On the Size of a Radial Set

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Abstract. Let G be a finite set in the plane. A point $x \notin G$ is called a *radial point* of G, if every line through x and a point from G includes at least two points of G. In this paper we show that for any line l not passing through the convex hull of G there are at most $(\frac{9}{10} + o(1))|G|$ radial points separated from G by l. As a consequence we prove two nice geometric applications in the plane.

1 Introduction

Definition 11 *Let G be a finite set of points in the real affine plane. We say that a point $x \notin G$ is a* radial point *of G, if every line through x and a point of G includes at least two points of G. A set of points R is called a* radial set *of G if every $x \in R$ is a radial point of G.*

The main result in this paper is the following theorem.

Theorem 12 *Let G be a set of n points in the plane, not contained in a line. Let R be a radial set of G. If R and G can be separated by a line then $|R| < (\frac{9}{10} + o(1))n$.*

It is very significant the the constant multiplier in front of n in the formulation of Theorem 12 is strictly less than 1 (at least when n is large enough). The importance of this will soon become clear when we discuss possible applications of this theorem. In a sense, bringing the constant multiplier to be less than 1 is the principle contribution of this paper and the reason for the cumbersome proof.

A line l is said to be *determined* by a set of points, if it contains at least two points from this set. For every set of non-collinear points in the plane, an old and celebrated theorem conjectured by Sylvester ([S93]) and proved by Gallai ([G44]), guarantees the existence of a line passing through exactly two points of G. In [F96], K. Fukuda conjectured that Gallai-Sylvester theorem can be generalized in the following nice way (this conjecture appears also in [DSF98]).

Conjecture 13 (Da-Silva, Fukuda) *Let G be a set of n green points and m red points in the plane. Assume that the green points can be separated by a line from the red points. If $|m - n| \leq 1$, then G determines a line which includes exactly one green point and exactly one red point.*

It turned out that this conjecture is in fact false, at least for small values of m, n. L. Finschi and K. Fukuda found a counterexample for $n = 4, m = 5$.

A weaker form of Conjecture 13 was proved by Pach and Pinchasi in [PP00]. It is shown there that any set of n red points and n green points, which is not contained in a line, determines a bichromatic line passing through at most two red points and at most two green points.

Definition 14 *Let G be a set of red and green points in the plane. An* almost red *line is a line, determined by G, which includes exactly one green point and at least one red point. Similarly we define an* almost green *line.*

In terms of this definition, Conjecture 13 is equivalent to finding a line which is almost green and almost red at the same time.

One corollary of Theorem 12 is that if G satisfies the conditions in Conjecture 13 and $|G|$ is large enough, then G determines an almost green line and also an almost red line (but those two lines may be different).

Corollary 15 *Let R be a set of m red points and B a set of n green points in the plane. Assume that B is not contained in a line and that R and B can be separated by a line. If $n \geq n(\epsilon)$ is large enough and $m \geq (\frac{9}{10} + \epsilon)n$, where $\epsilon > 0$ is arbitrary, then $G = R \cup B$ determines an almost red line.*

Proof: If G does not determine an almost red line, then R is a radial set for B. Therefore, by Theorem 12, $m < (\frac{9}{10} + o(1))n$. ∎

Corollary 16 *Let R be a set of m red points and B a set of n green points, such that none of these two sets is contained in a line. Assume that R and B can be separated by a line and that $|m - n| \leq 1$. Then for large enough n there must exist an almost red line and also an almost green line.* ∎

As for constructions, it is easy to come up with a construction of a set G and a radial set R for G so that R and G can be separated by a line and $|R| = \frac{1}{2}|G|$. For this, take G to be the set of vertices of a regular $2n$-gon, and let R be the n points at infinity that correspond to the directions of the edges of this regular $2n$-gon.

We should also mention here a closely related result by Pach and Sharir. In [PS99], Pach and Sharir show that without any restrictions, if R is a radial set for G, then

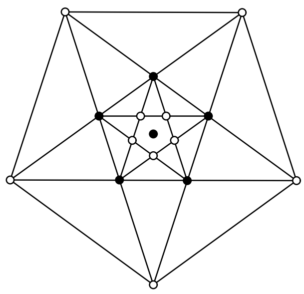

Fig. 1. a 6-point set with a 10-point radial set

$|R| = O(|G|)$. The (small) example, shown in Figure (1), in which the white points form a radial set for the black points, shows that in general $|R|$ can be larger than $|G|$, if we don't require that R and G can be separated by a line.

