Gallai-Sylvester Theorem for Pairwise Intersecting Unit Circles

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Abstract

For every finite family of (at least 5) pairwise intersecting unit circles in the plane, there is an intersection point that lies on exactly two circles. This proves a conjecture of A. Bezdek.

1 Introduction

The celebrated Gallai-Sylvester theorem ([S93],[G44]), in its dual form, asserts that for every finite collection of lines in the real projective plane that do not form a pencil (i.e, do not share a common point) there is a point that lies on exactly two lines. A possible analogue for circles could be: In any finite collection of (at least 2) unit circles that are pairwise intersecting (i.e, every two circles meet) there is an intersection point that belongs to exactly two circles.

Unfortunately, this statement is false. A simple counterexample consists of three distinct unit circles through a common point P, each two meeting in two points, and a fourth unit circle that passes through the remaining three intersection points (see Figure 1). A. Bezdek [B90] conjectured that this is the only counterexample to the statement above. He confirmed this under the additional condition that the distance between the centers of any two unit circles is less than $\sqrt{3}$.

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Definition 1.1. *A family $\mathcal{C}$ of unit circles in the plane is called* exceptional *if it consists of four circles C, C_1, C_2, C_3 where C_1, C_2, and C_3 pass through a point P_0, each two meeting in two points, and C passes through the three remaining intresection points of C_1, C_2, C_3 (see Figure 1).*

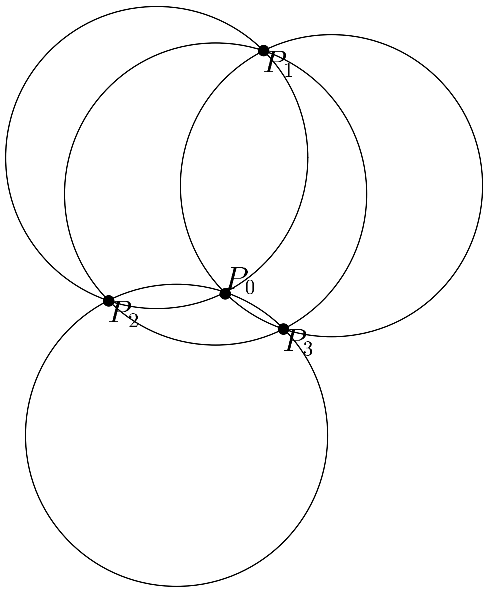

Figure 1: An exceptional configuration

The aim of this paper is to prove Bezdek's conjecture, namely, we prove the following theorem.

Theorem 1.2. *Let $\mathcal{C}$ be a finite family of (at least two) pairwise intersecting unit circles in the plane. Then there is a point through which exactly two circles pass, unless $\mathcal{C}$ is exceptional.*

2 Definitions and Notations

Throughout this paper $\mathcal{C}$ denotes a finite family of at least three (clearly, Theorem 1.2 is valid when $\mathcal{C}$ consists of just two circles) unit circles in the Euclidean plane. We assume that the diameter of the set of centers of those circles is ≤ 2, or in other words, that every two (distinct) circles in $\mathcal{C}$ either intersect: they either *cross* (i.e., have two points in common) or *touch* (i.e., have a unique point in common, and a common tangent at this point). We denote by $\mathcal{A}(\mathcal{C})$ the planar map induced by $\mathcal{C}$. Its vertices $\mathcal{V}(\mathcal{C})$ are the intersection points of circles in $\mathcal{C}$, its edges are the colsures of the components of $\bigcup \mathcal{C} \setminus \mathcal{V}(\mathcal{C})$, and its faces are the closures of the components of $\mathrm{R}^2 \setminus \bigcup \mathcal{C}$. We denote the (planar) graph of $\mathcal{A}(\mathcal{C})$ by $G_{\mathcal{C}}$. $G_{\mathcal{C}}$ may have multiple edges, but no loops (unless $\mathcal{C}$ consists of just two touching circles, a case which we excluded).

We usually denote a unit circle by C, possibly with some modifier (subscript or superscript). The closed disk bounded by that circle is denoted by D, and its center by O, with the same modifier.

Definition 2.1. *The* degree $d(P)$ *of a point* $P \in \mathcal{V}(\mathcal{C})$ *is the number of circles in* $\mathcal{C}$ *that pass through* P. *number of circles in* $\mathcal{C}$ *that pass through* P. *We call* P *a* simple *intersection point when* $d(P) = 2$.

For $k \geq 2$ denote by t_k the number of intersection points of degree k. The number V of vertices is $\sum_{k \geq 2} t_k$. Denote by f_k the number of faces of $G_{\mathcal{C}}$ which have k edges (we regard also the unbounded face). The number F of faces is $\sum_{k \geq 2} f_k$. Denote by E the number of edges of $G_{\mathcal{C}}$. We have

$$2E = \sum_{k \geq 2} k f_k \quad \text{as well as} \quad E = \sum_{k \geq 2} k t_k.$$

Therefore

$$2E = \sum_{k \geq 2} k f_k = 3F + \sum_{k \geq 2} (k-3) f_k.$$

Clearly, $G_{\mathcal{C}}$ is connected and has no loops. Hence, by Euler's formula, $V - E + F = 2$. Therefore,

$$\begin{aligned} -6 &= -3V + 3E - 3F = -3V + E + (2E - 3F) = \\ &= -3\sum_{k \geq 2} t_k + \sum_{k \geq 2} k t_k + \sum_{k \geq 2} (k-3) f_k = \\ &= \sum_{k \geq 2} (k-3) t_k + \sum_{k \geq 2} (k-3) f_k. \end{aligned}$$

Rearranging, we get

$$t_2 = 6 + \sum_{k \geq 3} (k-3) t_k + \sum_{k \geq 3} (k-3) f_k - f_2 \tag{1}$$

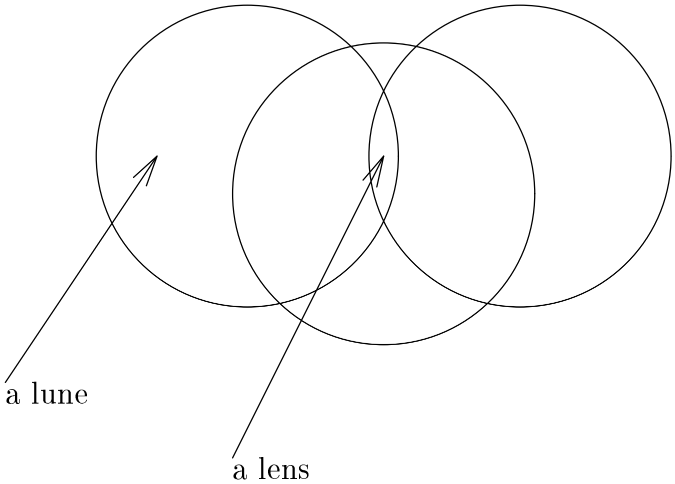

Figure 2: A lens and a lune

According to this notation Theorem 1.2 asserts that $t_2 > 0$ unless $\mathcal{C}$ is exceptional. We prove Theorem 1.2 by showing that the right hand side of 1 is positive unless $\mathcal{C}$ is exceptional.

