Crossing by lines all edges of a line arrangement

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Abstract

Let $\mathcal{L}$ be a family of n blue lines in the real projective plane. Suppose that $\mathcal{R}$ is a collection of m red lines, different from the blue lines, and it is known that every edge in the arrangement $\mathcal{A}(\mathcal{L})$ is crosses by a line in $\mathcal{R}$. We show that $m \geq \frac{n-1}{3.5}$. Our result is more general and applies to pseudo-line arrangements $\mathcal{A}(\mathcal{L})$, and even weaker assumptions are required for $\mathcal{R}$.

1 Introduction

Let $\mathcal{L}$ be a collection of n blue lines in the real projective plane. We consider the arrangement $\mathcal{A}(\mathcal{L})$ of all the lines in $\mathcal{L}$ and we wish to find another set of red lines $\mathcal{R}$, as small as possible, such that $\mathcal{L} \cap \mathcal{R} = \emptyset$ and every (blue) edge in the arrangement $\mathcal{A}(\mathcal{L})$ is crossed by a curve in $\mathcal{R}$. Clearly, if all the lines in $\mathcal{L}$ are concurrent, then one line can cut through all the n edges in $\mathcal{A}(\mathcal{L})$. We therefore assume that not all the lines in $\mathcal{L}$ are concurrent.

We denote by $f(n)$ the minimum number such that there exists a collection $\mathcal{L}$ of n non-concurrent blue lines in the real projective plane and another set $\mathcal{R}$ of $f(n)$ red lines such $\mathcal{L} \cap \mathcal{R} = \emptyset$ and every (blue) edge in $\mathcal{A}(\mathcal{L})$ is crossed by a line in $\mathcal{R}$.

It is not hard to see that $f(n) \leq n$. In fact for every arrangement $\mathcal{A}(\mathcal{L})$ of n blue lines one can find n red lines that together cross every blue edge of $\mathcal{A}(\mathcal{L})$. Indeed, assume we are given a collection $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ of n non-concurrent blue lines in the real projective plane. Let h be any line in general position with respect to $\mathcal{L}$. For every $1 \leq i \leq n$ let x_i be the intersection point of h and ℓ_i and let r_i be a line through x_i that is obtained from ℓ_i by a very small rotation about x_i in the counterclockwise direction. We leave it to the reader to verify that every edge in $\mathcal{A}(\mathcal{L})$ is crossed by (at least) one of the lines $r_1, \ldots, r_n$.

We note that sometimes n red lines are necessary to cut through all the blue edges in an arrangement of n blue lines. Indeed, this is the situation when the set $\mathcal{L}$ of n blue lines is in general position, namely, if no three lines in $\mathcal{L}$ are concurrent. It is immediate to see that in this case $n-1$ red lines are necessary just because there are $n-1$ blue edges on each blue line and every red line cuts through (at most) one blue edge on every blue line (or alternatively, there are $n(n-1)$ blue edges in total and every red line cuts through only n of them). To see that in fact n red lines are necessary and $n-1$ are not enough, notice that if $n-1$ red lines are enough to cut through all the $n(n-1)$ blue edges, then no two red line cross the same blue edge and every red line crosses n

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blue edges (rather than passing through some intersection point of blue lines). However, there is a triangular face (in fact many) in every line arrangement of non-concurrent lines. Every red line that crosses an edge of that triangle must cross another edge of the triangle. Therefore if all three edges of the triangle are crossed by red lines, it must be that there is one edge of the triangle that is crossed by two distinct red lines, implying that $n-1$ red lines are not enough to cross all the edges in the arrangement $\mathcal{A}(\mathcal{L})$.

In this paper we show that $f(n) \geq \frac{n}{3.5}$. We will show it in a more general setting of pseudo-lines and even weaker assumptions will be made on the set $\mathcal{R}$. In view of the above argument about arrangements $\mathcal{L}$ in general position, the question of bounding from below $f(n)$ is difficult because of arrangements of lines that are not in general position, that is, arrangements that contain three or more lines through the same point.