The proof of Theorem 12 is based on the careful analysis of the *flip array* (also called *allowable sequences*) associated with the set $R \cup G$. The method of flip arrays, invented by Goodman and Pollack, is described in Section 2. For further discussion of this method consult [GP93].

The proof of Theorem 12 has two parts. In the first part, described in Section 3, we show that $|R| \leq \frac{9}{5}n$. In the second part, described in Section 4, we further improve this bound and obtain $|R| < (\frac{9}{10} + o(1))n$. But first let us recall the notion of a flip array of a finite planar set.

2 Flip Arrays - Notations

Let G be a set of n points in the plane. We assume that the points of G have pairwise different x-coordinates, and we number them from 1 to n according to the order of their x coordinates. A *flip array* of G, usually denoted by S_G, is a sequence of permutations in S_n. Each permutation is obtained from G by projecting its points on a directed line l (which is in general position with respect to G), thus getting an ordering of the points according to the order of their projections on l. S_G are all possible permutations obtained this way from G ordered by the slope of the line l on which they were obtained. For a detailed description see Section 2 of [P03].

The first permutation is always the identity, while the last one is $(n, n-1, \ldots, 1)$.

It is important to note that we think of a permutation P as a sequence of n elements namely, $(P(1), \ldots, P(n))$. We then say that the *element* $P(i)$ is at the *place* i in the permutation P. The relative order of two elements i, j depends on whether $P^{-1}(i)$ is greater or less than $P^{-1}(j)$. If $P^{-1}(i) < P^{-1}(j)$ we say that i is *to the left* of j and that j is *to the right* of i.

A *block* in a permutation P is a sequence of consecutive elements in P. We some times refer to the block as a region (containing certain *places* in a permutation) and some times we refer to its content (the *elements* which are in that region). We say that a block B is *monotone increasing* if the elements in that block form a monotone increasing sequence from left to right. We define a *monotone decreasing* block similarly.

Notation 21 *Let $1 \leq a < b \leq n$. We denote by $[a, b]$ the block which consists of the places $a, a+1, \ldots, b$ in a general permutation (considered as a sequence of n elements).*

Every permutation in S_G is obtained from its predecessor by flipping a monotone increasing block of elements.

Assume $\sigma \in S_G$ is a permutation in the flip array S_G. For two elements $1 \leq x < y \leq n$, we say that x and y *change order* in σ, if in σ x is to the right of y (that is $\sigma^{-1}(x) > \sigma^{-1}(y)$) and in the permutations which are prior to σ in S_G, x is to the left of y.

If $P_1, P_2 \in S_G$ are two consecutive permutation so that P_2 is obtained from P_1 by a flip F, then we denote $P_F^- = P_1$ and $P_F^+ = P_2$.

We say that two elements $x, y \in \{1, 2, \ldots, n\}$ *change order* in a flip F, if x and y change order in P_F^+.

Let S_G be a flip array of a finite set G. For $P_1, P_2 \in S_G$, we say that P_1 is *previous* to P_2, if P_1 comes before P_2 in S_G. We then say that P_2 is later than P_1.

Similarly, we say that a flip F_1 is *previous* to a flip F_2 if $P_{F_1}^+$ is previous to $P_{F_2}^+$. We then say that F_2 is *later* than F_1. We say that a flip F *occurs between* a flip F_1 and a flip F_2 (where F_1 is previous to F_2), if F is later than F_1 and F_2 is later than F. In this case we sometimes say that F is between F_1 and F_2.

For $P_1, P_2 \in S_G$. We denote by $[P_1, P_2]$ the permutations in S_G which are not previous to P_1 and not later than P_2. For a flip F and $P_1, P_2 \in S_G$, we say that F is *between* P_1 and P_2 if there are two consecutive permutations $\sigma, \sigma' \in [P_1, P_2]$ so that σ' is obtained from σ by the flip F.

We need the following two simple observations.

