A note on coloring line arrangements

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Abstract

We show that the lines of every arrangement of $n$ lines in the plane can be colored with $O(\sqrt{n/\log n})$ colors such that no face of the arrangement is monochromatic. This improves a bound of Bose et al. [1] by a $\Theta(\sqrt{\log n})$ factor. Any further improvement on this bound would also improve the best known lower bound on the following problem of Erdős: estimate the maximum number of points in general position within a set of $n$ points containing no four collinear points.

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1 Introduction

Given a simple arrangement $A$ of a set $L$ of lines in $\mathbb{R}^2$ (no parallel lines and no three lines going through the same point), decomposing the plane into the set $C$ of cells (i.e. maximal connected components of $\mathbb{R}^2 \setminus L$), Bose et al. [1] defined a hypergraph $H_{\text{line-cell}} = (L, C)$ with the vertex set $L$ (the set of lines of $A$), and each hyperedge $c \in C$ being defined by the set of lines forming the boundary of a cell of $A$. They initiated the study of the chromatic number of $H_{\text{line-cell}}$, and proved that for $|L| = n$, $\chi(H_{\text{line-cell}}) = O(\sqrt{n})$ and $\chi(H_{\text{line-cell}}) = \Omega\left(\frac{\log n}{\log \log n}\right)$. In other words, they proved that the lines of every simple arrangement of $n$ lines can be colored with $O(\sqrt{n})$ colors so that there is no monochromatic face; furthermore, they provided an intricate construction of a simple arrangement of $n$ lines that requires $\Omega\left(\frac{\log n}{\log \log n}\right)$ colors.

In this short note, we improve their upper bound by a $\Theta(\sqrt{\log n})$ factor, and extend it to not necessarily simple arrangements.

Theorem 1. The lines of every arrangement of $n$ lines in the plane can be colored with $O(\sqrt{n/\log n})$ colors so that no face of the arrangement is monochromatic.

A set of points in the plane is in general position if it does not contain three collinear points. Let $\alpha(S)$ denote the maximum number of points in general position in a set $S$ of points in the plane, and let $\alpha_4(n)$ be the minimum of $\alpha(S)$ taken over all sets $S$ of $n$ points in the plane with no four point on a line. Erdős pointed out that $\alpha_4(n) \leq n/3$ and suggested the problem of determining or estimating $\alpha_4(n)$. Füredi [3] proved that $\Omega(\sqrt{n \log n}) \leq \alpha_4(n) \leq o(n)$.

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We observe that any improvement of the bound in Theorem 1 would immediately imply a better lower bound for $\alpha_4(n)$. Indeed, suppose that $\chi(A) \leq k(n)$ for any arrangement of $n$ lines, and let $P$ be a set of $n$ points, no four on a line. Let $P^*$ be the dual arrangement of a slightly perturbed $P$ (according to the usual point-line duality, see, e.g., [2 § 8.2]). Color $P^*$ with $k(n)$ colors such that no face is monochromatic, let $S^* \subseteq P^*$ be the largest color class, and let $S$ be its dual point set. Observe that the size of $S$ is at least $n/k(n)$ and it does not contain three collinear points, since the three lines that correspond to any three collinear points in $P$ bound a face of size three in $P^*$.

2 Proof of Theorem 1

Let $\mathcal{A}$ be an arrangement of a set $L$ of $n$ lines, decomposing the plane into the set $C$ of cells, and let $H_{\text{line-cell}}$ be the corresponding hypergraph (defined as in the previous section). We show that $\chi(H_{\text{line-cell}}) = O\left(\sqrt{\frac{n}{\log n}}\right)$.

An independent set in $H_{\text{line-cell}}$ is a set $S \subseteq L$ such that for every $c \in C$, $c$ is not a subset of $S$ (in other words, no cell of $\mathcal{A}$ has its boundary formed only by lines in $S$). The proof is based on the following fact.

**Theorem 2.** There is an absolute constant $c > 0$ such that the size $\alpha(H_{\text{line-cell}})$ of the maximum independent set is at least $c\sqrt{n \log n}$.

We color the lines in $\mathcal{A}$ so that no face is monochromatic by following the same method as in [1] (where they used the weaker version of Theorem 2 stating $\alpha(H_{\text{line-cell}}) = \Omega(\sqrt{n})$). That is, we iteratively find a large independent set of lines (whose existence is guaranteed by Theorem 2), color them with the same (new) color, and remove them from $\mathcal{A}$.