Definition 2.2. *Let C_1, C_2 be two circles in $\mathcal{C}$. In case C_1 and C_2 cross, we call $L = D_1 \cap D_2$ a* lens *if it is a face with two edges in the planar graph $G_{\mathcal{C}}$. We then say that both C_1 and C_2* support *L (see Figure 2). The two intersection points of C_1 and C_2 are called the* vertices *of L. If C_1 and C_2 touch at P, we call P a* lens *if at most one circle from $\mathcal{C} \setminus \{C_1, C_2\}$ passes through P. Also in this case we say that C_1 and C_2 support P.*

We call $D_1 \setminus \operatorname{int} D_2$ a lune *if it is a face with two edges of the planar graph $G_{\mathcal{C}}$. We then call $C_1 \setminus \operatorname{int} D_2$ the longer arc of the lune and say that C_1 supports the* longer arc *of that lune.*

Definition 2.3. *A lens that is not a singal point is called a* proper *lens.*

Observation 2.4. *The length of the longer arc of a lune is greater than π.*

Notation 2.5. *For a circle C and points A, B on C that are not antipodal, we denote by $\operatorname{arc}_C(AB)$ the closed shorter arc on C between A and B.*

Notation 2.6. *For two distinct points A and B in the plane we denote by $\overline{AB}$ the line through A and B. We denote by $\overrightarrow{AB}$ the closed ray which starts at A and includes B. $[AB]$ denotes the closed interval between A and B.*

Notation 2.7. *Let P, Q, R be three different points in the plane, which are not collinear. We denote by $\Delta(PQR)$ the closed triangle with vertices P,Q, and R. We denote by $\angle PQR$ the closed convex region bounded by the rays $\overrightarrow{QP}$ and $\overrightarrow{QR}$. $\measuredangle PQR$ denotes the angular measure of $\angle PQR$. Therefore $0 < \measuredangle PQR < \pi$.*

3 Some Lemmata

Lemma 3.1. *Assume $C, C_1, C_2 \in \mathcal{C}$, C_1 and C_2 cross, and* $\text{int }(D \cap D_1 \cap D_2) = \emptyset$. *Then $\measuredangle O_1OO_2 < \frac{\pi}{2}$.*

Proof. Denote by P the midpoint of $[O_1O_2]$. Since C_1 and C_2 cross, $P \in \text{int } D_1 \cap \text{int } D_2$, and $\measuredangle O_1OO_2 \geq \frac{\pi}{2}$ would imply that $P \in \text{int } D$ as well. ∎

Lemma 3.2. *Let $C, C_1, C_2 \in \mathcal{C}$. If $D \cap D_1$ and $D \cap D_2$ are distinct lenses, then*

1. $(D \cap D_2) \cap \text{int } D_1 = \emptyset$,
2. $(D_1 \cap D_2) \cap \text{int } D = \emptyset$.

Proof. Denote $l_1 = D \cap D_1$ and $l_2 = D \cap D_2$.

We first show 1. Since l_1 and l_2 are distinct lenses $\text{int }(l_1 \cap l_2) = \emptyset$. Now

$$\emptyset = \text{int }(l_1 \cap l_2) = \text{int }(D_1 \cap D \cap D_2) = \text{int } D_1 \cap \text{int }(D \cap D_2).$$

If $l_2 = D \cap D_2$ is a proper lens, then it follows that also $(D \cap D_2) \cap \text{int } D_1 = \emptyset$.

Assume $D \cap D_2 = \{P\}$ is not a proper lens. If 1 is not true then $P \in \text{int } D_1$. It follows that $D \cap D_1$ is a proper lens. But one of its edges, $C \cap D_1$, contains the intersection point P in its relative interior. This means that $D \cap D_1$ is not a lens (i.e., not a face with two edges in $G_{\mathcal{C}}$.

To prove 2 we argue similarly

$$\emptyset = \text{int }(l_1 \cap l_2) = \text{int }(D \cap D_1 \cap D_2) = \text{int } D \cap \text{int }(D_1 \cap D_2).$$

If C_1 and C_2 cross then $D_1 \cap D_2$ is two dimensional and it follows that $(D_1 \cap D_2) \cap \text{int } D = \emptyset$.

If C_1 and C_2 touch at P and 2 is not true, then $P \in \text{int } D$. It follows that $l_1 = D \cap D_1$ is a proper lens. But P lies in the relative interior of the edge $D \cap C_1$ of l_1, a contradiction. ∎

Lemma 3.3. *Assume $C, C_1, C_2 \in \mathcal{C}$. If $D \cap D_1$ and $D \cap D_2$ are distinct lenses, then*

1. $0 < \measuredangle O_1OO_2 \leq \frac{\pi}{2}$. *Equality holds only if C_1 and C_2 touch at a point P, and C passes through P.*

2. $0 < \measuredangle OO_1O_2 \leq \frac{\pi}{2}$, *and* $0 < \measuredangle O_1O_2O \leq \frac{\pi}{2}$.

Proof. We first prove 1. If $\measuredangle O_1OO_2 = 0$ then O, O_1, O_2 are colinear, and without loss of generality O_1 is between O and O_2. It follows that C_1 and C cross and C_2 meets relint $(C \cap D_1)$. Therefore, $D \cap D_1$ is not a face with two edges in $\mathcal{A}(\mathcal{C})$ and we obtain a contradiction. By Lemma 3.2, $D_1 \cap D_2 \cap \text{int } D = \emptyset$. If C_1 and C_2 cross then by Lemma 3.1, $\measuredangle O_1OO_2 < \frac{\pi}{2}$.

If C_1 and C_2 touch at P and $\measuredangle O_1OO_2 > \frac{\pi}{2}$ then $P \in \text{int } D$. This contradicts Lemma 3.2. If, on the other hand, $\measuredangle O_1OO_2 = \frac{\pi}{2}$, then the length of $[OP]$ is 1, and C passes through P.

To prove 2, we restrict our attention to the subfamily $\mathcal{C}' = \{C, C_1, C_2\}$. 2 will follow from 1 if we show that within $\mathcal{C}'$, $D_1 \cap D_2$ is a lens.

If C_1 and C_2 touch, then $D_1 \cap D_2$ is a lens within $\mathcal{C}'$. Assume that C_1 and C_2 cross. If $D_1 \cap D_2$ is not a lens, then it is not a face with two edges in $\mathcal{A}(\mathcal{C}')$. It follows that C meets either relint $(C_1 \cap D_2)$ or relint $(C_2 \cap D_1)$. Without loss of generality assume that the former case happens. Then C and C_2 cross, and C_1 meets relint $(D \cap D_2)$. This is a contradiction for we assume that $D \cap D_2$ is a lens. ∎

Lemma 3.4. *Assume C, C_1, C_2 are distinct circles in $\mathcal{C}$, and $D_1 \cap D_2$ is a lens with vertices A, B (possibly $A = B$, if $D_1 \cap D_2$ is not a proper lens). Denote by Q the intersection point of the line $\overline{O_1O_2}$ with C_1 which is not in D_2. Let $\overrightarrow{r} \subset \overline{O_1O_2}$ be the ray which starts at Q and does not include O_1 (see Figure 3). Then*

1. *D does not meet $\overrightarrow{r}$.*

2. *If $\text{int } D$ is disjoint from $D_1 \cap D_2$, then the arc $C_1 \cap D$ is included either in $\text{arc}_{C_1}(QA)$ or in $\text{arc}_{C_1}(QB)$. (If $A = B$ then by $\text{arc}_{C_1}(QA)$ and $\text{arc}_{C_1}(QB)$, we mean to the two semicircles of C_1 from both sides of $[A, Q]$.)*

