On the number of distinct directions of planes determined by n points in $\mathbb{R}^3$

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Abstract

We show that any set of n points in $\mathbb{R}^3$, that is not contained in a plane, determines at least $\lfloor n/2 \rfloor + 1$ two-dimensional affine hyper-planes, no two of which are parallel.

1 Introduction

In 1970, Scott [S70] raised the following questions: What is the minimum number of different directions assumed by the connecting lines of (1) n points in the plane, not all on a line, (2) n points in 3-space, not all on a plane?

In 1982, Ungar [U82] solved the first problem, by verifying Scott's conjecture that in the plane the above minimum is equal to $2\lfloor n/2 \rfloor$, for any $n > 3$. Ungar's proof is a real gem, a brilliant application of the method of *allowable sequences* invented by Goodman and Pollack [GP81], [GP93].

Scott's conjecture regarding a set of n points in $\mathbb{R}^3$ was recently settled in [PPS07] as follows:

Theorem 1 ([PPS07]). *Every set of $n \geq 6$ points in $\mathbb{R}^3$, not all of which are on a plane, determines at least $2n - 5$ different directions if n is odd, and at least $2n - 7$ different directions if n is even. This bound is sharp for every odd $n \geq 7$.*

In this paper we consider another natural analogue of Scott's conjecture in $\mathbb{R}^2$ to three dimensions. Namely, we consider distinct directions of two-dimensional planes determined by n points in $\mathbb{R}^3$.

Let P be a finite set of points in $\mathbb{R}^3$. A (affine) plane H is said to be *determined* by P if it passes through (at least) three non-collinear points in P. Similarly, we say that a line ℓ is

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determined by P if ℓ passes through (at least) two points of P. A straight line segment e is said to be determined by P, if e is delimited by two points in P.

We consider all the planes determined by a set P of n points in $\mathbb{R}^3$ and we wish to bound from below the number of distinct directions of these planes in terms of n. Here, two planes are of the same direction if they are parallel in the affine space $\mathbb{R}^3$, or in other words if the normal vectors to these planes are the same.

Clearly, if the set P is coplanar, then it determines at most one plane and hence we will assume that the set P is not contained in a plane. By considering small examples it is natural to conjecture the following:

Conjecture 1. *Let P be a set of n points in $\mathbb{R}^3$ that is not contained in a plane. Then P determines at least n planes with pairwise distinct directions.*

If one cares only about distinct planes determined by P, regardless of their affine position, then it was shown by Hanani ([H54]) that a non-coplanar set of n points in $\mathbb{R}^3$ determines at least n distinct planes. For every $n \geq 4$ it is not hard to come up with a construction of n points that determine precisely n planes with pairwise distinct directions. One can take for example $n-2$ points on one line ℓ and two points on another line that is not coplanar with ℓ. Then the resulting set of n points determines only n planes, with pairwise distinct direction.

In Section 3, we make a first step towards giving an affirmative answer to Conjecture 1. We prove the following partial result:

Theorem 2. *Let P be a set of n points in $\mathbb{R}^3$ that is not contained in a plane. Then P determines at least $\lfloor n/2 \rfloor + 1$ planes, no two of which are parallel.*

2 Some preliminary notions and results

A generalization and strengthening of the result in [U82] about the minimum number of distinct directions of lines determined by n points in the plane, is given in [PPS03]. In order to introduce this result we need the following definition that will be important also throughout the rest of the paper.

Definition 1. Two segments in $\mathbb{R}^2$ are called *convergent* if they are opposite edges in a convex quadrilateral, or if they are collinear. See Figure 1

In other words, two non-collinear segments are not convergent if and only if a line through one of the segments meets the other segment. The notion of convergent segments in the plane was first introduced by Kupitz (see [K84]), who conjectured that a geometric graph on n vertices with no pair of convergent edges has at most $2n-2$ edges. Kupitz conjecture was proved by Katchalski, Last, and Valtr in [KL98] and [V98]. This result is in fact tight for $n \geq 4$, by a construction of Kupitz ([K84]).

