On the Degenerate Crossing Number

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Abstract

The *degenerate crossing number* $\mathrm{cr}^*(G)$ of a graph G is the minimum number of crossing *points* of edges in any drawing of G as a simple topological graph in the plane. This notion was introduced by Pach and Tóth who showed that for a graph G with n vertices and $e \geq 4n$ edges $\mathrm{cr}^*(G) = \Omega(e^4/n^4)$. In this paper we completely resolve the main open question about degenerate crossing numbers and show that $\mathrm{cr}^*(G) = \Omega(e^3/n^2)$, provided that $e \geq 4n$. This bound is best possible (apart for the multiplicative constant) as it matches the tight lower bound for the standard crossing number of a graph.

1 Introduction

The graphs considered in this paper contain no loops or parallel edges. A *topological graph* is a drawing of a graph in the plane such that the vertices are drawn as distinct points and the edges are drawn as Jordan arcs connecting corresponding points without passing through other vertices of the graph. Two edges in a topological graph may intersect at a finite number of points, where in each intersection point they either share a common endpoint or properly cross each other. If every pair of edges intersect at most once, then the topological graph is called *simple*. One sometimes assumes that in a topological graph there are no three edges that cross each other at the same point. However, in this paper we are interested in topological graphs in which such crossings are allowed.

For a topological graph D we denote by $\mathrm{cr}(D)$ the number of crossings of pairs of edges in D. The *crossing number* of an abstract graph G, denoted by $\mathrm{cr}(G)$, is the minimum value of $\mathrm{cr}(D)$ taken over all drawings D of G as a topological graph. It is not hard to see that if $\mathrm{cr}(G) = \mathrm{cr}(D)$, then D is a simple topological graph.

Ajtai, Chvátal, Newborn, Szemerédi [1] and, independently, Leighton [3] proved that there is an absolute constant $c > 0$ such that for every abstract graph G with n vertices and e edges $\mathrm{cr}(G) \geq c\frac{e^3}{n^2}$, provided that $e \geq 4n$. This result is referred to as the *Crossing Lemma* and has numerous applications in combinatorial and computational geometry, number theory, and other fields of mathematics.

Several works have considered different crossing numbers (see [2, § 9.4], [4, § 5.3], and the references therein). One such example is the *degenerate* crossing number. This notion was first introduced in [5] following questions by G. Rote, M. Sharir, and others who asked what happens

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if we count crossing *points*, rather than crossings of pairs of edges? For example, if $k \geq 2$ edges in a topological graph D cross each other at the same point, then we count this point only once, instead of $\binom{k}{2}$ times as in $\mathrm{cr}(D)$.

Pach and Tóth [5] proved that in any drawing of a graph G, the number of crossing points is at least a constant times the number of edges of G, and that this bound cannot be improved by much. Namely, they showed that any graph G with e edges can be drawn as a topological graph containing at most $e-1$ crossing points. However, in their construction a pair of edges may cross many times. Therefore, they have also considered the minimum number of crossing points in any possible drawing of a graph as a *simple* topological graph.

For a topological graph D we denote by $\mathrm{cr}^*(D)$ the number of crossing points of edges of D. For an abstract graph G we denote by $\mathrm{cr}^*(G)$ the *degenerate* crossing number of G. That is, the minimum value of $\mathrm{cr}^*(D)$ taken over all possible drawings of G as a simple topological graph D. In [5] it is shown that $\mathrm{cr}^*(G) = \Omega(\frac{e^4}{n^4})$, and the question remained whether the behavior of $\mathrm{cr}^*(G)$ is very much different from $\mathrm{cr}(G)$. Or, in the words of [4], "it is a challenging question to decide whether the $\frac{e^4}{n^4}$ term can be replaced by $\frac{e^3}{n^2}$, just like in the Crossing Lemma." In this paper we answer this question on the affirmative.

Theorem 1. *Let G be a simple topological graph with n vertices and $e \geq 4n$ edges. Then the number of points in which two or more edges of G cross each other is at least $\Omega(\frac{e^3}{n^2})$.*

2 The Proof

Before proving Theorem 1, we start with few auxiliary lemmas.

