frac{d}{2^d}n.$$

Observe that $\binom{m}{d-1} \leq (\frac{me}{d-1})^{d-1}$ which implies that

$$m \geq \frac{d-1}{2e}n^{1/(d-1)}.$$

This proves the theorem. ■

Proof of Theorem 1.2: Let m denote the number of directions of the edges of P. By Theorem 1.1, $m \geq \frac{d-1}{2e}n^{1/(d-1)}$. Minimizing the right hand side when d is the variable gives $d-1 = \log n$. Therefore, $m \geq \frac{1}{2}\log n$. ■

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