Observation 22 *Let S_G be a flip array of a set G. Every two elements change order at some point (permutation) in the flip array S_G. From that point on (i.e., in all permutations that come afterwards in S_G) they are always in inverted order.*

Observation 23 *Let S_G be a flip array of a set G of n points in the plane. If a line M determined by G is represented by a flip of the block $[a, b]$, then there are exactly $a - 1$ points of G in one open half plane bounded by M, and $n - b$ points in the other half plane bounded by M.*

3 Part I of the Proof of Theorem 12

Let $S_{R\cup G}$ be a flip array of the set $R \cup G$. We consider the points of R to be red points, and the points of G to be green points. We can therefore consider red elements and green elements when analyzing the flip array $S_{R\cup G}$. Since R and G can be separated by a line, we may assume that in the initial state of the flip array all the red elements are to the left of the green elements. In other words, the red elements are $\{1, 2, \ldots, |R|\}$ and the green elements are $\{|R|+1, |R|+2, \ldots, |R|+|G|\}$. Therefore, if T is a flip whose block B includes red elements as well as green elements, then in P_T^- the red elements in B are to the left of the green elements in B.

Observe that the fact that R is a radial set of G translates, in terms of the flip array $S_{R\cup G}$, to that there is no flip whose block includes exactly one green element and at least one red element.

Definition 31 *A flip T is called* interesting *if the block of T includes the leftmost green element in P_T^- and at least one red element. If T is an interesting flip, then the* red size *of T, which is denoted by $rs(T)$, is the number of red elements in the block of T.*

Claim 32 *Every interesting flip T represents a line which includes an edge of* conv G.

Proof: Let l be the line which is represented by T. As T is an interesting flip, the block of T includes a green element and a red element. Hence, it includes at least two green elements, which means that l is determined by G. The green elements in the block of T are the leftmost green elements in P_T^-. This means that if we project G on a line M which is perpendicular to l, then the images, under this projection, of the points of $G \setminus l$ are on one side of the image of l on M. This shows that l supports conv G. ∎

Claim 33 *Every red element r takes part in exactly one interesting flip T. T is the first flip whose block includes r and a green element.*

Proof: Let r be any red element. Let T be the first flip whose block B includes r and some green element. Such a flip must exist since r changes order with all green elements during the flip array. In P_T^-, all the green elements are to the right of r and therefore there are no green elements to the left of the block B. It follows that B includes the leftmost green element in P_T^-. r is included in B and hence T is an interesting flip.

Assume that r takes part in an interesting flip T' which is later than T. Let b denote the leftmost green element included in B in the permutation P_T^-. Let B' denote the block of T'. Clearly, b is not included in B', for b and r change order already in T. It follows that in $P_{T'}^-$, b is to the left of the block B' (for it is to the left of r). Therefore, B' does not include the leftmost green element in $P_{T'}^-$. This is a contradiction to the assumption that T is an interesting flip. ∎

Assignment of Green Elements to Every Interesting Flip

Fix T, an interesting flip. In this section we describe how to assign to T a list of size $rs(T)$ of distinct green elements. Let $B = [a, b]$ denote the block of T. Denote by

$$r_{rs(T)}, r_{rs(T)-1}, \ldots, r_1$$

the red elements in B, from left to right, as they appear in P_T^+.

In P_T^+, B includes the leftmost green elements. B does not include all of the green elements, for otherwise G contained in a line (which is represented by the flip T). Therefore, $r_{rs(T)}$ and some green element must have their order changed at a flip which occurs after T. In particular, there must be a flip, later than T, whose block includes elements from B.

Claim 34 *Let F be the first flip later than T whose block includes $r_{rs(T)}$. If T' is an interesting flip which is later than T. Then T' is also later than F.*

Proof: Assume to the contrary that T' is not later than F. Let $b_1, \ldots, b_k$ (note that $k \geq 2$) denote the green elements in the block of T. In P_T^+, $b_1, \ldots, b_k$ are the leftmost green elements. In every $P \in [P_T^+, P_F^-]$, $r_{rs(T)}$ remains untouched and does not change order with any other element. It follows that in every $P \in [P_T^+, P_F^-]$ $b_1, \ldots, b_k$ are the leftmost green elements. In particular, this is true for $P_{T'}^-$. T' is an interesting flip and hence its block includes the two leftmost green elements in $P_{T'}^-$. Those two elements must be from $\{b_1, \ldots, b_k\}$. This is a contradiction for every two elements from $\{b_1, \ldots, b_k\}$ change order already in T. ∎

Claim 35 *The first element of B which takes part in a flip which is later than T is r_1 alone.*