Lemma 1. *Let G be a simple topological graph with e edges and let X be a set of $|X|$ distinct points. The number of incidences $I(X, E(G))$ between the points of X and the edges of G satisfies $I(X, E(G)) = O(|X|^{2/3}(cr(G))^{1/3} + |X| + e)$.*

Proof. We define a simple topological graph H whose vertices are the points of X. We connect two points of X by an edge if they are consecutive points of X along an edge w of G. We draw this edge along the portion of w delimited by these two vertices.

Clearly, we have $I(X, E(G)) \leq |E(H)| + e$. Observe that H is a simple topological graph on $|X|$ vertices and has at most $\mathrm{cr}(G)$ pairs of crossing edges. It follows from the Crossing Lemma that $(|E(H)| - 4|X|)^3/|X|^2 = O(\mathrm{cr}(G))$. Therefore,

$$I(X, E(G)) \leq |E(H)| + e = O(|X|^{2/3}\mathrm{cr}(G)^{1/3} + |X| + e),$$

as claimed. □

For a topological graph G and an integer $k \geq 2$ we denote by $t_k(G)$ the number of crossing points of precisely k edges in G. The following lemma is a direct corollary of Lemma 1 and the fact that $\mathrm{cr}(G) = \sum_{k\geq 2}\binom{k}{2}t_k(G)$.

Lemma 2. *There is a constant c_1 such that for every simple topological graph G with e edges and for every integer $k' \geq 2$ we have:*

$$\sum_{k\geq k'} t_k(G) \leq c_1 \cdot \left(\frac{\sum_{k\geq 2}\binom{k}{2}t_k(G)}{k'^3} + \frac{e}{k'} \right).$$

Proof. Let X denote the set of all intersection points of edges of G through which at least k' edges of G pass. We have $|X| = \sum_{k \geq k'} t_k(G)$. Therefore,

$$k'|X| \leq I(X, E(G)) = O(|X|^{2/3}\mathrm{cr}(G)^{1/3} + |X| + e) = O\left(|X|^{2/3}\left(\sum_{k\geq 2}\binom{k}{2}t_k(G)\right)^{1/3} + |X| + e\right).$$

From here we deduce:

$$\sum_{k\geq k'} t_k(G) = |X| = O\left(\frac{\sum_{k\geq 2}\binom{k}{2}t_k(G)}{k'^3} + \frac{e}{k'}\right).$$

□

Recall that our goal is to give a lower bound for the number of crossing points in G, that is, for $\sum_k t_k(G)$. The following lemma shows that it is enough to obtain a lower bound for $\sum_k kt_k(G)$.

Lemma 3. *Let G be a (connected) simple topological graph with n vertices and $e \geq 8n$ edges. Then $t_2(G) + t_3(G) \geq \frac{1}{8}\sum_{k\geq 2} kt_k(G)$.*

Proof. Denote by $d_1, \ldots, d_n$ the degrees of the n vertices of G. Consider the planar map obtained by adding the crossing points of G as new vertices and subdividing the edges accordingly. Let V, E, and F, be the numbers of vertices, edges, and faces, respectively, of this planar map, and let f_k denote its number of faces with precisely k edges. Then: $V = n + \sum_k t_k(G)$, $\sum_i d_i + \sum_k 2kt_k(G) = 2E = \sum_k kf_k$, and $F = \sum_k f_k$.

From Euler's polyhedral formula in the plane ($V - E + F = 2$) we have:

$$6 = 3V - 3E + 3F = 3n + 3\sum_k t_k(G) - \frac{1}{2}\sum_i d_i - \sum_k kt_k(G) - \sum_k kf_k + 3\sum_k f_k.$$

Rearranging this equality, keeping in mind that every face in this planar map has at least 3 edges because G is simple (and has more than two vertices, as $e \geq 8n$), we get

$$2t_2(G) + 2t_3(G) = t_2(G) + 2t_3(G) + \sum_{k\geq 4}(k-3)t_k(G) + \sum_{k\geq 3}(k-3)f_k + e - 3n + 6 \geq \frac{1}{4}\sum_{k\geq 2} kt_k(G),$$

and the lemma is proved. □

We are now ready to prove Theorem 1.