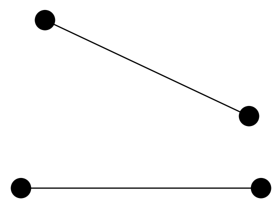

Figure 1: Two convergent segment

In [PPS03] it is shown that any set P of n non-collinear points in the plane determines at least $2\lfloor n/2 \rfloor$ segments, no two of which are convergent. This result clearly implies the result in [U82], as two parallel segments are always convergent.

The result in [PPS03] follows very easily if P is a set of points no three of which are collinear. In fact, under this condition P determines n segments no two of which are convergent. The interesting and more difficult cases are when more than two points in P may be collinear.

In this respect the result in [KL98] and [V98] is formulated for sets of points that are in general position, i.e., contain no three points on a line. However, it is an easy consequence that the same result is true for any set of points. As we will need this more general version, we give a short argument showing how it can be deduced from the result in [KL98] and [V98]:

Theorem 3 (Simple corollary of [KL98] and [V98]). *Let P be a set of n points in the plane and let E be a collection of segments determined by P. If no two segments in E are convergent, then E consists of at most $2n-2$ segments.*

Proof. Recall that two segments are not convergent exactly when a line through one segment meets the other segment. Hence, if two segments e and f are not convergent, and e' and f' are two segments containing e and f, respectively, then e' and f' are not convergent. Therefore, we can assume that all the segments in E are maximal segments determined by P. That is, each segment e in E is delimited by the two extreme points of P on the line containing e.

Consider now a small perturbation of a point $x \in P$. By a small perturbation we refer to a continuous perturbation that does not cause new collinearities among the points of P. This perturbation effects those segments delimited by x. We claim that a small enough perturbation cannot cause two segments in E to become convergent. Indeed, let e and f be two segments in E. The assertion is clear if e and f share a common endpoint. Therefore, assume that they do not. Since e and f are not convergent, assume without loss of generality that the line ℓ through e meets the segment f. If ℓ meets the relative interior of f, then no small perturbation of a point $x \in P$ can change this. It is left to consider the case where ℓ meets a point $y \in P$ delimiting the segment f. Because e is maximal, and e and f do not share a common endpoint, it follows that y is contained in the relative interior of e. But then the line through f meets that relative interior of e and this cannot be changed by a small perturbation of a point in P.

To complete the proof consider small perturbations of the points in P such that the resulting set of points is in general position. Then apply the theorem of Katchalski, Last, and Valtr to conclude that E consists of at most $2n-2$ edges. ∎

3 Proof of the main theorem

Lemma 1. *Let $P=\{p_1,\dots,p_n\}$ be a non-coplanar set of n points in $\mathbb{R}^3$, contained in the open half-space $\{z>0\}$. We assume also that the points of P are in general position with respect to the origin in the sense that there is no plane determined by P that passes through O. Let $e_1,\dots,e_m$ be a set of m straight line segments determined by P. Assume that for every $1\le i<j\le m$ either the plane through O and e_i meets e_j, or the plane through O and e_j meets e_i, or both. Then P determines at least $\frac{1}{2}m+1$ planes with pairwise distinct directions.*

Remark: Note that because $P\subset\{z<0\}$, the condition on $e_1,\dots,e_m$ is equivalent to that the central projection through O of the segments $e_1,\dots,e_m$, on a plane parallel to $\{z=0\}$, results in a collection of segments no two of which are convergent.

Proof. It will be convenient for us to assume that the set P lies not only in the half-space $\{z>0\}$ but in fact in the open quadrant $\{y>0,z>0\}$. To see that there is no loss of generality in this assumption, we show that one can find a linear transformation which takes P to be in the quadrant $\{y>0,z>0\}$. Consider the linear transformation which takes a point $(x,y,z)\in\mathbb{R}^3$ to the point $(\epsilon x,\epsilon y,z)$, for a very small, but positive, ϵ. This mapping takes P to a set of points P' contained in a narrow cone around the positive part of the z-axis. A rotation which takes the positive part of the z-axis to the ray $\{x=y=z>0\}$ will take P' to a set of points P'' that lies in the open quadrant $\{y>0,z>0\}$, as desired. Observe that applying a linear transformation to P does not effect the other conditions in Lemma 1, nor the property of two planes determined by P to have distinct directions.