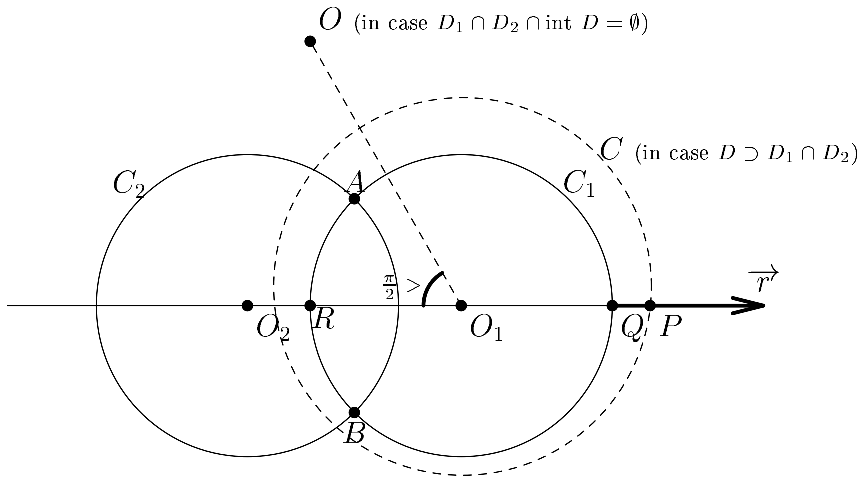

Figure 3: Lemma 3.4

Proof. 1. Assume C meets $\overrightarrow{r}$ at a point P that is either equal to Q or lies beyond Q (on $\overrightarrow{r}$). $D_1 \cap D_2$ is a lens, and therefore either $D \supset D_1 \cap D_2$ or $(D_1 \cap D_2) \cap \text{int } D = \emptyset$.

If $D \supset D_1 \cap D_2$ then in particular D include the point $R \in C_1$ that lies opposite to Q (see figure 3). It follows (since $P, R \in C \neq C_1$) that The diameter of D is then greater than 2, which is impossible.

If, on the other hand, $(D_1 \cap D_2) \cap \text{int } D = \emptyset$, then, within the family $\mathcal{C}' = \{C_1, C_2, C\}$, $D \cap D_2$ is also a lens. By Lemma 3.3, $\measuredangle O_2 O_1 O \leq \frac{\pi}{2}$ (see Figure 3). This implies that the distance between P and O is greater than 1, i.e, $P \notin C$.

2. Let a denote the arc $C_1 \cap D$ of the circle C_1. Since we assume $\text{int } D \cap (D_1 \cap D_2) = \emptyset$, $\text{arc}_{C_1}(AB) \cap \text{int } D = \emptyset$. Moreover, by 1 $Q \notin D$. It follows that a is included either in $\text{arc}_{C_1}(QA)$ or in $\text{arc}_{C_1}(QB)$. ∎

Lemma 3.5. *Assume $C_1, C_2, C_3, C_4 \in \mathcal{C}$ are four distinct circles. If the four intersections $D_1 \cap D_3, D_1 \cap D_4, D_2 \cap D_3,$ and $D_2 \cap D_4$ are lenses, then C_1 and C_2 touch at a point P, and C_3 and C_4 touch at P as well.*

Proof. We restrict our attention to the subfamily $\mathcal{C}' = \{C_1, C_2, C_3, C_4\}$. Note that if the intersection $D_i \cap D_j$ is not a lens in $\mathcal{C}'$, then it not a lens in $\mathcal{C}$ as well. Assume that all four intersections are lenses. By Lemma 3.3, all twelve angles $\angle O_i O_j O_k$ $(i, j, k \in \{1, 2, 3, 4\})$ are less or equal to $\frac{\pi}{2}$. However, this is easily seen to be possible only if O_1, O_2, O_3, O_4 are the vertices of a rectangle

R. Let P the the center of R. The four intervals $[P, O_i]$, $i = 1, 2, 3, 4$, all have the same length d. If $d < 1$, then $P \in \text{int } (D_1 \cap D_2 \cap D_3 \cap D_4)$. This is impossible since already int $(D_1 \cap D_2 \cap D_3) = \emptyset$, by Lemma 3.2. If $d > 1$ then the circles centered at opposite vertices of R are disjoint. There remains the case $d = 1$. In this case all four circles pass through P, and the two circles centered at opposite vertices of R touch at P. P is not a lens, because it lies on four circles. It follows that the vertex of R opposite to O_1 is O_2, and O_4 lies opposite to O_3. ∎

Lemma 3.6. *Let $C, C_1, C_2, C_3 \in \mathcal{C}$ be four distinct circles. Assume that $D \cap D_i$ ($i = 1, 2, 3$) are lenses. Then for some $j \in \{1, 2, 3\}$, C_j supports no lens other than $D \cap D_j$.*

Proof. By Lemma 3.3 $\measuredangle O_i O O_j \leq \frac{\pi}{2}$ for $1 \leq i < j \leq 3$. Assume, without loss of generality, that $O_2 \in \text{int } \angle O_1 O O_3$. Rotate (and if neccesary flip) the plane so that the line $\overline{OO_2}$ is vertical, with O_2 above O, O_1 is to the left of $\overline{OO_2}$ and O_3 is to the right of $\overline{OO_2}$. Denote by Q the intersection point of $\overline{OO_2}$ with C_2 which is outside D, and denote by P the intersection point of $\overline{OO_2}$ with C which is outside D_2. Denote the intersection points of C and C_2 by A, B. If $D \cap D_2$ is not a proper lens, then $A = B \in \overline{OO_2}$. Otherwise, assume that A is to left of $\overline{OO_2}$. In the sequel, when $A = B$ $\text{arc}_{C_2}(AQ)$ denotes the semicircle of C_2 which is to the left of $\overline{OO_2}$, and $\text{arc}_{C_2}(BQ)$ denotes the semicircle of C_2 which is to the right of $\overline{OO_2}$.

We show that C_2 supports no lens other than $D \cap D_2$. Assume to the contrary, that $C_0 \in \mathcal{C}$, $C_0 \neq C$ and $D_0 \cap D_2$ is a lens.

By Lemma 3.3, $\Delta(OO_0O_2)$ is a proper triangle, hence $O_0 \notin \overline{OO_2}$. Assume, without loss of generality, that O_0 is to the right of $\overline{OO_2}$. We restrict our attention to the subfamily $\mathcal{C}' = \{C, C_0, C_1, C_2\}$ and intend to show that $D_0 \cap D_1$ is a lens in $\mathcal{C}'$, contradicting Lemma 3.5. (Indeed, by Lemma 3.5, this is possible only if C and C_0 touch at a point V, and C_1, C_2 touch at V as well. This is impossible, since O_1 is to the left of $\overline{OO_2}$ and O_0 is to the right of $\overline{OO_2}$.)