Proof: We recall that $B = [a, b]$ and r_1 is at the place b (which is the right-most place in B) in P_T^+. Let T' be the first flip later than T whose block $B' = [a', b']$ includes an element from B. Clearly, B and B' share exactly one place in common, for otherwise there would be two elements which change order both in T and in T'. Therefore, either $a' = b$ and $b' > b$ in which case r_1 is included in B', or $b' = a$ and $a' < a$. Assume to the contrary that the latter case happens. In P_T^+, the element at the place a (the leftmost element in B) is the leftmost green element. This is true also in $P_{T'}^-$, because T' is the first flip later than T which includes elements from B. Therefore B' includes one green element (which is the leftmost green element) and some nonzero number of red elements, contradicting our assumption that R is a radial set. ∎

Claim 36 *The element which is next to the right of r_1 in P_T^+ is a green element.*

Proof: Assume to the contrary that the element next to the right of r_1 in P_T^+ is a red element which we denote by r. Let $b_1, \ldots, b_k$ denote the green elements in B, from left to right, as they appear in P_T^-. In P_T^-, r is next to the right of b_k. It follows that r already changed order with each one of $b_1, \ldots, b_k$. However, $b_1, \ldots, b_k$ are the leftmost green elements in P_T^- and hence those are the only green elements with which r changed order until right before T. Since R is a radial set of G, at every flip in which r changes order with some green element it changes order with at least two green elements. It follows that there are at least two green elements in B which change order already before T, which is a contradiction. ∎

Let x_0^T denote the green element next to the right of r_1 in P_T^+. In P_T^+, x_0^T is the leftmost green element which is not in the block of T. Let T_1 be the first flip later than T, whose block B_1 includes r_1. We show that in $P_{T_1}^-$ the element x, next to the right of r_1, is a green element.

Indeed, if $x = x_0^T$, then we are done. Assume $x \neq x_0^T$. In P_T^+, x is to the right of x_0^T for it is to the right of r_1 and x_0^T is the the green element next to the right of r_1 in P_T^+. In $P_{T_1}^-$, x_0^T is to the right of x, for otherwise, x_0^T is to the left of r_1 which means that r_1 and x_0^T changed order between T and T_1, contradicting the assumption that T_1 is the first flip later than T whose block includes r_1. It follows now that x and x_0^T change order before T_1 and that $x > x_0^T$. Since x_0^T is green and $x > x_0^T$, we conclude that that x is also green.

Since in $P_{T_1}^-$, the element next to the right of r_1 is green, it follows that all the elements in B_1 except for r_1 are green. Let x_1^T denote the rightmost element of B_1 in $P_{T_1}^-$. Observe that in $P_{T_1}^+$, x_1^T is at the place b and it is the leftmost green element which is not in the block of T.

We now inductively define $x_2^T, \ldots, x_{rs(T)}^T$ and $T_2, \ldots, T_{rs(T)}$. Let $k \geq 1$ and assume we already defined $x_1^T, \ldots, x_k^T$ and $T_1, \ldots, T_k$. Assume also that x_i^T is at the place $b-k+1$ in $P_{T_k}^+$ and it is the leftmost green element which is not in the block of T. Moreover, assume that in $P_{T_k}^+$, the content of the block $B' = [a, b-k]$ is the same as right after T.

The element at the place $b-k$ in $P_{T_k}^+$ is r_{k+1}. Similar to Claim 35, the first element of B' which takes part in a flip which is later than T_k is r_{k+1} alone. Let T_{k+1} be the first flip, which is later than T_k, whose block includes r_{k+1}. Let B_{k+1} denote the block of T_{k+1}. Then $B_{k+1} = [b-k+1, c]$ for some $c > b-k+1$. Again we can show (since in $P_{T_k}^+$ the element next to the right of r_{k+1} is a green element, namely x_k^T) that the rightmost element of B_{k+1} in $P_{T_{k+1}}^-$ is green. We denote that element by x_{k+1}^T. In $P_{T_{k+1}}^+$ x_{k+1}^T is at the place $b-k+1$ and it is the leftmost green element which is not in the block of T. The content of the block $[a, b-k]$ in $P_{T_{k+1}}^+$ is the same as in P_T^+. Thus, the inductive construction is completed.

Next, we define x_{-1}^T to be the leftmost green element (in B) in P_T^+. Note that by Claim 34, $T_1, \ldots, T_{rs(T)}$ occur before the first interesting flip which is later than T. Moreover, for every $0 \leq k \leq rs(T)$, x_k^T is the leftmost green element in $P^+_{T_k}$ which is not in the block of T.

Claim 37 *$x_0^T < x_1^T < x_2^T < \ldots < x_{rs(T)}^T$, and every two of them change order in one of the permutations in $[P_T^+, P^+_{T_{rs(T)}}]$.*

Proof: It is enough to show that for every $0 \leq i < rs(T)$, $x_i^T < x_{i+1}^T$ and that they change order in some $\sigma \in [P_T^+, P^+_{T_{rs(T)}}]$. Denote for convenience $T_0 = T$.