We will therefore assume that $P\subset\{y>0,z>0\}$. Let K_1 be the plane $\{z=1\}$. Let T denote the central projection of $\mathbb{R}^3$ through O on the plane K_1. That is, T takes a point $p\in\mathbb{R}^3\setminus\{z=0\}$ to the intersection point of the line through O and p with the plane K_1. The image of a point $p=(a,b,c)\in\mathbb{R}^3\setminus\{z=0\}$ is the point $T(p)=(\frac{a}{c},\frac{b}{c},1)$.

For every $i=1,\dots,n$, let (a_i,b_i,c_i) denote the Cartesian coordinates of the point $p_i\in P$. T takes each point $p_i\in P$ to the point $q_i=(\frac{a_i}{c_i},\frac{b_i}{c_i},1)$ on K_1. By the conditions of the lemma, the segments $e_1,\dots,e_m$, determined by P, are projected by T to m segments on K_1, no two of which are convergent. We conclude that $q_1,\dots,q_n$ determine m segments on K_1, no two of which are convergent.

Let $\mathcal{D}'$ be the duality on the plane K_1 which takes a point $(a,b,1)$ to the line $\{ax+by=1\}$ on K_1, and which takes a line $\{ax+by=1\}$ on K_1 to the point $(a,b,1)\in K_1$. For every $1\le i\le n$, $\mathcal{D}'$ takes the point q_i to the line $\ell_i=\{a_ix+b_iy=c_i\}$ on K_1. Let $\mathcal{L}$ denote the collection of these n lines on K_1.

Observe that the y-coordinate of each point q_i is positive. This is by the definition of T

and because of the assumption that $P \subset \{y > 0, z > 0\}$. Let ℓ_i and ℓ_j be two lines in $\mathcal{L}$. Because the y-coordinates of both q_i and q_j are positive, it follows that $\mathcal{D}'$ takes the segment delimited by q_i and q_j to the set of lines passing through the intersection point of ℓ_i and ℓ_j with a slope that lies in the range between the slope of ℓ_i and the slope of ℓ_j. We call the union of this set of lines the *double-wedge* determined by ℓ_i and ℓ_j. The intersection point of ℓ_i and ℓ_j is called the *apex* of the double-wedge (see Figure 2). Because the points $q_1, \ldots, q_n$ determine m segments no two of which are convergent, then the line arrangement $\mathcal{L}$ contains m double-wedges such that among each two, there is one double wedge that contains the apex of the other double-wedge. We say that two such double-wedges are *not convergent*. See Figure 2.

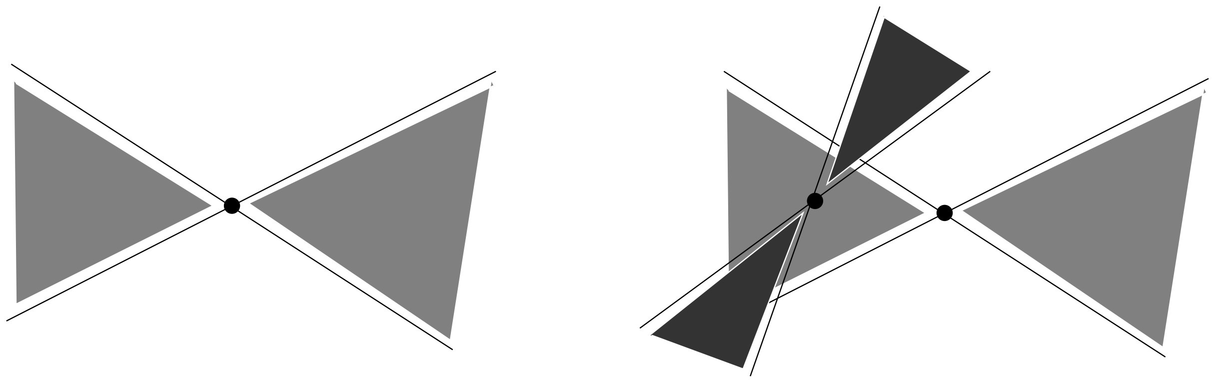

Figure 2: On the left: a double wedge. On the right: two double wedges that are not convergent.