We start with two applications of Lemma 3.2. Since $D \cap D_2$ and $D \cap D_2$ are lenses,

$$D \cap D_2 \cap \text{int } D_0 = \emptyset. \tag{2}$$

Since $D \cap D_1$ and $D \cap D_2$ are lenses,

$$D \cap D_2 \cap \text{int } D_1 = \emptyset. \tag{3}$$

Next come four applications of Lemma 3.4. With C_0, C_2, C standing for C, C_1, C_2, respectively, $C_2 \cap D_0 \subseteq \text{arc}_{C_2}(QA) \setminus \{Q\}$, or $C_2 \cap D_0 \subseteq \text{arc}_{C_2}(QB) \setminus \{Q\}$. Since O_0 is to the right of $\overline{OO_2}$, we conclude that

$$C_2 \cap D_0 \subseteq \text{arc}_{C_2}(QB) \setminus \{Q\}. \tag{4}$$

With C_0, C, C_2 standing for C, C_1, C_2 respectively, in Lemma 3.4 (repeating the above argument), we find that

$$C \cap D_0 \subseteq \text{arc}_C(PB) \setminus \{P\}. \tag{5}$$

With $C1, C_2, C$ standing for C, C_1, C_2, respectively, in Lemma 3.4, we obtain

$$C_2 \cap D_1 \subseteq \text{arc}_{C_2}(QA) \setminus \{Q\}. \tag{6}$$

With C_1, C, C_2 standing for C, C_1, C_2, respectively, in Lemma 3.4, we obtain

$$C \cap D_1 \subseteq \text{arc}_C(PA) \setminus \{P\}. \tag{7}$$

Now note that

$$\text{arc}_{C_2}(QA) \cap \text{arc}_{C_2}(QB) = \begin{cases} \{Q\} & \text{if } A \neq B \\ \{Q, A\} & \text{if } A = B, \end{cases} \tag{8}$$

and similarly,

$$\text{arc}_C(PA) \cap \text{arc}_C(PB) = \begin{cases} \{P\} & \text{if } A \neq B \\ \{P, A\} & \text{if } A = B. \end{cases} \tag{9}$$

If $D \cap D_2$ is a proper lens, i.e., if $A \neq B$, then we conclude from 4,6, and 8 that

$$C_2 \cap D_0 \cap D_1 = \emptyset, \tag{10}$$

and from 5,7, and 9 that

$$C \cap D_0 \cap D_1 = \emptyset. \tag{11}$$

This means that $D_0 \cap D_1$ is indeed a lens in $\mathcal{C}'$.

If $D \cap D_2$ is an improper lens, i.e., if $A = B$, then the same arguments lead only to the inclusions

$$C_2 \cap D_0 \cap D_1 \subseteq \{A\}, \tag{12}$$
$$C \cap D_0 \cap D_1 \subseteq \{A\}. \tag{13}$$

$D_2 \cap D_0$ is a lens, and $D_2 \cap D = \{A\}$ is a lens. Therefore, by Lemma 3.2,

$$A \notin \text{int } D_0. \tag{14}$$

Similarly, $D \cap D_1$ is a lens, and $D \cap D_2 = \{A\}$ is a lens. Therefore, by Lemma 3.2,

$$A \notin \text{int } D_1. \tag{15}$$

In order to show that $D \cap D_0$ is a lens, it suffices to astablish equalities 10 and 11. These, in turn, follow from the known inclusions 10 and 11, provided that we know that $A \notin D_0 \cap D_1$.

Assume, to the contrary, that

$$A \in D_0 \cap D_1. \tag{16}$$

From 14,15, and 16 we conclude that $A \in C_0 \cap C_1$. But this is impossible since $\{A\} = D \cap D_2 = C \cap C_2$ is an improper lens and therefore A can belong to at most one circle in $\mathcal{C}' \setminus \{C, C_2\}$. (See Definition 2.2.) ∎

Lemma 3.7. *If $\#(\mathcal{C}) = n$, then there are at most n lenses in $\mathcal{A}(\mathcal{C})$.*

Proof. By induction on n. The statement is clearly true for $n = 2, 3$. If every circle in $\mathcal{C}$ supports at most two lenses, then the number of lenses is at most n, since every lens is supported by exactly two circles. Assume $n \geq 4$. If a circle $C \in \mathcal{C}$ supports three or more lenses, then, by Lemma 3.6, another circle $C' \in \mathcal{C}$ supports just one lens. Define $\mathcal{C}' = \mathcal{C} \setminus \{C'\}$. By the induction hypothesis, $\mathcal{C}'$ has at most $n-1$ lenses. Adding C' to $\mathcal{C}'$, will produce just one more lens (and could possibly kill a few lenses of $\mathcal{A}(\mathcal{C}')$). ∎

Our next aim is to study the lunes in $\mathcal{A}(\mathcal{C})$, and more specifically the relationsheep between lunes and lenses in $\mathcal{A}(\mathcal{C})$.

Let us start with some simple observations. If D is a unit disk, A, B are distinct points in D, and C^* is any circle of radius ≥ 1 that passes through

A and B, then $\mathrm{arc}_{C^*}(AB) \subset D$. The same holds, of course, if we replace the disk D by an arbitrary intersection of unit disks in the plane. It follows that if C, C' are distinct circles in $\mathcal{C}$, then $D \cap D'$ is the union of all short unit circle arcs with endpoints on $C \cap D'$. Therefore if C, C', C'' are distinct circles in $\mathcal{C}$, then $D \cap D' \subset D \cap D''$ iff $C \cap D' \subset C \cap D''$. Moreover, $D' \cap D'' \subset D$ iff $C \cap C' \subset D$, i.e., iff the two arcs $C' \cap D$ and $C'' \cap D$ have two points in common.

Now consider the relative position of the two arcs $C \cap D'$ and $C \cap D''$ on C. There are essentially three possible cases:

1. $C \cap D' \subset C \cap D''$ (or $C \cap D'' \subset C \cap D'$).
2. The arcs $C \cap D'$ and $C \cap D''$ overlap, but no one includes the other.
3. The arcs $C \cap D'$ and $C \cap D''$ are disjoint, or have just one endpoint in common.

In case 1 the arcs $D \cap C'$ and $D \cap C''$ are either disjoint, or they have just one endpoint in common.

In case 2 the points of $C \cap C'$ separate the points of $C \cap C''$ on C, and the circles C' and C'' cross once inside D and once outside D. In case 3 the intersection $D' \cap D''$ maybe either entirely inside D, or entirly outside D, possibly with one vertex on C.

Thus we see that the inclusion $D' \cap D'' \subset D$ is possible only in case 3. Next we show that this inclusion is impossible when both arcs $C \cap D'$ and $C \cap D''$ lie in a semicircle of C. But first we need one more technical lemma. We denote the length of a circular arc S by $|S|$.

Lemma 3.8. *Let C, C' be distinct circles in $\mathcal{C}$, with $C \cap C' = \{A, B\}$ (possibly $A = B$, if C and C' tangent). Assume $|\mathrm{arc}_C(AB)| = |\mathrm{arc}_{C'}(AB)| = \alpha$. Denote by L, L' the rays tangent to $\mathrm{arc}_C(AB)$ and $\mathrm{arc}_{C'}(AB)$ at A. Then:*

1. *the angle between L and L' is α.*
2. *the intersection of C (or C') with the convex angular region bounded by L and L' has length 2α.*

Proof. See Figure 4. Note that the ray $\overrightarrow{AB}$ makes an angle $\frac{1}{2}\alpha$ with each of the tangents L, L'. ∎