Theorem 1. *Let $\mathcal{L}$ be a collection of n bi-infinite x-monotone non-concurrent blue pseudo-lines in the real projective plane. Let $\mathcal{R}$ be a collection of m bi-infinite x-monotone curves such that every curve in $\mathcal{R}$ meets every curve in $\mathcal{L}$ precisely once. Assume that every (blue) edge in the arrangement $\mathcal{A}(\mathcal{L})$ is crossed by a red curve in $\mathcal{R}$, then $m \geq \frac{n-1}{3.5}$.*

Notice that Theorem 1 is valid for the case where $\mathcal{L}$ and $\mathcal{R}$ are collections of lines. An $\Omega(n)$ bound for $f(n)$ follows from the so called "weak Dirac's conjecture". In 1951 Dirac [4] conjectured that in any set of n non-concurrent lines there exists a line incident to at least $\frac{n}{2} - O(1)$ intersection points with other lines in the set. Szemerédi and Trotter [12] and Beck [1] proved a weaker result which is that under the above conditions there exists a line incident to $\Omega(n)$ intersection points. This result is known as the "weak Dirac's conjecture". The proofs in [1, 12] used the upper bound of Szemerédi and Trotter [12] for the number of incidences between points and lines in the plane.

Notice that any lower bound for the problem of Dirac implies immediately the same lower bound for $f(n)$. Indeed, if there exists a blue line ℓ incident to k intersection points with other blue lines, then clearly k distinct red lines are required just to ensure that every blue edge on ℓ is crossed by a red line.

In the original papers [12] and [1] lower bounds of $10^{-186}n$ and $2^{-1000}n$, respectively, are shown for the "weak Dirac's conjecture". Today much better bounds in terms of the multiplicative constant are known for the number of incidences between points and lines in the plane (see [8] for the best bound). Consequently also the constant in the "weak Dirac's conjecture" is improved. Very recently, Payne and Wood [9] did carry out this calculation of the best constant in the "weak Dirac's conjecture". They combined the above mentioned progress on the number of point-line incidences in the plane and with some more ideas showed that the lower bound in the "weak Dirac's conjecture" can be improved to $\frac{n-3}{76}$.

On the other hand and also very recently, Lund, Purdy, and Smith [7] showed that Dirac's conjecture is false if we replace lines by pseudo-lines. They constructed examples of n pseudo-lines where each one is incident to at most $\frac{4}{9}n$ intersection points.

Finally, we note that Theorem 1 is very close in spirit to the following problem that was raised independently by Grünbaum and Motzkin [6] and by Erdős and Purdy [5].

Problem A. *Given a set $\mathcal{L}$ of n non-concurrent blue lines in the real projective plane we wish to find a set $\mathcal{R}$ of m red lines different from the blue ones such that every intersection point of blue lines is incident to a red line. Give a lower bound for m in terms of n.*

An $\Omega(n)$ bound for m can be deduced in Problem A from the "weak Dirac's conjecture" in exactly the same way as for bounding from below $f(n)$. Until very recently, nothing beyond that was known on Problem A. In a recent work [10] it is shown that $m \geq \frac{n-1}{3}$ in Problem A. In the current paper we use a variant of the method in [10] to prove Theorem 1.

The advantage in Theorem 1 upon a bound for Problem A lies in possible implication of the result in Theorem 1 towards solving Dirac's conjecture. This is the main motivation for this paper. Given an arrangement $\mathcal{A}(\mathcal{L})$ of blue lines, let $g(\mathcal{L})$ denote the maximum number of edges on a line in $\mathcal{L}$. Notice that $g(\mathcal{L})$ is also the maximum number of vertices in the arrangement $\mathcal{A}(\mathcal{L})$ on a line in $\mathcal{L}$. Dirac's conjecture is that $g(\mathcal{L}) \geq \frac{n}{2} - O(1)$ for any non-concurrent family $\mathcal{L}$ of n lines. $g(\mathcal{L})$ is a natural lower bound for the cardinality of a set $\mathcal{R}$ of red lines that cross all blue edges in $\mathcal{A}(\mathcal{L})$. Suppose one can show by a direct argument (possibly algorithmic construction) that $c \cdot g(\mathcal{L})$ red lines (or pseudo-lines) are enough to cross all blue edges of $\mathcal{A}(\mathcal{L})$, where $c > 0$ is an absolute constant (hopefully not too big), then Theorem 1 implies that $g(\mathcal{L}) \geq \frac{n}{3.5c}$. Hence, if c is not too big (in fact, assuming Dirac's Conjecture is true, then $c = 2$ is indeed enough here), then we can improve significantly the best known multiplicative constant in the weak Dirac's conjecture.

