A note on coloring line arrangements

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Abstract

We show that the lines of every arrangement of n lines in the plane can be colored with $O(\sqrt{n/\log n})$ colors such that no face of the arrangement is monochromatic. This improves a bound of Bose et al. [1] by a $\Theta(\sqrt{\log n})$ factor. Any further improvement on this bound would also improve the best known lower bound on the following problem of Erdős: estimate the maximum number of points in general position within a set of n points containing no four collinear points.

Keywords: Arrangements of lines, chromatic number, sparse hypergraphs.

1 Introduction

Given a *simple* arrangement $\mathcal{A}$ of a set L of lines in $\mathbb{R}^2$ (no parallel lines and no three lines going through the same point), decomposing the plane into the set C of cells (i.e. maximal connected components of $\mathbb{R}^2 \setminus L$), Bose et al. [1] defined a hypergraph $\mathcal{H}_{\text{line-cell}} = (L, C)$ with the vertex set L (the set of lines of $\mathcal{A}$), and each hyperedge $c \in C$ being defined by the set of lines forming the boundary of a cell of $\mathcal{A}$. They initiated the study of the chromatic number of $\mathcal{H}_{\text{line-cell}}$, and proved that for $|L| = n$, $\chi(\mathcal{H}_{\text{line-cell}}) = O(\sqrt{n})$ and $\chi(\mathcal{H}_{\text{line-cell}}) = \Omega\left(\frac{\log n}{\log\log n}\right)$. In other words, they proved that the lines of every simple arrangement of n lines can be colored with $O(\sqrt{n})$ colors so that there is no monochromatic face; furthermore, they provided an intricate construction of a simple arrangement of n lines that requires $\Omega\left(\frac{\log n}{\log\log n}\right)$ colors.

In this short note, we improve their upper bound by a $\Theta(\sqrt{\log n})$ factor, and extend it to not necessarily simple arrangements.

Theorem 1. *The lines of every arrangement of n lines in the plane can be colored with $O(\sqrt{n/\log n})$ colors so that no face of the arrangement is monochromatic.*

A set of points in the plane is in *general position* if it does not contain three collinear points. Let $\alpha(S)$ denote the maximum number of points in general position in a set S of points in the plane, and let $\alpha_4(n)$ be the minimum of $\alpha(S)$ taken over all sets S of n points in the plane with no four point on a line. Erdős pointed out that $\alpha_4(n) \leq n/3$ and suggested the problem of determining or estimating $\alpha_4(n)$. Füredi [3] proved that $\Omega(\sqrt{n\log n}) \leq \alpha_4(n) \leq o(n)$.

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We observe that any improvement of the bound in Theorem 1 would immediately imply a better lower bound for $\alpha_4(n)$. Indeed, suppose that $\chi(A) \leq k(n)$ for any arrangement of n lines, and let P be a set of n points, no four on a line. Let P^* be the dual arrangement of a slightly perturbed P (according to the usual point-line duality, see, e.g., [2, § 8.2]). Color P^* with $k(n)$ colors such that no face is monochromatic, let $S^* \subseteq P^*$ be the largest color class, and let S be its dual point set. Observe that the size of S is at least $n/k(n)$ and it does not contain three collinear points, since the three lines that correspond to any three collinear points in P bound a face of size three in P^*.

2 Proof of Theorem 1

Let $\mathcal{A}$ be an arrangement of a set L of n lines, decomposing the plane into the set C of cells, and let $\mathcal{H}_{\text{line-cell}}$ be the corresponding hypergraph (defined as in the previous section). We show that $\chi(\mathcal{H}_{\text{line-cell}}) = O\left(\sqrt{\frac{n}{\log n}}\right)$.

An independent set in $\mathcal{H}_{\text{line-cell}}$ is a set $S \subset L$ such that for every $c \in C$, c is not a subset of S (in other words, no cell of $\mathcal{A}$ has its boundary formed only by lines in S). The proof is based on the following fact.

Theorem 2. *There is an absolute constant $c > 0$ such that the size $\alpha(\mathcal{H}_{\textit{line-cell}})$ of the maximum independent set is at least $c\sqrt{n \log n}$.*

We color the lines in $\mathcal{A}$ so that no face is monochromatic by following the same method as in [1] (where they used the weaker version of Theorem 2 stating $\alpha(\mathcal{H}_{\text{line-cell}}) = \Omega(\sqrt{n})$). That is, we iteratively find a large independent set of lines (whose existence is guaranteed by Theorem 2), color them with the same (new) color, and remove them from $\mathcal{A}$.