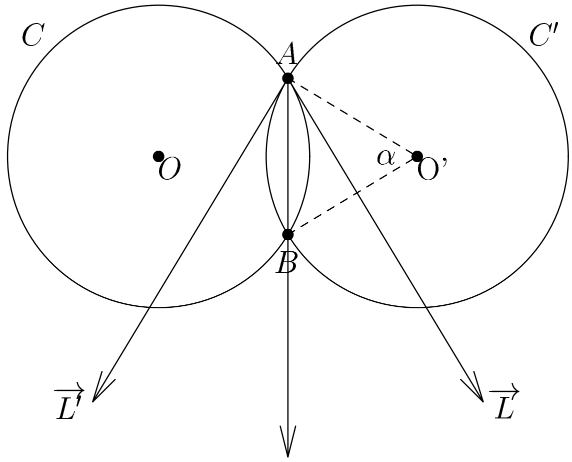

Figure 4: Lemma 3.8

Lemma 3.9. *Let C, C', C'' be distinct circles in $\mathcal{C}$. Assume that the arcs $C \cap D'$ and $C \cap D''$ are disjoint, or have just one endpoint in common, and both lie on a open semicircle $S \subset C$. Then the intersections $D \cap D'$ and $D \cap D''$ are disjoint or share just one point on C. In the latter case $D' \cap D''$ is not a point, i.e., C' and C'' are not tangent.*

Proof. Assume $C \cap C' = \{A', B'\}$ (possibly $A' = B'$) and $C \cap C'' = \{A'', B''\}$ (possibly $A'' = B''$), and that the points $A'B', A''B''$ appear in this order on S (possibly $B' = A''$), with $|C \cap D'| = \alpha \geq 0$, $|\text{arc}_C(B'A'')| = \beta \geq 0$, $|C \cap D''| = \gamma \geq 0$, and $\alpha + \beta + \gamma < \pi$.

Draw the ray L' tangent to $C' \cap D$ at B', and the ray L'' tangent to $C'' \cap D$ at A''. Since $2\alpha + \beta + 2\gamma < 2\pi$, by Lemma 3.8, these rays do not cross inside D, and they separate $D \cap D'$ from $D \cap D''$. If the intersection of $D \cap D'$ and $D \cap D''$ share one point on C then this point must be $B' = A''$. If C' and C'' tangent at B' then L and L' coincide. In this case $\beta = 0$ and $\alpha + \gamma = \pi$ condradicting the assumption that $\alpha + \beta + \gamma < \pi$. ▮

Lemma 3.10. *Suppose $C \in \mathcal{C}$ supports the longer arc of a lune. Then for any two distinct circles $C_1, C_2 \in \mathcal{C} \setminus \{C\}$. $D_1 \cap D_2 \not\subseteq D$.*

Proof. Suppose $D \setminus \text{int } D'$ is a lune. Then the arc $C \cap D'$ is included in a open semicircle S, and for any circle $C_i \in \mathcal{C} \setminus \{C\}$ we have $C \cap D_i \subset C \cap D' \subset S$. We conclude that $D_1 \cap D_2 \not\subseteq D$. This is obvious if the arcs $C \cap D_1$ and $C \cap D_2$

do have interior points in common, and follows from Lemma 3.9 if they do not. ∎

Corollary 3.11. *Suppose $C \in \mathcal{C}$ supports the longer arc of a lune. Then every lens that is included in D must be supported by C. Moreover, if C_1 and C_2 are two circles which are tangent at a point $P \in D$ then either $C = C_1$ or $C = C_2$.*

Proof. If the lens $D_1 \cap D_2$ is included in D but not supported by C (i.e, both C_1 and C_2 are different from C), then $D_1 \cap D_2 \subset D$ contradicting Lemma 3.10.

For the second part, assume that $C_1, C_2 \in \mathcal{C}$ are two distinct circles which are tangent at a point $P \in D$ and both C_1 and C_2 are distinct from D. Then $D \supset D_1 \cap D_2$ which contradicts Lemma 3.10 ∎

From this point on we assume that every intersection point of circles in $\mathcal{C}$ has degree ≥ 3.

Lemma 3.12. *Let $C_1, C_2 \in \mathcal{C}$ be two different circles, then either $D_1 \cap D_2$ includes a lens or there is a circle which is tangent to C_1 or to C_2 at a point on the boundary of $D_1 \cap D_2$.*

Proof. Clearly, if C_1 and C_2 touch, then there is nothing to prove. Assume then that C_1 and C_2 cross and $C_1 \cap C_2 = \{A, B\}$. We assume that no circle in $\mathcal{C}$ is tangent to C_1 or to C_2 at a point on the boundary of $D_1 \cap D_2$.

Let $G'_{\mathcal{C}}$ be the planar subgraph of $G_{\mathcal{C}}$ which is induced by the vertices of $G_{\mathcal{C}}$ which are inside $D_1 \cap D_2$. We regard the complement of $D_1 \cap D_2$ as the unbounded face of $G'_{\mathcal{C}}$.

For every $k \geq 2$ let u_k denote the number of vertices inside int $(D_1 \cap D_2)$ through which exactly k circles pass. Let v_k denote the number of vertices, different from A and B, on the boundary of $D_1 \cap D_2$ through which exactly k circles pass. Denote by w_k the number of faces in $G'_{\mathcal{C}}$ which have exactly k edges.

Let V, E, F denote the number of vertices, edges, and faces of $G'_{\mathcal{C}}$ respectively. Then $V = 2 + \sum_{k \geq 2} u_k + v_k$ and $F = \sum_{k \geq 2} w_k$.
Counting the number of edges in two different ways we get

$$2E = \sum_{k \geq 2} k w_k = 3F + \sum_{k \geq 2} (k-3) w_k. \tag{17}$$

Since we assume that $D_1 \cap D_2$ includes no lens, all faces of $G'_{\mathcal{C}}$ have at least 3 edges. Note that the unbounded face has exactly $2 + \sum_{k \geq 2} v_k$ edges. Therefore,

$$\sum_{k \geq 2} (k-3) w_k \quad \geq \quad \sum_{k \geq 2} v_k - 1. \tag{18}$$

Observe that the degree in $G'_{\mathcal{C}}$ of a vertex counted by u_k is $2k$ and the degree of a vertex counted by v_k is $k+1$. Let $d(A)$ and $d(B)$ denote the degrees in $G'_{\mathcal{C}}$ of A and B respectively. Counting the sum of the degrees of all vertices in two different ways we obtain

$$2E \quad = \quad d(A) + d(B) + \sum_{k \geq 2} 2k u_k + (k+1) v_k. \tag{19}$$

By Euler's formula $V - E + F = 2$, so that pluging in 17, 18, and 19 we obtain

$$\begin{aligned}
-12 \quad &= \quad -6V + 6E - 6F = -6V + 2E + 2(2E - 3F) = \\
&= \quad -6(2 + \sum_{k \geq 2} u_k + v_k) + d(A) + d(B) + \\
&+ \quad \sum_{k \geq 2} (2k u_k + (k+1) v_k) + 2 \sum_{k \geq 2} (k-3) w_k = \\
&= \quad -12 + d(A) + d(B) + \sum_{k \geq 2} ((2k-6) u_k + (k-5) v_k) + \\
&+ \quad 2 \sum_{k \geq 2} (k-3) w_k \geq \\
&\geq \quad -12 + d(A) + d(B) + \sum_{k \geq 2} ((2k-6) u_k + (k-5) v_k) + \\
&+ \quad 2(\sum_{k \geq 2} v_k - 1) = \\
&= \quad -14 + d(A) + d(B) + \sum_{k \geq 2} ((2k-6) u_k + (k-3) v_k).
\end{aligned}$$