Consider now again the original set P in $\mathbb{R}^3$. Let $\mathcal{D}$ be the duality which takes a point $(a, b, c) \in \mathbb{R}^3 \setminus \{O\}$ to the plane $\{ax+by+cz = 1\}$ and which takes a plane $\{ax+by+cz = 1\}$ to the point $(a, b, c) \in \mathbb{R}^3$. This duality respects incidence relations and sends parallel planes to points collinear with O. Let $H_1, \ldots, H_n$ denote the duals of $p_1, \ldots, p_n$, respectively, under the duality $\mathcal{D}$.

For every $i = 1, \ldots, n$, we have $p_i = (a_i, b_i, c_i)$ and therefore, $\mathcal{D}(p_i) = H_i = \{a_i x + b_i y + c_i z = 1\}$. Let K denote the plane $\{z = z_0\}$ for large enough number z_0, so that K lies above all the vertices in the hyper-planes arrangement $\mathcal{H}$ consisting of $H_1, \ldots, H_n$. The intersection of H_i with K is the line $L_i = \{a_i x + b_i y + c_i z_0 = 1\}$ on K. Let $T' : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation which takes a point (x, y, z) to the point $\frac{1}{z_0}(-x, -y, z)$. For each i, T' takes the line L_i to the line $\ell'_i = \{a_i x + b_i y - c_i = -\frac{1}{z_0}\}$ on the plane K_1. Note that for z_0 large enough, this line is arbitrarily close to the line $l_i \in \mathcal{L}$ on the plane K_1. Moreover, for any three indices i, j, k, the lines ℓ'_i, ℓ'_j, and ℓ'_k are concurrent whenever ℓ_i, ℓ_j, and ℓ_k are. This is easy to see because if ℓ_i, ℓ_j, and ℓ_k are concurrent, then $T(p_i)$, $T(p_j)$, and $T(p_k)$ are collinear. But this means that p_i, p_j, and p_k are collinear, for otherwise they determine a plane passing through O, which is impossible. The collinearity of p_i, p_j, and p_k now easily implies that ℓ'_i, ℓ'_j, and ℓ'_k are concurrent. We denote by $\mathcal{L}'$ the set of lines $\{\ell'_1, \ldots, \ell'_n\}$.

We would like to show that the set P determines "many" planes with pairwise distinct directions. This is equivalent to showing that the $\mathcal{D}$-dual arrangement $\mathcal{H}$ which consists of the planes $H_1, \ldots, H_n$ determines "many" vertices, no two of which are collinear with the origin O. Because T' is a non-singular linear transformation, it is enough to show that in the arrangement $T'(\mathcal{H})$ there are "many" vertices, no two of which are collinear with the origin.

We saw already that the arrangement $T'(\mathcal{H})$ intersects K_1 in an arrangement of lines $\ell'_1, \ldots, \ell'_n$, and that if z_0 is large enough, then each ℓ'_i is arbitrarily close to ℓ_i and any concurrency of lines from $\mathcal{L}$ corresponds to concurrency of corresponding lines from $\mathcal{L}'$. Because $\mathcal{L}$ determines m double-wedges no two of which are convergent, it follows that $\ell'_1, \ldots, \ell'_n$ as well determine m double-wedges, no two of which are convergent.