It is more likely that one can show that $c \cdot g(\mathcal{L})$ red pseudo-lines are enough to cross all blue edges of $\mathcal{A}(\mathcal{L})$ rather that to show that $c \cdot g(\mathcal{L})$ red lines (or pseudo-lines) are enough to visit all blue vertices of the arrangement $\mathcal{A}(\mathcal{L})$. This is because *any* line crosses n distinct blue edges in the arrangement $\mathcal{A}(\mathcal{L})$, while it is highly nontrivial (and, as far as we know, open) to find even a single curve r passing through linearly many vertices of $\mathcal{A}(\mathcal{L})$ such that r does not intersect any line in $\mathcal{L}$ more than once. Hence it is easier to generate "small" families of red lines crossing all blue edges in $\mathcal{A}(\mathcal{L})$, than to generate "small" families of red (pseudo-)lines passing together through all vertices of the arrangement $\mathcal{A}(\mathcal{L})$.

2 Proof of Theorem 1

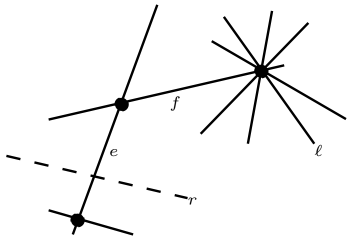
Throughout the proof we will call the curves in $\mathcal{L} \cup \mathcal{R}$ lines rather than 'curves' or 'pseudo-lines'. We will not assume anything on the algebraic structure of these curves. We will need only that $\mathcal{L}$ is an arrangement of bi-infinite $x$monotone pseudo-lines, for which we can apply the zone theorem (see below). As for the curves in $\mathcal{R}$, we will only assume that every such curve is bi-infinite and x-monotone and crosses every curve in $\mathcal{L}$ at most once.

The idea of the proof is to estimate in two different ways the cardinality of the following set Q of special quadruples (r, e, f, ℓ) such that:

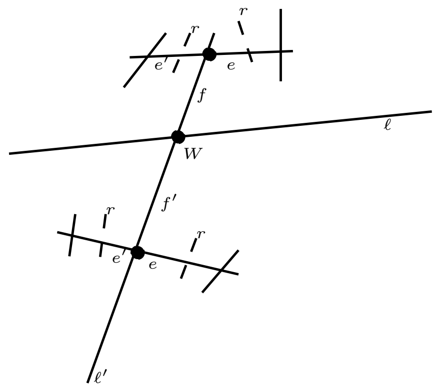
- e is an edge in the arrangement $\mathcal{A}(\mathcal{L})$ crossed by the red line $r \in \mathcal{R}$.
- f is an edge in $\mathcal{A}(\mathcal{L})$ adjacent to e in some face of $\mathcal{A}(\mathcal{L})$.
- ℓ is a line in $\mathcal{L}$, not containing e, passing through the vertex of f that is not on e.

See Figure 1(a) for an example of a quadruple in Q. We note that throughout all of our drawings below, lines in $\mathcal{L}$ are drawn solid while lines in $\mathcal{R}$ are drawn dashed. A simple lower bound for $|Q|$ is argued as follows. Consider any two lines ℓ and ℓ' in $\mathcal{L}$. Let W be the intersection point of ℓ and ℓ'. Then there are precisely two edges f, f' of $\mathcal{A}(\mathcal{L})$ on the line ℓ' that are incident to W (here we use the fact that not all the lines in $\mathcal{L}$ pass through the same point. Notice also that the two edges incident to W may have the same other vertex Z, in the case where all the lines in $\mathcal{L}$, but one, are concurrent). For each of these two edges there are two edges e, e', of which W is not a vertex, that are adjacent to it in some face of $\mathcal{A}(\mathcal{L})$. Each of the edges e, e' in both cases are crossed by some red line r in $\mathcal{R}$. Therefore, for every (ordered) pair of blue lines ℓ and ℓ' we obtain four distinct quadruples in Q, so that no quadruple arises more than once in this manner (see Figure 1(b)). This implies that

$$|Q| \geq 4n(n-1). \tag{1}$$



(a) A quadruple (r, e, f, ℓ)



(b) ℓ and ℓ' give rise to four quadraples

Figure 1: The definition of the quadruples in Q.