Proof of Theorem 1: Let G be a simple topological graph with n vertices and $e \geq 4n$ edges. If $4n \leq e < 8n$, then the claimed bound follows from the next weaker lower bound. This bound is also a slight improvement, for the case of simple topological graphs, of the result in [5], according to which the number of crossing points of edges in G is at least $e/3 - n + 2$.

Lemma 4. *Let G be a simple topological graph with $n > 2$ vertices and e edges. Then $cr^*(G) \geq e - 3n + 6$.*

Proof. Let G' be the plane graph we obtain by turning every crossing point of G into a new vertex and subdividing the edges accordingly. Denote by n' and e' the number of vertices and edges of G', respectively, and let $x = n' - n$ be the number of crossing points in G. Then $e' \geq e + 2x$, since every new vertex we add subdivides at least two edges. The graph G' has no parallel edges since G is a simple topological graph, therefore $e + 2x \leq e' \leq 3n' - 6 = 3(n + x) - 6$. Hence, $x \geq e - 3n + 6$. $\square$

Therefore, we henceforth assume that $e \geq 8n$. Notice that we may also assume that G has at least n distinct crossing points of edges. Indeed, otherwise turn all crossing points of edges in G into vertices and G becomes a planar graph G' with at most $2n - 1$ vertices. The number of edges in G is at most the number of edges in G'. This in turn is at most $3(2n-1) - 6 < 8n$ contradicting our assumption that the number of edges in G is at least $8n$.

Let $P_1, P_2, P_3, \ldots$ be the crossing points of edges in G, and let g_i be the number of edges that cross each other at P_i. Assume without loss of generality that $g_1 \geq g_2 \geq g_3 \geq \ldots$. We start by preforming the following change in the graph G: Each of the n crossing points $P_1, P_2, \ldots, P_n$ becomes a vertex and subdivides the edges containing it accordingly. Denote the resulting graph by G' and note that the number n' of vertices in G' satisfies $n' = 2n$. Notice also that with each P_i we added at least $B = g_n$ new edges to the graph and therefore the number e' of edges in G' satisfies $e' \geq e + nB$. It is very important to notice that no more than B edges in G' may cross at the same point.

Observe that $\sum_{k \geq 2} \binom{k}{2} t_k(G') = \sum_{k=2}^{B} \binom{k}{2} t_k(G')$ counts the number of pairs of crossing edges of G' and therefore,

$$\sum_{k \geq 2} \binom{k}{2} t_k(G') \geq c \frac{e'^3}{n'^2}, \tag{1}$$

where c is the constant from the Crossing Lemma (observe that $e' \geq 4n'$ as $e' \geq e \geq 8n = 4n'$).

Let $c_0 > 0$ be an absolute constant to be determined later. If $B \leq c_0$, then (1) implies that

$$\binom{c_0}{2} \sum_{k \geq 2} t_k(G') = \binom{c_0}{2} \sum_{k=2}^{B} t_k(G') \geq \sum_{k=2}^{B} \binom{k}{2} t_k(G') \geq c \frac{e'^3}{n'^2}.$$

Hence,

$$\mathrm{cr}^*(G) \geq \mathrm{cr}^*(G') = \sum_{k \geq 2} t_k(G') \geq \frac{c}{\binom{c_0}{2}} \frac{e'^3}{n'^2} \geq \frac{c}{\binom{c_0}{2}} \frac{e^3}{4n^2}$$

and thus Theorem 1 is proved in this case.

We therefore assume that $B > c_0$. We now analyze the sum $\sum_{k=2}^{B} \binom{k}{2} t_k(G')$ in a different way. Recall that by Lemma 2 there is a constant c_1 such that for every $k' \geq 2$ we have

$$\sum_{k \geq k'}^{B} t_k(G') \leq c_1 \left(\frac{\sum_{k=2}^{B} \binom{k}{2} t_k(G')}{k'^3} + \frac{e'}{k'} \right).$$