Rearranging, we get

$$\begin{aligned}
2u_2 + v_2 \quad &\geq \quad -2 + d(A) + d(B) + \sum_{k \geq 3} ((2k-6) u_k + (k-3) v_k) \geq \\
&\geq \quad -2 + d(A) + d(B) \geq 2.
\end{aligned} \tag{20}$$

The last inequality is because $d(A), d(B) \geq 2$. Every vertex counted by u_2 is an intersection point of precisely two circles in $\mathcal{C}$. Moreover, since we assume that no circle in $\mathcal{C}$ touches C_1 or C_2 at a point on the boundary of $D_1 \cap D_2$, every vertex counted by v_2 is an intersection point of precisely two circles in $\mathcal{C}$. We assume that through every intersection point in $\mathcal{A}(\mathcal{C})$ at least 3 circles pass and hence $u_2 = v_2 = 0$. In view of 20, this is a contradiction. ∎

Corollary 3.13. *Assume that $C_1, C_2 \in \mathcal{C}$ and each of C_1, C_2 supports the longer arc of a lune, then either $D_1 \cap D_2$ is a proper lens or C_1 and C_2 touch.*

Proof. By Lemma 3.12, either $D_1 \cap D_2$ includes a lens l or it includes a tangency point P of two circles $C_3, C_4 \in \mathcal{C}$. In the first case, using Corollary 3.11, l is supported by C_1 as well as by C_2 and hence $l = D_1 \cap D_2$. In the second case since $P \in D_1$, by the second part of Corollary 3.11, C_1 is one of C_3, C_4, and similarly C_2 is one of C_3, C_4. Hence, C_1 and C_2 touch at P. ∎

Corollary 3.14. *There are at most three lunes in $\mathcal{A}(\mathcal{C})$.*

Proof. Assume $C_1, C_2, C_3, C_4 \in \mathcal{C}$ and each of them supports the longer arc of some lune. By Corollary 3.13, for every $1 \leq i < j \leq 4$ either $D_i \cap D_j$ is a proper lens or C_i and C_j touch.

We restrict our attention to the subfamily $\mathcal{C}' = \{C_1, C_2, C_3, C_4\}$. Every tangency point of two circles in $\mathcal{C}'$ must be a lens (in $\mathcal{C}'$). To see this assume, without loss of generality, that C_1 and C_2 touch at a point P. We show that no other circle from $\mathcal{C}'$ passes through P (and hence P is a lens in $\mathcal{C}'$). Assume to the contrary that C_3 passes through P. By the second part of Corollary 3.11, C_3 must coinside with C_1 or C_2.

We conclude that within the family $\mathcal{C}'$ all six intersection $D_i \cap D_j$ are lenses. This contradicts Lemma 3.7, for we obtained six lenses in a family of just four circles. ∎

Lemma 3.15. *Suppose that $D_1 \cap D_2$ is a lens, then C_1 includes an edge of the unbounded face.*

Proof. Denote by Q the intersection point of $\overline{O_1 O_2}$ with C_1 which is not in D_2. By Lemma 3.4, the ray $\overrightarrow{r} \subset \overline{O_1 O_2}$ which start at Q and does not include O_1, meets no circle of $\mathcal{C}$ except for C_1 at Q. But that means that Q belongs to the interior of some edge of the unbounded face, which is included in C_1. ∎

Corollary 3.16. *The number of edges of the unbounded face is at least the number of lenses.*

Proof. Let $C_1, \ldots, C_m$ be all the circles in $\mathcal{C}$ that support a lens. By Lemma 3.15, the unbounded face has at least m edges. By Lemma 3.7, there are at most m lenses. ∎

Denote by m the number of lenses and by l the number of lunes. The faces of $G_{\mathcal{C}}$ counted by f_2 are exactly the *proper* lenses and the lunes. Therefore $f_2 \leq m + l$.

Substituting this in 1 we get

$$t_2 = 6 + \sum_{k\geq 3}(k-3)t_k + \sum_{k\geq 3}(k-3)f_k - f_2 \geq 6 + (m-3) - (m+l) = 3 - l$$

where the inequality is because $f_2 \leq m + l$, $\sum_{k\geq 3}(k-3)t_k \geq 0$, and $\sum_{k\geq 3}(k-3)f_k \geq m - 3$ due to the unbounded face.

Since we assume $t_2 = 0$ we must have $l = 3$ (because of Corollary 3.14). Note also that all inequalities must be equalities. Hence we deduce the following equalities. $f_2 = m + l$, implying that all lenses are proper lenses. $\sum_{k\geq 3}(k-3)t_k = 0$, implying that every intersection point has degree 3. $\sum_{k\geq 3}(k-3)f_k = m - 3$, implying that the unbounded face has exactly m edges and hence (by the proof of Corollary 3.16) every circle which includes an edge of the unbounded face must support a lens. Moreover, every bounded face which is and not a lens nor a lune must be a triangle (that is, a face with just three edges). An important corollary is that no two circles in $\mathcal{C}$ touch. For if two circles in $\mathcal{C}$ touch at a point P, then $d(P) = 3$ implies that P is a lens which is not a proper lens and we obtain a contradiction.

From now on we assume that every two circles in $\mathcal{C}$ cross.

Lemma 3.17. *Let C, C_1, C_2 be three distinct unit circles. Assume that C_1 and C_2 cross, and $D \supset D_1 \cap D_2$. Denote the intersection points of C_1 and C_2 by M, N. Denote the intersection points of C and C_1 by X_1, Y_1 and those of C and C_2 by X_2, Y_2 (as indicated in Figure 5). Let $\theta = |\mathrm{arc}_C(X_1X_2)|$ and let $\alpha = |\mathrm{arc}_{C_1}(MN)|$. Then*

$$|\mathrm{arc}_{C_1}(Y_1M)| + |\mathrm{arc}_{C_2}(Y_2M)| = \pi + \alpha - \theta.$$

Proof. Rotate (and if neccesary flip) the plane so that $\overline{O_1O_2}$ is horizontal, O_1 is to the right of O_2 and X_1, X_2 are below $\overline{O_1O_2}$. We show how to deal with the case where O is on or above $\overline{O_1O_2}$. The other case can be treated

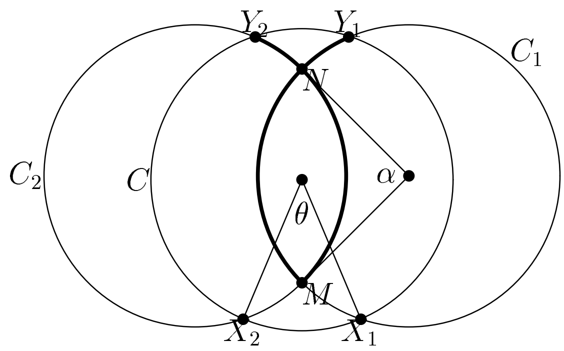

Figure 5: Lemma 3.17

similarly. Note that each of the quadilaterals $OX_1O_1Y_1$ and $OX_2O_2Y_2$ is a rhombus.

$$\begin{aligned}
|\mathrm{arc}_{C_2}(MY_2)| \quad + & \quad |\mathrm{arc}_{C_1}(MY_1)| = \alpha + \measuredangle O_1O_2Y_2 + \measuredangle O_2O_1Y_1 = \\
= & \quad \alpha + \measuredangle O_1O_2O + \measuredangle OO_2Y_2 + \measuredangle O_2O_1O + \measuredangle OO_1Y_1 = \\
= & \quad \alpha + \measuredangle O_1O_2O + \measuredangle O_2OX_2 + \measuredangle O_2O_1O + \measuredangle O_1OX_1 = \\
= & \quad \alpha + \measuredangle O_1O_2O + \measuredangle O_2O_1O + \measuredangle O_2OO_1 - \theta = \\
= & \quad \pi + \alpha - \theta.
\end{aligned}$$