We first show that $x_i^T \neq x_{i+1}^T$. Assume to the contrary that $x_i^T = x_{i+1}^T$. Let $B_{i+1} = [c, d]$ be the block of T_{i+1}. In $P^-_{T_{i+1}}$, r_{i+1} is at the place c and $x_i^T = x_{i+1}^T$ is at the place d. $d > c+1$, for otherwise B_{i+1} includes only two elements one green and one red, contradicting our assumption that R is a radial set of G. Let x denote the green element at the place $c+1$ in $P^-_{T_{i+1}}$. In $P^+_{T_i}$, x_i^T is the leftmost green element not in B, and therefore is to the left of x. In $P^-_{T_{i+1}}$, x is to the left of x_i^T (which is then at the place d). It follows that x and x_i^T change order before T_{i+1}. This is a contradiction for their order changes in T_{i+1}. Hence $x_i^T \neq x_{i+1}^T$.

In $P^+_{T_i}$, x_i^T is the leftmost green element which is not in B. In particular, it is to the left of x_{i+1}^T. In $P^+_{T_{i+1}}$, x_{i+1}^T is the leftmost green element which is not in B. In particular it is to the left of x_i^T. This shows that $x_i^T < x_{i+1}^T$ and that x_i^T and x_{i+1}^T change order in some $\sigma \in [P^+_{T_i}, P^+_{T_{i+1}}]$. ∎

We are now ready to define the list of $rs(T)$ green elements which we assign to T. We take this list to be

$$x_{-1}^T, x_1^T, x_2^T, \ldots, x_{rs(T)-1}^T.$$

Definition 38 *Define $Green(r_j) = x_j^T$ for $1 \leq j \leq rs(T) - 1$ and $Green(r_{rs(T)}) = x_{-1}^T$.*

Remark: $Green(r_j)$ is well defined, for every red element takes part in exactly one interesting flip (Claim 33).

Counting the Number of Assigned Green Elements

Claim 39 *Let T be an interesting flip. Then x_{-1}^T is not a member of any list which is assigned to any interesting flip T' which is later than T.*

Proof: Let T' be an interesting flip, later than T. In P_T^+, x_{-1}^T is the leftmost green element. Denote $b = x_{-1}^T$. Let F be any flip, later than T, whose block B' includes b. Since R is a radial set of G, B' includes at least one more green element b'. In P_T^+, b' is to the right of b. Since b and b' change order at F, it follows that $b < b'$ and that in P_F^+, b' is to the left of b.

In particular, it follows that in $P_{T'}^+$, b is not the leftmost green element, and hence $b \neq x_{-1}^{T'}$. Assume $b = x_j^{T'}$ for some $1 \leq j \leq rs(T') - 1$. By Claim 37, $x_{j-1}^{T'} < b$ and they change order at a flip F' which is later than T'. This is a contradiction for, as we have seen in the previous paragraph, we should have $b < b'$, for every green element b' in the block of F'. ∎

Claim 310 *Let T be an interesting flip. If $rs(T) \geq 3$ then none of $x_1^T, \ldots, x_{rs(T)-2}^T$ is a member a list which is assigned to any interesting flip T', which later than T.*

Proof: Let T' be an interesting flip which occurs later than T. Let $1 \leq i \leq rs(T) - 2$ and assume that $x_i^T = x_j^{T'}$ where $j = -1$ or $1 \leq j \leq rs(T') - 1$. By Claim 37, x_{i+1}^T and x_{i+2}^T change order with x_i^T and move to its left already before T'. We consider two possible cases.

Case 1. $j \neq -1$. In some $\sigma \in S_{R \cup G}$ which is later than or equal to $P_{T'}^+$, $x_j^{T'}$ is the leftmost green element which is not in the block of T (recall the construction of $x_0^{T'}, \ldots, x_{rs(T')}^{T'}$). Hence x_{i+1}^T and x_{i+2}^T must both be in the block of T' which is impossible since they change order already before T'.