We now define a geometric graph G on the plane K_1. The vertices of G will be vertices of the arrangement $T'(\mathcal{H})$ centrally projected on K_1 through O. Each edge of G will correspond to one of the m double-wedges, determined by $\ell'_1, \ldots, \ell'_n$, no two of which are convergent. We will then show that no two edges in G are convergent. Hence, by Theorem 3, the number m of edges in G is at most $2|V(G)| - 2$, where $V(G)$ is the set of vertices of G. It will then follow that $|V(G)| \geq m/2 + 1$. This will prove the lemma because each vertex of G corresponds to a central projection of a vertex of $T'(\mathcal{H})$.

To construct G, suppose that ℓ'_i and ℓ'_j determines one of the above mentioned m double-wedges on K_1. The apex of this double-wedge is in fact the intersection of $T'(H_i)$, $T'(H_j)$, and K_1. Let A and B be the two extreme vertices of $T'(\mathcal{H})$ on the line of intersection of $T'(H_i)$ and $T'(H_j)$. We consider the line segment $[A, B]$, delimited by A and by B, and centrally project it on K_1 to obtain an edge of G. Hence, A and B are projected to vertices of G.

It is left to show that no two edges of G are convergent. Let e and f be two edges of G. Assume that e corresponds to the double-wedge W_{ij} determined by ℓ'_i and ℓ'_j, and that f corresponds to the double-wedge W_{kl} determined by ℓ'_k and ℓ'_l. Without loss of generality assume that W_{ij} contains the apex of the double-wedge W_{kl}. We will show that the line through e meets the edge f on K_1, implying that e and f are not convergent.

Recall that ℓ'_i is the line $\{a_i x + b_i y - c_i = -\frac{1}{z_0}\}$ and similarly, $\ell'_j = \{a_j x + b_j y - c_j = -\frac{1}{z_0}\}$, both on the plane K_1.

H_i is the plane $\{a_i x + b_i y + c_i z = 1\}$. As T' is the mapping defined by $T'(x, y, z) = \frac{1}{z_0}(-x, -y, z)$, $T'(H_i)$ is the plane $\{a_i x + b_i y = c_i z + \frac{1}{z_0}\}$. Similarly, $T'(H_j) = \{a_j x + b_j y = c_j z + \frac{1}{z_0}\}$. It is now easy to verify that the line of intersection of $T'(H_i)$ and $T'(H_j)$ is projected through O to the line $t_{ij} = \{(a_i - a_j)x + (b_i - b_j)y = c_i - c_j\}$ on K_1.

Since $-\frac{a_i - a_j}{b_i - b_j}$, the slope of t_{ij} on K_1, is never between $-\frac{a_i}{b_i}$ and $-\frac{a_j}{b_j}$, the slopes of ℓ'_i, and ℓ'_j on K_1, respectively, it follows that t_{ij} is not contained in W_{ij} (see Figure 3).

Let R denote the plane passing through O and t_{ij}. Let R^+ denote the closed half-space bounded by R which contains the apex of W_{kl}. R^- will denote the other closed half-space bounded by R. For convenience define $T'(H_i)^+ = T'(H_i) \cap R^+$ and $T'(H_i)^- = T'(H_i) \cap R^-$. Similarly, denote $T'(H_j)^+ = T'(H_j) \cap R^+$ and $T'(H_j)^- = T'(H_j) \cap R^-$.

The apex of W_{kl} is in the convex portion of $\mathbb{R}^3$ bounded by $T'(H_i)^+$ and $T'(H_j)^+$.

Case 1. The apex of W_{kl} is neither on $T'(H_i)+$ nor on $T'(H_j)^+$. Because the plane K_1