▮

Lemma 3.18. *Let C, C_1, C_2 be three pairwise crossing unit circles. If $D \supset D_1 \cap D_2$, then $\measuredangle O_1OO_2 > \frac{\pi}{2}$.*

Proof. Denote by A, B the intersection points of C_1 and C_2. Let S_A, S_B be the unit disks centered at A, B respectively. Note that $O \in S_A \cap S_B$. Observe that the boundaries of S_A and S_B intersect at O_1 and O_2. $S_A \cap S_B$ is included in the disk whose diameter is $[O_1O_2]$. Hence $\measuredangle O_1OO_2 > \frac{\pi}{2}$. ▮

4 Proof of the Main Theorem

Proof of Theorem 1.2. Assume to the contrary that every intersection point $P \in \mathcal{A}(\mathcal{C})$ has $d(P) \geq 3$. As we have shown already, it follows from here that there are exactly three lunes in $\mathcal{A}(\mathcal{C})$ and every intersection point P has

$d(P) = 3$. Moreover, every circle which includes an edge of the unbounded face must support some lens, and every bounded face which is not a lens or a lune must be a triangle. We also noted that no two circles in $\mathcal{C}$ touch, and every lens in $\mathcal{C}$ is a proper lens.

Let C_1, C_2, C_3 be the three circles which support the longer arcs of the three lunes in $\mathcal{A}(\mathcal{C})$. By Corollary 3.13, for every $1 \leq i < j \leq 3$ $D_i \cap D_j$ is a proper lens. Denote $l_1 = D_2 \cap D_3, l_2 = D_1 \cap D_3, l_3 = D_1 \cap D_2$.

Claim 4.1. *Let $C \in \mathcal{C}$. Then D includes at least one of l_1, l_2, l_3.*

Proof. If C is one of C_1, C_2, C_3 then clearly D includes two of l_1, l_2, l_3. Let $C \in \mathcal{C} \setminus \{C_1, C_2, C_3\}$ and assume that D does not include any of l_1, l_2, l_3. Hence, D is disjoint from the interior of each of l_1, l_2, l_3.

$D \cap \text{int } l_2 = \emptyset$ implies that int $D \cap \text{int } D_1 \cap \text{int } D_3 = \emptyset$. It follows that $D_3 \cap \text{int } (D \cap D_1) = \emptyset$. Similarly, $D \cap \text{int } l_3 = \emptyset$ implies $D_2 \cap \text{int } (D \cap D_1) = \emptyset$. In other words, if we restrict our attention to the subfamily $\mathcal{C}' = \{C, C_1, C_2, C_3\}$ then $D \cap D_1$ is a lens.

In the same way we obtain that also $D \cap D_2$ and $D \cap D_3$ are lenses (within $\mathcal{A}(\mathcal{C}')$). This is a contradiction to Lemma 3.5. ∎

By Lemma 3.2, $D_1 \cap D_2 \cap D_3$ has an empty interior so that it is either empty or a singal point.

Case 1. $D_1 \cap D_2 \cap D_3$ is a point P. Let $C \in \mathcal{C} \setminus \{C_1, C_2, C_3\}$. By Claim 4.1, D must include at least one of l_1, l_2, l_3. Hence $P \in D$. We claim that $P \in \text{int } D$. Assume to the contrary that $P \in C$. Then D includes either one or two lenses of l_1, l_2, l_3. If D includes just one lens, say l_1, then it must be disjoint from the interiors of l_2 and l_3. This is possible only if C touches C_1 at P, which is impossible since every two circles in $\mathcal{C}$ cross. If D includes two lenses, say l_2 and l_3, and is disjoint from the interior of the third, l_1, then it is easily seen by inspection that C must coinside with C_1, contradicting our assumption on C.

Hence $P \in \text{int } D$. It follows now that D includes all lenses l_1, l_2, l_3. Denote the remaining intersection points of the circles C_1, C_2, and C_3 by P_1, P_2, P_3 (see figure 1).

Lemma 4.2. *$\Delta(P_1P_2P_3)$ overlaps $\Delta(O_1O_2O_3)$. Moreover, the circle which passes through P_1, P_2, and P_3 is a unit circle.*

Proof. We ragard the affine plane as the two dimensional vector space $\mathbf{R}^2$

with P as its origin. Then

$$\begin{aligned} P_1 &= O_2 + O_3, \\ P_2 &= O_1 + O_3, \\ P_3 &= O_1 + O_2. \end{aligned}$$

Therefore the points P_1, P_2, and P_3 are at unit distance from the point $O_1 + O_2 + O_3$, which shows that the circle through P_1,P_2, and P_3 is a unit circle.

To see that $\Delta(O_1O_2O_3)$ and $\Delta(P_1P_2P_3)$ overlap observe that

$$\Delta(P_1P_2P_3) = (O_1 + O_2 + O_3) - \Delta(O_1O_2O_3).$$

∎

Let $C \in \mathcal{C} \setminus \{C_1, C_2, C_3\}$. We know that D includes $l_1 \cup l_2 \cup l_3$ and hence $D \supset \Delta(P_1P_2P_3)$. By Lemma 3.1, the triangle $\Delta(O_1O_2O_3)$ is acute. Therefore, by Lemma 4.2, $\Delta(P_1P_2P_3)$ is also acute. Let C_0 be the circle through P_1,P_2, and P_3. C_0 is the smallest circle which includes $\Delta(P_1P_2P_3)$. By Lemma 4.2, C_0 is a unit circle. Hence C must coinside with C_0. In other words $\#(\mathcal{C}) \leq 4$. If $\mathcal{C} = \{C_0, C_1, C_2, C_3\}$ then $\mathcal{C}$ is exceptional. Otherwise, $\mathcal{C} = \{C_1, C_2, C_3\}$ and each of P_1, P_2, and P_3 has degree 2.

Case 2. $D_1 \cap D_2 \cap D_3 = \emptyset$. Denote by B_1, B_2, B_3 and A_1, A_2, A_3 the intersection points of C_1, C_2, and C_3 as indicated in figure 6.

Claim 4.3. *Assume $C \in \mathcal{C}$ and D includes l_1 but not l_2 or l_3. Then C intersects C_1 in two points on* $\mathrm{arc}_{C_1}(B_2B_3)$.

Remark. By simetry, Claim 4.3 is true if we change the role of l_1 by l_2 or l_3, and correspondently, C_1 by C_2 or C_3.

Proof. We show that O is inside $\angle O_2O_1O_3$. D is disjoint from int $l_2 =$ int $(D_1 \cap D_3)$ and from int $l_3 =$ int $(D_1 \cap D_2)$. Therefore, by Lemma 3.1, the triangles $\Delta(O_1O_2O)$ and $\Delta(O_1O_3O)$ have all their angles acute. Moreover, by Lemma 3.2, all the angles of $\Delta(O_1O_2O_3)$ are acute. If $O \notin \angle O_2O_1O_3$, then either $O \in \angle O_1O_3O_2$ or $O \in \angle O_1O_2O_3$. We assume, without loss of generality, that $O \in \angle O_1O_3O_2$. Since $O \notin \angle O_2O_1O_3$, it follows that $O_3 \in \angle O_1OO_2$. By Lemma 3.18, $\measuredangle O_2OO_3 > \frac{\pi}{2}$. We obtain a contradiction since now $\measuredangle O_2OO_3 \leq \measuredangle O_2OO_1 < \frac{\pi}{2}$.