To obtain a good lower bound for $|\mathcal{R}|$ in terms of n it will therefore be helpful to bound from above the number of quadruples in Q to which a given red line r belongs. For $r \in \mathcal{R}$ we denote by $Q(r)$ the set of quadruples in Q in which the red line is r.

Lemma 1. *For every line $r \in \mathcal{R}$ we have $|Q(r)| \leq 14n$.*

Proof. Let r be the given red line and assume without loss of generality that r is horizontal. That is, we consider an affine picture of the projective plane in which r is horizontal. We can also assume, by applying a suitable projective transformation, that no two lines in $\mathcal{L} \cup \mathcal{R}$ are parallel in this affine picture.

Let $e_1, \ldots, e_n$ denote the edges of $\mathcal{A}(\mathcal{L})$ crossed by r from left to right. For every $1 \leq i \leq n$ let U_i and V_i denote the vertices of e_i below and above r, respectively (see Figure 2). We denote by f_i and f'_i the two edges adjacent to e_i at V_i in some faces of $\mathcal{A}(\mathcal{L})$. We denote by g_i and g'_i the two edges adjacent to e_i at U_i in some faces of $\mathcal{A}(\mathcal{L})$. W_i and W'_i will denote the endpoints of f_i and f'_i, respectively, different from V_i. Z_i and Z'_i will denote the endpoints of g_i and g'_i, respectively, different from U_i (see Figure 2). Finally, for every vertex B of $\mathcal{A}(\mathcal{L})$ let $d(B)$ denote the number of lines in $\mathcal{L}$ passing through B.

Therefore, the number of quadruples in $Q(r)$ is precisely

$$\sum_{i=1}^{n} (d(W_i) - 1) + (d(W'_i) - 1) + (d(Z_i) - 1) + (d(Z'_i) - 1).$$

For an integer $k \geq 1$ we say that a vertex B of $\mathcal{A}(\mathcal{L})$ is k-interesting if it equals to k of the points in the list: $W_1, W'_1, \ldots, W_n, W'_n, Z_1, Z'_1, \ldots, Z_n, Z'_n$.

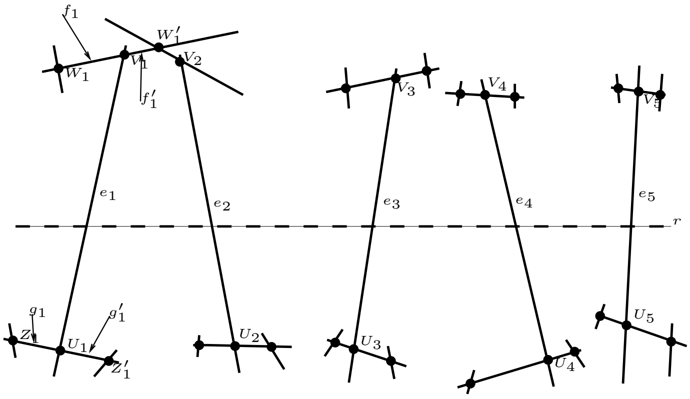


Figure 2: Notation in Lemma 1.

In this new terminology we have:

$$|Q(r)| = \sum_{k\geq 1} \sum_{B \text{ is } k\text{-interesting}} k(d(B)-1). \tag{2}$$

Notice that it may happen that some vertex B in $\mathcal{A}(\mathcal{L})$ is k-interesting for $k > 1$. We now show that no vertex B can be k-interesting for $k \geq 5$.

Claim 1. *If a vertex B of $\mathcal{A}(\mathcal{L})$ is k-interesting, then $k \leq 4$.*

Proof. We claim that there are no three indices i_1, i_2, and i_3 such that B equals W_{i_j} or W'_{i_j} for every $j = 1, 2, 3$. Assume there are such indices $i_1 < i_2 < i_3$. Let Y_{i_1}, Y_{i_2}, and Y_{i_3} denote the three intersection points of e_{i_1}, e_{i_2}, and e_{i_3}, respectively, with r. Notice that for every $1 \leq j \leq 3$ the three points Y_{i_j}, V_{i_j}, and B must belong to the (boundary of the) same face in $\mathcal{A}(\mathcal{L})$. Consider the line $\ell \in \mathcal{L}$ containing the edge e_{i_2}. The line ℓ does not pass through B and it separates Y_{i_1} and Y_{i_3} (here we use the assumption that r crosses every line in $\mathcal{L}$ at most once). Therefore, either B and Y_{i_1} do not belong to the same face in $\mathcal{A}(\mathcal{L})$, or B and Y_{i_3} do not belong to the same face in $\mathcal{A}(\mathcal{L})$. In both cases we reach a contradiction (see Figure 3).