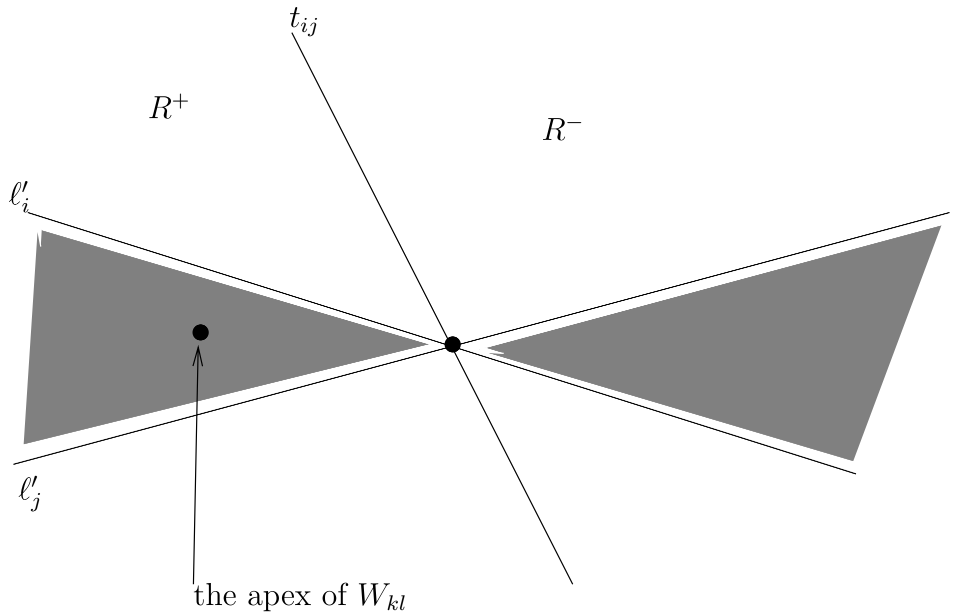

Figure 3: The projection on K_1. A view from the origin O.

lies above all vertices of the arrangement $T'(\mathcal{H})$, then all the vertices of $T'(\mathcal{H})$ on the line of intersection of $T'(H_k)$ and $T'(H_l)$ lie on the part of this line below K_1. Let us denote this ray by $\overrightarrow{r}$. Without loss of generality assume that $\overrightarrow{r}$ crosses $T'(H_i)$ before it crosses $T'(H_j)$. It follows that $\overrightarrow{r}$ intersects $T'(H_i)$ on $T'(H_i)^+$. The intersection point of $\overrightarrow{r}$ and $T'(H_i)^+$ is a vertex of $T'(\mathcal{H})$ that lies in R^+. The intersection point of $\overrightarrow{r}$ with $T'(H_j)$ must be therefore on $T'(H_j)^-$, as $\overrightarrow{r}$ cannot cross twice the plane $T'(H_i)$. The intersection point of $\overrightarrow{r}$ and $T'(H_j)^-$ is a vertex of $T'(\mathcal{H})$ that lies in R^-. It follows that the edge f of G, which corresponds to the double-wedge W_{kl} must cross the line t_{ij} that contains the edge e.

Case 2. The apex of W_{kl} is either on $T'(H_i)+$ or on $T'(H_j)^+$. Without loss of generality assume that the apex of W_{kl} is on $T'(H_i)+$. Because K_1 lies above all vertices in the arrangement $T'(\mathcal{H})$, it follows that the line of intersection of $T'(H_k)$ and $T'(H_l)$ is contained in the plane $T'(H_i)$. Therefore, the intersection point of $T'(H_k)$, $T'(H_l)$, $T'(H_i)$, and $T'(H_j)$ is a vertex in $T'(\mathcal{H})$. The projection of this vertex through O on K_1 lies on both e and f. It follows that e and f are not convergent. ∎

We are now ready to proceed to the proof of Theorem 2. As mentioned already, it is shown in [PPS03] that any non-collinear set of n points in the plane determines at least $2\lfloor n/2 \rfloor$ segments no two of which are convergent. In view of this, Theorem 2 is a corollary of Lemma 1:

Proof of Theorem 2. Translate P so that it is contained in the half-space $\{z > 0\}$ of $\mathbb{R}^3$, and so that no plane determined by P is incident to the origin O. Consider now the central projection of the points of P through O on the plane $K_1 = \{z = 1\}$. The points of P are projected to n non-collinear points on K_1. By the result in [PPS03], one can find at least $m = 2\lfloor n/2 \rfloor$ segments on K_1 determined by the images of the points of P such that no two of these segments are convergent. The corresponding segments determined by the points of P satisfy the conditions in Lemma 1. The corollary now follows. ∎

4 Suggestions and concluding remarks

Let P be a set of points in $\mathbb{R}^3$, that is not contained in a plane. Call a point O *good* with respect to P if O lies outside of the convex hull of P and it is in general position with respect to P in the sense that O does not lie on any plane determined by P.