Case 2. $j = -1$. Then $x_j^{T'}$ is leftmost green element in $P_{T'}^+$. This is a contradiction for x_{i+1}^T and x_{i+2}^T are both to the left of $x_j^{T'}$ in $P_{T'}^+$. ∎

Claim 311 *Let T be an interesting flip with $rs(T) \geq 2$. $x_{rs(T)-1}^T$ may be a member of at most two more lists that are assigned to interesting flips which are later than T.*

Proof: Assume to the contrary that $x_{rs(T)-1}^T$ in included in the lists assigned to S_1, S_2, and S_3, which are interesting flips that occur in that order later than T.

Denote $x = x_{rs(T)-1}^T$ and $b = x_{rs(T)}^T$. By Claim 37, $x < b$, and they change order already before S_1. Therefore, in $P_{S_1}^-$, b is to the left of x and this is true also for every permutation which is later than $P_{S_1}^-$.

Assume by contradiction that $x = x_i^{S_1} = x_j^{S_2} = x_k^{S_3}$.

First note that none of i, j, k equals -1. This is because x cannot be the leftmost green element once b moves to its left (which happens before S_1).

Since $i \neq -1$ then $x = x_i^{S_1}$ is, in some $\sigma \in S_{R \cup G}$ which is later than S_1, the leftmost green element which is not in the block of S_1. It follows that b is included in the block of S_1. Similarly, b is included also in the blocks of S_2 and S_3. By Claim 32, S_1, S_2 and S_3 represent three lines which include three edges of conv G. This is a contradiction, for there is no point which belongs three different edges of conv G. ∎

Claim 312 *Let T be an interesting flip with $rs(T) \geq 2$. If $x^T_{rs(T)-1} = x^S_j$ for some interesting flip S which is later than T, then $j = 1$.*

Proof: In the proof of Claim 311 we saw that $j \neq -1$. Assume that $j > 1$. We first show that x^S_1 cannot be in the block B of T. For assume it is, then by Claim 37, $x^S_0 < x^S_1$ and they change order after S. Therefore, in P^-_T, x^S_0 is to the left of x^S_1. Since T is an interesting flip, B includes the leftmost green elements in P^-_T. It follows that $x^S_0 \in B$ as well. This is a contradiction for x^S_0 and x^S_1 cannot change order in T, as they change order in some permutation which is later than P^+_S.

Let $\sigma = P^+_{T_{rs(T)-1}}$ (recall the definition of $T_1, \ldots, T_{rs(T)}$). σ is previous to the permutation P^-_S and in σ $x^T_{rs(T)-1}$ is the leftmost green element which is not in the block of T. In particular, x^S_1 is to the right of $x^T_{rs(T)-1}$ in σ. By Claim 37, $x^T_{rs(T)-1} = x^S_j > x^S_1$. It follows that x^S_1 and x^S_j change order already before S, contradicting Claim 37. ∎

Definition 313 *A green element x is called* labeled *if it is included in a list assigned to some interesting flip T. A labeled element is called* A-labeled *if it is x^T_{-1} for some T, and it is called* B-labeled *otherwise.*

Claim 314 *The number of labeled elements is at least $\frac{5}{9}|R|$.*

Proof: Let T be an interesting flip. From Claims 39, 310, and 312, it follows that except for x^T_1 and $x^T_{rs(T)-1}$, every other green element which is included in the list assigned to T is not a member of any list assigned to any other interesting flip $S \neq T$. x^T_1 and $x^T_{rs(T)-1}$ may be included in at most three different lists each.

Denote by $T^1, T^2, \ldots, T^l$ the interesting flips in the flip array $S_{R\cup G}$. By Claim 33, every red element takes part in exactly one interesting flip. Therefore,

$$\sum_{i=1}^{l} rs(T^i) = |R|.$$

Define

$$l_1 = \#\{1 \leq i \leq l | rs(T^i) = 1\},$$
$$l_2 = \#\{1 \leq i \leq l | rs(T^i) = 2\}.$$

If T is an interesting flip and $rs(T) \geq 3$ then $x^T_{-1}, x^T_2, \ldots x^T_{rs(T)-2}$ appear only in the list assigned to T. Regardless of $rs(T)$, x^T_1 and $x^T_{rs(T)-1}$ may be members of at most two more lists assigned to other interesting flips. It follows that the number of labeled elements is at least

$$l_1 + 1\frac{1}{3}l_2 + \sum_{rs(T)\geq 3} (rs(T) - 2 + \frac{2}{3}),$$

which is greater than or equal to

$$\frac{1\frac{2}{3}}{3}(l_1 + 2l_2 + \sum_{rs(T)\geq 3} rs(T)) = \frac{5}{9}\sum_{i=1}^{l} rs(T^i) = \frac{5}{9}|R|.$$

∎

4 Part II of the Proof of Theorem 12

In Section 3, we saw that the set of labeled elements is of size at least $\frac{5}{9}|R|$. Roughly speaking, we are going to show that at most $\frac{1}{2} + o(1)$ of the green elements are labeled and that will clearly imply Theorem 12.