Clearly, this algorithm produces a valid coloring. We verify, by induction on n, that at most $\frac{2}{c}\sqrt{n/\log n}$ colors are used in this coloring. We assume the bound is valid for all $n \leq 256$ (by taking sufficiently small $c > 0$). For $n > 256$, we have $\log 4 < \frac{1}{4}\log n$. Let i be the smallest integer such that after i iterations the number of remaining lines is at most $n/4$. Since in each of these iterations at least $c\sqrt{\frac{n}{4}\log\frac{n}{4}} \geq c\sqrt{\frac{n}{8}\log n}$ vertices (lines) are removed, $i \leq \frac{n/4}{c\sqrt{\frac{n}{8}\log n}} \leq \frac{1}{\sqrt{2}c}\sqrt{n/\log n}$. Therefore, by the induction hypothesis the number of colors that the algorithm uses is at most

$$i+\frac{2}{c}\sqrt{\frac{\frac{n}{4}}{\log\frac{n}{4}}} \leq \frac{1}{\sqrt{2}c}\sqrt{\frac{n}{\log n}}+\frac{1}{c}\sqrt{\frac{n}{\log n - \frac{1}{4}\log n}} < \frac{1}{\sqrt{2}c}\sqrt{\frac{n}{\log n}}+\frac{\sqrt{4/3}}{c}\sqrt{\frac{n}{\log n}} < \frac{2}{c}\sqrt{\frac{n}{\log n}} \quad \square$$

The proof of Theorem 2 is based on a result on independent sets in sparse hypergraphs. Given a hypergraph $\mathcal{H}$ on a vertex set V, the sub-hypergraph $\mathcal{H}[X]$ induced by $X \subset V$ consists of all edges of $\mathcal{H}$ that are contained in X. A hypergraph $\mathcal{H} = (V, E)$ is k-uniform if every edge $e \in E$ has size k. Given a k-uniform hypergraph $\mathcal{H}$ and a set $S \subset V$ with $|S| = k-1$, the co-degree of S is the number of all vertices $v \in V$ such that $S \cup \{v\} \in E$. Kostochka et al. [4] proved that if $\mathcal{H}$ is a k-uniform hypergraph, $k \geq 3$, with all co-degrees at most d, $d < n/(\log n)^{3(k-1)^2}$, then $\alpha(\mathcal{H}) \geq c_k\left(\frac{n}{d}\log\frac{n}{d}\right)^{\frac{1}{k-1}}$, where $c_k > 0$.

In fact, a careful look at their proof reveals the following result, that we state for 3-uniform hypergraphs, since this is the case that we need.

Lemma 2.1 ([4])**.** *Let $\mathcal{H} = (V, E)$ be a 3-uniform hypergraph on $|V| = n$ vertices with all co-degrees at most d, $d < n/(\log n)^{12}$. Let X be a random subset of V, obtained by choosing each vertex of V independently with probability $p = \frac{n^{-2/5}}{(d\log\log\log\log n)^{3/5}}$. Let Z be a set chosen uniformly at random among all the independent sets of $\mathcal{H}[X]$. Then, with high probability $|Z| = \Omega(\sqrt{n\log n})$.*

With Lemma 2.1 in hand we can now prove Theorem 2.

Proof of Theorem 2: A cell of an arrangement $\mathcal{A}$ is called an r-cell, if r lines of L are forming its boundary. Let $\mathcal{H}_\triangle \subset \mathcal{H}_{\text{line-cell}}$ be the 3-uniform hypergraph with the vertex set L being the set of lines, and each hyperedge defined by the triple of lines forming the boundary of a 3-cell of $\mathcal{A}$. Since any two lines can participate in the boundaries of at most four 3-cells of $\mathcal{A}$, all co-degrees of $\mathcal{H}$ are at most $d = 4$. Now, as in Lemma 2.1, let X be a random subset of L, obtained by choosing each line in L independently with probability $p = \frac{n^{-2/5}}{(4\log\log\log n)^{3/5}}$. Since there are $O(n^2)$ faces in $\mathcal{A}$ and $O(n)$ of them are 2-cells (since every line can bound at most four such faces), expected number of 2-cells of $\mathcal{A}$ in $\mathcal{H}_{\text{line-cell}}[X]$ is $O(p^2n) = o(\sqrt{n\log n})$, and expected number of r-cells, $r \geq 4$, of $\mathcal{A}$ in $\mathcal{H}_{\text{line-cell}}[X]$ is $O(p^4n^2) = o(\sqrt{n\log n})$. From Lemma 2.1 it follows that there exists a set $Z \subset X \subset L$ of size $\Omega(\sqrt{n\log n})$, that is an independent set of $\mathcal{H}_\triangle[X]$, and such that the number of r-cells, $r \neq 3$, of $\mathcal{A}$ in $\mathcal{H}_{\text{line-cell}}[Z]$ is $o(\sqrt{n\log n})$. Removing from Z one vertex (line) for each such r-cell, we obtain an independent set of $\mathcal{H}_{\text{line-cell}}$ of size $\Omega(\sqrt{n\log n})$. $\square$

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