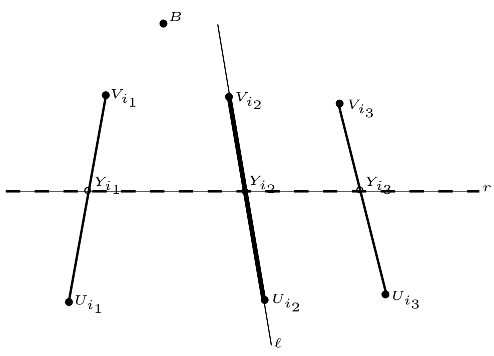


Figure 3: Illustration of the proof of Claim 1.

In exactly the same way one can show that there are no three distinct indices i_1, i_2, and i_3 such that B equals Z_{i_j} or Z'_{i_j} for every $1 \leq j \leq 3$. This conclude the proof of Claim 1. (Figure 4 shows a case in which a vertex B is 4-interesting.) ∎

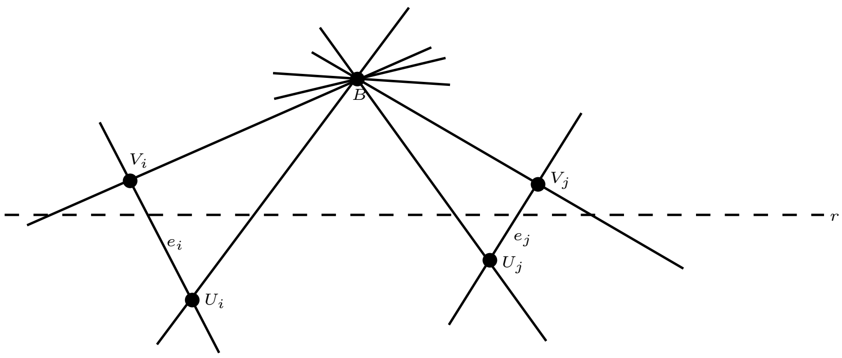


Figure 4: The vertex B is 4-interesting.

From Claim 1 and from (2) it follows that

$$|Q(r)| \leq 4 \cdot \sum_{B \text{ is interesting}} (d(B) - 1). \tag{3}$$

Recall the notion of a *zone* of a line r with respect to a line arrangement $\mathcal{A}(\mathcal{L})$. This is the collection of all faces in $\mathcal{A}(\mathcal{L})$ that are crossed by r. The complexity of a zone is the total sum of all sizes of faces of the zone, where the size of a face is the number of its edges. The notion of a zone was first introduced in [3], where an upper bound of $6n$ is established for the complexity of a zone of an arrangement $\mathcal{A}(\mathcal{L})$ of n lines. This bound was later improved in [2] to $5.5n - 1$, which is tight up to the additive factor. In [3] and [2] the zone was considered in the *affine* plane rather than the *projective* plane. This difference is not significant. The maximum complexity of a zone in the projective plane is smaller by two units than in the affine plane. We will take as an upper bound for the complexity of a zone in an arrangement of n lines in the projective plane the bound of $5.5n$. Observe that in the projective plane the complexity of a zone is also the sum of all sizes of faces in the zone where the size of the face is the number of its *vertices* rather than the number of its edges (this is because we do not need to handle unbounded faces, as in the affine plane).

The importance of the zone for bounding $|Q(r)|$ is because of the simple observation that all the interesting vertices must belong to faces of the zone of the line r with respect to the arrangement $\mathcal{A}(\mathcal{L})$.