Definition 41 *If x is B-labeled and T is the first interesting flip so that $x = x_j^T$ for some $1 \leq j \leq rs(T) - 1$, then we say that x is labeled at $P_{T_j}^+$ (recall the definition of $T_1, \ldots, T_{rs(T)}$).*

If x is A-labeled, say $x = x_{-1}^T$ for some interesting flip T, then we say that x is labeled at P_T^+.

Claim 42 *If x is B-labeled, then x cannot belong to any block of an interesting flip.*

Proof: Suppose that $x = x_j^T$ ($j \neq -1$) and that x belongs to the block of an interesting flip S. Clearly $S \neq T$. We split into two cases.

Case 1. S occurs before T. By Claim 37, $x_0^T < x_j^T$ and they change order after T. It follows that in P_S^-, x_0^T is to the left of x_j^T and therefore also belongs to the block of S. This is a contradiction, for x_0^T and x_j^T cannot change order in S, as they change order after T.

Case 2. S occurs after T. By Claim 37, $x_j^T < x_{rs(T)}^T$ and they change order before S. It follows that in P_S^-, $x_{rs(T)}^T$ is to the left of x_j^T and therefore also belongs to the block of S. This is a contradiction, for $x_{rs(T)}^T$ and x_j^T cannot change order in S, as they change order before S. ∎

Claim 43 *Assume that x is A-labeled, $x = x_{-1}^T$ for an interesting flip T. Let F be any flip which occurs before T, then x is the rightmost element of the block of F in P_F^-.*

Proof: Assume not. Let b be the rightmost element of the block of F in P_F^-. Since x is a green element, so is b. In P_F^+, b moves to the left of x. This is a contradiction for $x = x_{-1}^T$ is the leftmost green element in P_T^+. ∎

Claim 44 *Suppose that x is B-labeled, say $x = x_j^T$ where $j \neq -1$ and T is an interesting flip. Let F be a flip which occurs before T and whose block B_F includes x. Then in P_F^-, x is in one of the two rightmost places in B_F. Moreover, let w be the rightmost element of B_F in P_F^-. If $w \neq x$ then w cannot be B-labeled in any permutation which is later than P_F^+.*

Proof: Assume to the contrary that x is not in one of the two rightmost places in B_F. Then the green elements y, z in the two rightmost places of B_F, move to the left of x by the flip F. Since $x = x_j^T$, then for some σ which is later than T, x is the leftmost green element which is not in the block of T. Therefore y and z must be in the block of T. This is a contradiction for y and z change order already in F.

To see the second assertion of the claim, assume that x is not the leftmost element of B_F in P_F^- and that w is B-labeled at a permutation which is later than P_F^+. w moves to the left of x by the flip F. Let σ be the permutation at which x is labeled. We know that σ is later than F. In σ, x is the leftmost green element which is not in the block of T. Therefore w must belong to the block of T, contradicting Claim 42. ∎

Claim 45 *Suppose that x is B-labeled, say $x = x_j^T$ where $j \neq -1$ and T is an interesting flip. Then there are at most two A-labeled elements that move to the left of x before it is labeled.*

Proof: Assume to the contrary that y_1, y_2, y_3 are three A-labeled elements which move to the left of x before it is labeled. Let σ be the permutation at which x is labeled. In σ, x is the leftmost green element which is not in the block of T. Therefore y_1, y_2, y_3 belong to the block of T. Without loss of generality assume that in P_T^-, y_1, y_2, y_3 are in that order from left to right in the block of T. y_2 cannot be A-labeled before T, because it moves to the left of y_1 by the flip T. y_2 cannot be A-labeled after T, because y_3 moves to the left of y_2 by the flip T. Clearly, y_2 is not A-labeled in T, for it is not the leftmost green element in P_T^+. This contradicts the assumption that y_2 is A-labeled. ∎

Fix k, a positive integer. Let $r_1, \dots r_k$ be the first k red elements that take part in an interesting flip (if $r < r'$ are two red elements which take part in the same interesting flip T, we regard r as a red element which takes part in T before r' does).