It could be tempting to conjecture that for every set P of n points in $\mathbb{R}^3$, that is not contained in a plane, there is a good point O such that the central projection through O of the set P on a plane determines $2n-2$ segments no two of which are convergent. This, together with Lemma 1, would immediately imply Conjecture 1. However, there are sets P for which such a good point O does not exist. This is strongly related to Scott's conjecture in $\mathbb{R}^3$, mentioned in the introduction. Indeed, let P be a set of n points (n odd) in $\mathbb{R}^3$ that does not determines more than $2n-5$ lines with pairwise distinct directions. One can take, for example, the vertices of a regular $(n-3)$-gon centered at the origin on the $x-y$ plane, together with the points $(0,0,0)$, $(0,0,1)$, and $(0,0,-1)$. Any central projection of such a point set P through a good point O will determine at most $2n-5$ segments no two of which are convergent. This is because two parallel segments will always project to two convergent segments.

It will therefore be interesting to determine the maximum number $g(n)$ such that for any set P of n non-coplanar points in $\mathbb{R}^3$ there exists a good point O such that the central projection of P through O on a plane determines at least $g(n)$ segments no two of which are convergent. From [PPS03] and the example above it follows that for every odd n, $n-1 \leq g(n) \leq 2n-5$. Similarly, one can show that for even n, $n \leq g(n) \leq 2n-3$.

Lemma 1 implies immediately that any set P of n non-coplanar points in $\mathbb{R}^3$ determines at least $\lceil g(n)/2 \rceil + 1$ planes with pairwise distinct directions.

One could hope to get a good lower bound for $g(n)$ from the constructive proof of Scott's conjecture in $\mathbb{R}^3$ presented in [PPS07]. Yet, a close examination of the resulting set of segments with pairwise distinct directions, produced by the proof in [PPS07], does not directly imply such a lower bound.

Finally, looking into the proof of Lemma 1 one realizes that the crucial step there is proving that the edges of the geometric graph G are pairwise non-convergent. The following simple observation can be used to strengthen Lemma 1 and perhaps lead to a complete solution of Conjecture 1. The geometric graph G is constructed in such a way that the edges in G correspond to line segments determined by P, and the vertices of G correspond to planes determined by P, no two of which are parallel. In fact, if e is a segment determined by P and $\tilde{e}$ is the corresponding edge in G, then every plane determined by P which passes through e corresponds to some point on $\tilde{e}$. Therefore, if e and f are two segments determined by P that happen to lie on the same plane H, then the corresponding edges $\tilde{e}$ and $\tilde{f}$ in G will meet at the point which corresponds to H. In particular, $\tilde{e}$ and $\tilde{f}$ are not convergent.

Therefore, Lemma 1 can be strengthened as follows:

Lemma 2. *Let $P = \{p_1, \ldots, p_n\}$ be a non-coplanar set of n points in $\mathbb{R}^3$, contained in the open half-space $\{z > 0\}$. We assume also that the points of P are in general position with respect to the origin in the sense that there is no plane determined by P that passes through O. Let $e_1, \ldots, e_m$ be a set of m straight line segments determined by P. Assume that for every $1 \leq i < j \leq m$ either the plane through O and e_i meets e_j, or the plane through O and e_j meets e_i, or e_i and e_j are coplanar. Then P determines at least $\frac{1}{2}m + 1$ planes with pairwise distinct directions.*

The following question now naturally arises: Is it true that any set of n non-coplanar points in $\mathbb{R}^3$ determines at least $2n - 3$ segments, every two of which are either coplanar, or their central projections through some fixed good point O are not convergent?

An affirmative answer to this question will imply that Conjecture 1 is true.

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