Clearly, this algorithm produces a valid coloring. We verify, by induction on $n$, that at most $\frac{2}{3}\sqrt{n/\log n}$ colors are used in this coloring. We assume the bound is valid for all $n \leq 256$ (by taking sufficiently small $c > 0$). For $n > 256$, we have $\log 4 < \frac{1}{4} \log n$. Let $i$ be the smallest integer such that after $i$ iterations the number of remaining lines is at most $n/4$. Since in each of these iterations at least $c\sqrt{\frac{n/4}{\log \frac{n}{4}}} \geq c\sqrt{\frac{n}{\log n}}$ vertices (lines) are removed, $i \leq \frac{n/4}{c\sqrt{\frac{n}{\log n}}} \leq \frac{\sqrt{\frac{n}{\log n}}}{\sqrt{2}c}$. Therefore, by the induction hypothesis the number of colors that the algorithm uses is at most

$$i + \frac{2}{c} \sqrt{\frac{n}{4}} \log \frac{n}{4} \leq \frac{1}{\sqrt{2}c} \sqrt{\frac{n}{\log n}} + \frac{1}{c} \sqrt{\frac{n}{\log n} - \frac{4}{3} \log n} < \frac{1}{\sqrt{2}c} \sqrt{\frac{n}{\log n} + \frac{4}{3}} + \frac{n}{c} \sqrt{\log n} < \frac{2}{c} \sqrt{\frac{n}{\log n}}.$$

The proof of Theorem 2 is based on a result on independent sets in sparse hypergraphs. Given a hypergraph $H$ on a vertex set $V$, the sub-hypergraph $H[X]$ induced by $X \subseteq V$ consists of all edges of $H$ that are contained in $X$. A hypergraph $H = (V,E)$ is $k$-uniform if every edge $e \in E$ has size $k$. Given a $k$-uniform hypergraph $H$ and a set $S \subseteq V$ with $|S| = k - 1$, the co-degree of $S$ is the number of all vertices $v \in V$ such that $S \cup \{v\} \in E$. Kostochka et al. [4] proved that if $H$ is a $k$-uniform hypergraph, $k \geq 3$, with all co-degrees at most $d$, $d < n/(\log n)^{(k-1)^2}$, then $\alpha(H) \geq c_k \left(\frac{n}{\log \frac{n}{4}}\right)^{1/2}$, where $c_k > 0$.

In fact, a careful look at their proof reveals the following result, that we state for 3-uniform hypergraphs, since this is the case that we need.

**Lemma 2.1 ([3]).** Let $H = (V,E)$ be a 3-uniform hypergraph on $|V| = n$ vertices with all co-degrees at most $d$, $d < n/(\log n)^{12}$. Let $X$ be a random subset of $V$, obtained by choosing each vertex of $V$ independently with probability $p = \frac{n^{-2/5}}{(d \log \log \log n)^{3/5}}$. Let $Z$ be a set chosen uniformly at random among all the independent sets of $H[X]$. Then, with high probability $|Z| = \Omega(\sqrt{n \log n})$. 

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With Lemma 2.1 in hand we can now prove Theorem 2.

Proof of Theorem 2. A cell of an arrangement $A$ is called an $r$-cell, if $r$ lines of $L$ are forming its boundary. Let $H_\triangle \subset H_{\text{line-cell}}$ be the 3-uniform hypergraph with the vertex set $L$ being the set of lines, and each hyperedge defined by the triple of lines forming the boundary of a 3-cell of $A$. Since any two lines can participate in the boundaries of at most four 3-cells of $A$, all co-degrees of $H$ are at most $d = 4$. Now, as in Lemma 2.1, let $X$ be a random subset of $L$ obtained by choosing each line in $L$ independently with probability $p = \frac{\sqrt{n^2}}{(4 \log \log \log n)^{3/5}}$. Since there are $O(n^2)$ faces in $A$ and $O(n)$ of them are 2-cells (since every line can bound at most four such faces), expected number of 2-cells of $A$ in $H_{\text{line-cell}}[X]$ is $O(p^2 n) = o(\sqrt{n \log n})$, and expected number of $r$-cells, $r \geq 4$, of $A$ in $H_{\text{line-cell}}[X]$ is $O(p^4 n^2) = o(\sqrt{n \log n})$. From Lemma 2.1 it follows that there exists a set $Z \subset X \subset L$ of size $\Omega(\sqrt{n \log n})$, that is an independent set of $H_\triangle[X]$, and such that the number of $r$-cells, $r \neq 3$, of $A$ in $H_{\text{line-cell}}[Z]$ is $o(\sqrt{n \log n})$. Removing from $Z$ one vertex (line) for each such $r$-cell, we obtain an independent set of $H_{\text{line-cell}}$ of size $\Omega(\sqrt{n \log n})$.

References