Define a graph $\mathcal{H}$ whose vertices are the green elements. Connect two green elements, x and y, by an edge if there is a flip F whose block B_F includes at least one of $r_1, \dots, r_k$ and in P_F^-, x and y are the two rightmost elements of B_F. Denote by A the set of A-labeled vertices, by B the set of B-labeled vertices, and by Z the set of vertices which are not labeled.

We recall the definition of $Green(r)$ (Definition 38) where r is a red element.

Claim 46 *Let x be a green element which is labeled at a permutation σ. Let $1 \leq i \leq k$ and suppose that x and r_i change order in a permutation which is later than σ. Then for some $j < i$, $Green(r_j) = x$.*

Proof: Let r be the red element such that $Green(r) = x$. It is enough to show that r takes part in an interesting flip before r_i does. In σ, r is to the right of x but r_i is to the left of x, because r_i and x change order in a permutation which is later than σ. We recall (Claim 33) that the first flip in which r gets flipped together with a green element is an interesting flip.

Case 1. x is A-labeled. In σ, x is the leftmost green element. It follows now that r takes part in an interesting flip before r_i does.

Case 2. x is B-labeled, and say $x = x_j^T$ where $j \neq -1$. In σ, x is the leftmost green element which is not in the block of T. Since in σ r_i is to the left of x and r is to the right of x, then r takes part in an interesting flip which is previous or equal to T and r_i takes part in an interesting flip which is later than or equal to T. If r and r_i both take part in T, then clearly $r < r_i$ (as r is to the right of r_i in σ). In either case it follows that r takes part in an interesting flip before r_i does. ∎

Corollary 47 (and a definition) *There are at most k green elements which get flipped together with some r_i after they are labeled. Denote the set of those elements by C.*

Notation 48 *Let $A' = A \setminus C$ and $B' = B \setminus C$.*

Corollary 49 *Assume F is a flip whose block includes some r_i where $1 \leq i \leq k$ and an element $x \in A' \cup B'$. Let σ be the permutation at which x is labeled. Then σ is later than P_F^+.*

By Claim 43, there are no edges between vertices of A'. By Claim 44, there are no edges between vertices of B'. By Claim 45, every vertex of B' is connected to at most two vertices of A'. It follows that there are at most $2|A' \cup B'|$ edges connecting two vertices from $A' \cup B'$.

By Claims 43 and 44, every flip whose block includes a vertex $x \in A' \cup B'$ and some r_i ($1 \leq i \leq k$), contributes 1 to the degree of that vertex and a total of at most 2 to the sum of the degrees in $\mathcal{H}$. There are at most $\binom{k}{2}$ flips the block of whom includes more than one of $\{r_1, \ldots, r_k\}$. Hence, the sum of the degrees of all vertices in $A' \cup B'$ is at least $k(|A' \cup B'|) - k^2$. It follows that the number E of edges between vertices of Z and $A' \cup B'$ is at least $k(|A' \cup B'|) - k^2 - 4|A' \cup B'| - E_C$, where E_C is the number of edges between a vertex from $A' \cup B'$ and a vertex from C. This is because there are at most $2|A' \cup B'|$ edges connecting two vertices from $A' \cup B'$.

For every two green elements which are connected by and edge in $\mathcal{H}$, we say that the edge (xy) is associated with r_i ($1 \leq i \leq k$), if there is a flip F in which x, y, r_i take part, and in P_F^-, the two right most elements of the block of F are x and y. By the definition of $\mathcal{H}$, for every edge is associated with some r_i (at least one). For every $1 \leq i \leq k$, there are at most $\frac{n}{2}$ edges in $\mathcal{H}$ associated with r_i. Indeed, at every flip F in which r_i takes part, it contributes at most one edge to $\mathcal{H}$. At every such flip r_i changes order with at least two green elements. It follows that the number of those flips is at most $\frac{n}{2}$.

We conclude that $E \le \frac{kn}{2} - E_C$. Combining the lower and upper bounds on E, we get

$$\frac{k-4}{k}|A' \cup B'| < k + \frac{n}{2}.$$

We conclude that

$$|A \cup B| = |A' \cup B'| + |C| \le |A' \cup B'| + k \le k + \frac{k}{k-4}(k + \frac{n}{2}).$$

Since this is true for every $k \le |R|$, this shows that $|A \cup B| < (\frac{1}{2} + o(1))n$. In view of Section 3, in which we proved that $|A \cup B| \ge \frac{5}{9}|R|$, this completes the proof of Theorem 12. ∎

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