Let $s = 8c_1$. We have

$$\begin{aligned}
\sum_{k=2}^{B}\binom{k}{2}t_k(G') &= \sum_{k=2}^{2^{\lfloor \log_2 s\rfloor}}\binom{k}{2}t_k(G') + \sum_{i=\lfloor \log_2 s\rfloor}^{\lceil \log_2 B\rceil}\sum_{k=2^i+1}^{2^{i+1}}\binom{k}{2}t_k(G') \\
&\le \frac{s}{2}\sum_{k\ge 2}kt_k(G') + \sum_{i=\lfloor \log_2 s\rfloor}^{\lceil \log_2 B\rceil}\binom{2^{i+1}}{2}\sum_{k\ge 2^i}t_k(G') \\
&\le \frac{s}{2}\sum_{k\ge 2}kt_k(G') + \sum_{i=\lfloor \log_2 s\rfloor}^{\lceil \log_2 B\rceil}\frac{2^{2i+2}}{2}c_1\left(\frac{\sum_{k\ge 2}\binom{k}{2}t_k(G')}{2^{3i}} + \frac{e'}{2^i}\right) \\
&\le \frac{s}{2}\sum_{k\ge 2}kt_k(G') + 4c_1\frac{\sum_{k\ge 2}\binom{k}{2}t_k(G')}{s} + 4c_1e'B \\
&\le 4c_1\sum_{k\ge 2}kt_k(G') + \frac{1}{2}\sum_{k\ge 2}\binom{k}{2}t_k(G') + 4c_1e'B. \qquad (2)
\end{aligned}$$

From (2) we conclude:

$$\sum_{k=2}^{B}\binom{k}{2}t_k(G') \le 8c_1\sum_{k\ge 2}kt_k(G') + 8c_1e'B. \qquad (3)$$

From (1), (3), and Lemma 3 we deduce:

$$t_2(G') + t_3(G') \ge \frac{1}{8}\sum_{k\ge 2}kt_k(G') \ge \frac{1}{64c_1}\sum_{k\ge 2}\binom{k}{2}t_k(G') - \frac{1}{8}e'B \ge \frac{c}{64c_1}\frac{e'^3}{n'^2} - \frac{1}{8}e'B. \qquad (4)$$

We claim that

$$\frac{1}{8}e'B \le \frac{1}{2}\cdot\frac{c}{64c_1}\frac{e'^3}{n'^2}. \qquad (5)$$

Indeed, if (5) is false, then $e' < 4n'\sqrt{c_1B/c}$. However, recall that $e' \ge Bn = Bn'/2$, and therefore we get that $B < 64c_1/c$. This leads to a contradiction once we choose $c_0 = 64c_1/c$, as we assume that $B > c_0$.

Once we established (5), then together with (4), we have

$$\mathrm{cr}^*(G) \ge t_2(G) + t_3(G) \ge t_2(G') + t_3(G') \ge \frac{c}{128c_1}\frac{e'^3}{n'^2} \ge \frac{c}{2^9c_1}\frac{e^3}{n^2},$$

where the last inequality is because $e' \ge e$ and $n' = 2n$. □

Remarks. In our proof of Theorem 1 we did not make any use of the fact that G is a *simple* topological graph except for when we argued that no face in the planar map induced by G is a digon, that is, a face with two edges. This implies that our Theorem 1 is valid also for non-simple topological graphs G, and $\mathrm{cr}^*(G) = \Omega(\frac{e^3}{n^2})$ also for those graphs, provided that there are no digons in the planar map induced by G. It follows that the construction in [5] of a topological graph with at most $e-1$ crossing points is possible only if many digons are introduced, and indeed, not very surprisingly, this is the case there.

As an easy corollary of Theorem 1 we get:

Corollary 1. *There is a constant c' such that for every integer $k > 0$ the following holds. If G is a simple topological graph with n vertices and e edges in which every edge contains at most k crossing points, then $e \leq c'n\sqrt{k}$.*

Proof. By choosing $c' > 4$ the upper bound holds trivially if $e < 4n$. Otherwise, it follows from Theorem 1 that there is a constant c^* such that $\mathrm{cr}^*(G) \geq c^* \frac{e^3}{n^2}$. On the other hand, every edge of G contains at most k crossing points, therefore $\mathrm{cr}^*(G) \leq ke$. Combining these two estimates, we conclude that $e \leq n\sqrt{k/c^*}$. □

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