Consider an interesting vertex B in $\mathcal{A}(\mathcal{L})$. The contribution of B to the upper bound for $|Q(r)|$ in (3) is $d(B) - 1$, while its contribution for the complexity of the zone of r may be only 1. The obvious way to fix this is to consider a face F in the zone of r of which B is a vertex. Then slightly perturb the $d(B)$ lines passing through B such that each of them contributes an edge to F. This way we increase the complexity of the zone of the line r and now the $d(B)$ lines through B contribute $d(B) - 1$ vertices to the face F in the zone of r (see Figure 5).

After performing this change for every interesting vertex B, the right hand side of (3) is bounded by four times the complexity of the zone of r in this perturbed arrangement of n lines and hence bounded from above by $4 \cdot 5.5n = 22n$. We can slightly improve on this bound by noticing that the

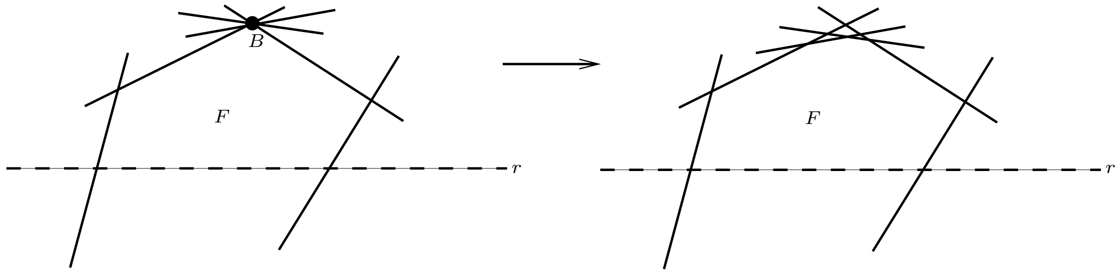


Figure 5: The small perturbation near vertex B.

two vertices of every edge crossed by r are counted in two distinct faces of the zone of r. Hence the total number of vertices that lie in the zone of r is bounded by $5.5n - 2n = 3.5n$. Each of these vertices is at most 4-interesting and therefore by (3), $|Q(m)| \leq 4 \cdot 3.5n = 14n$. This concludes the proof of Lemma 1. ∎

From Lemma 1 it follows that the cardinality of Q is bounded from above by $14n|\mathcal{R}|$. Combining this with (1) we have $4n(n-1) \leq 14n|\mathcal{R}|$ implying that $|\mathcal{R}| \geq \frac{n-1}{3.5}$. ∎

We end with a remark about upper bounds for the function $f(n)$. As we have seen, every arrangement of n blue lines in general position shows that $f(n) \leq n$, as n red lines are enough (and in fact also necessary) to cross all the blue edges. We are not aware of general constructions that could provide a better upper bound. Namely, we are not aware of general constructions of n blue lines for which less than n red lines (or pseudo-lines) are enough to cross all the blue edges in the arrangement of the blue lines. There are, however, some sporadic such constructions.

Figure 6 shows an arrangement of 7 blue lines such that all the edges in this arrangement are crossed (in the projective plane) by only 5 red lines. We believe that there should be arbitrary large such constructions.

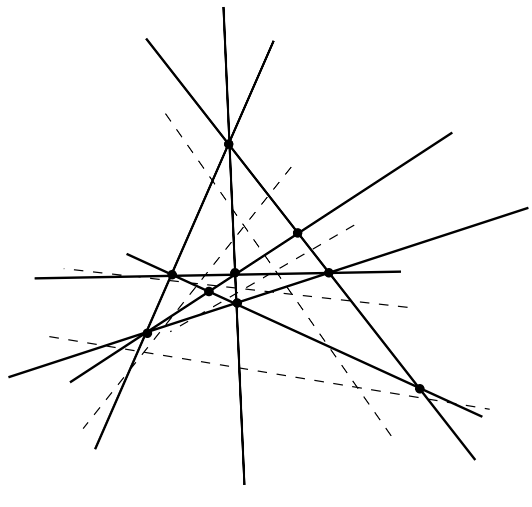


Figure 6: An arrangement of 7 blue lines and only 5 red lines crossing all blue edges.

It is highly possible that the bound in Theorem 1 can be improved with the same method but with a more careful analysis as for the bound of $|Q(r)|$ in Lemma 1. As this would damage the presentation, we did not make an attempt to see if this is indeed the case. Once a direct relation between the result in Theorem 1 and Dirac's conjecture will be established, such an effort will be more justified.

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