We conclude that $\overrightarrow{O_1O}$ must be between $\overrightarrow{O_1O_2}$ and $\overrightarrow{O_1O_3}$. Since D is disjoint from int l_2 and int l_3, $D \cap C_1$ must be disjoint from the relative

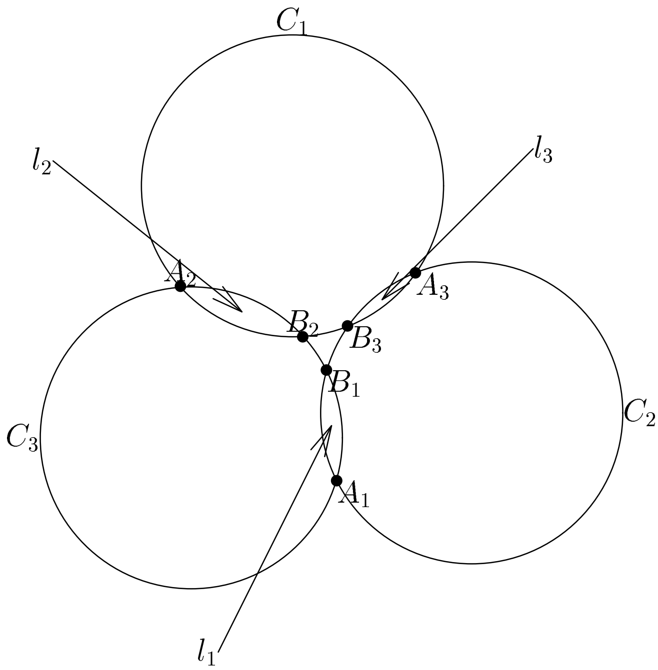

Figure 6:

interiors of both arcs $D_2 \cap C_1$ and $D_3 \cap C_1$. The center of the arc $C_1 \cap D$ is between the centers of $C_1 \cap D_2$ and $C_1 \cap D_3$. It follows now that $D \cap C_1$ must be included in $\text{arc}_{C_1}(B_2B_3)$. ∎

Claim 4.4. *Assume that $C \in \mathcal{C} \setminus \{C_1, C_2, C_3\}$. If C supports a lens then D includes exactly one of the lenses l_1, l_2, or l_3.*

Proof. By Claim 4.1, D includes at least one of l_1, l_2, or l_3. Assume, without loss of generality, that $D \supset l_1 \cup l_2$ and that $D \cap D'$ is a lens. D includes also $\text{arc}_{C_3}(B_1B_2)$. D' must include only l_3 (and not any of l_1, l_2) for otherwise $D \cap D'$ includes either l_1 or l_2 and therefore is not a lens.

By Claim 4.3, int D' meets the relative interior of $\text{arc}_{C_3}(B_1B_2)$, implying that also int $(D \cap D')$ meets it. This is a contradicting to the assumption that $D \cap D'$ is a lens. ∎

Claim 4.5. *Each of A_1, A_2, and A_3 is included in the interior of some unit disk bounded by a circle from $\mathcal{C}$.*

Proof. We prove the claim for A_1. Assume that the claim is not true for A_1. Let $C_2' \in \mathcal{C}$ be the circle so that $D_2 \setminus \text{int } D_2'$ is a lune. Since $A_1 \in D_2'$,

$A_1 \notin \text{int } D'_2$ implies that C'_2 passes through A_1. Similarly, let $C'_3 \in \mathcal{C}$ be the circle so that $D_3 \setminus \text{int } D'_3$ is a lune, then C'_3 passes through A_1. However, $d(A_1) = 3$ and C_2, C_3 pass through A_1. It follows that $C'_2 = C'_3$. Denote $C' = C'_2 = C'_3$. D' includes all lenses l_1, l_2, l_3. Denote the intersection points of C' with C_2 and C_3 other then A_1 by Y, Z respectively (see figure 7).

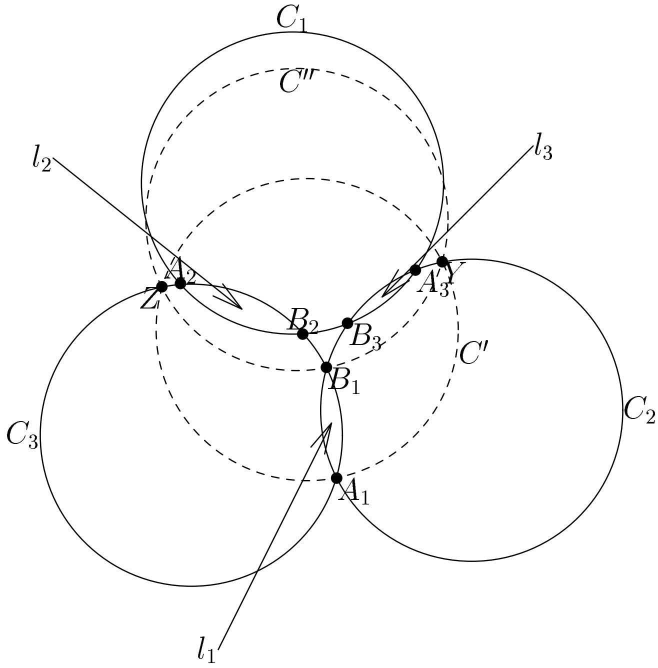

Figure 7:

The face whose two edges are $\text{arc}_{C_3}(A_1B_1)$and $\text{arc}_{C'}(A_1Y)$ must have exactly three edges so there must be a circle (other than C_2) in $\mathcal{C}$ which passes through Y and B_1. Similarly, there must be a circle (other than C_3) in $\mathcal{C}$ which passes through Z and B_1. Since $d(B_1) = 3$ and C_2 and C_3 pass through B_1, it follows that the same circle $C'' \in \mathcal{C}$ passes through Y,B_1, and Z.

Observe that $Y \neq A_3$ for otherwise there would be four circles through this point, namely C_1, C_2, C', and C''. It is easily seen now that the face whose edge is $\text{arc}_{C''}(YB_1)$ which is adjacent to the face A_1YB_1 has more than three edges (as no other circles pass through Y nor B_1), a contradiction. ∎

Let Δ_0 denote the closed rigion which is bounded by the arcs $\text{arc}_{C_1}(B_2B_3)$, $\text{arc}_{C_2}(B_3B_1)$, and $\text{arc}_{C_3}(B_1B_2)$.

Lemma 4.6. *Let $C \in \mathcal{C}$. Then $D \cap \Delta_0 \cap \overline{B_1A_1} \neq \emptyset$. (Similarly for $\overline{B_2A_2}$ and $\overline{B_3A_3}$.)*

Proof. If $D \supset l_1$ then clearly $B_1 \in D \cap \Delta_0 \cap \overline{B_1A_1}$. If $D \supset (l_2 \cup l_3)$ then D includes $\mathrm{arc}_{C_1}(B_2B_3)$. Therefore, $D \cap \Delta_0 \cap \overline{B_1A_1}$ includes the intersection point of $\mathrm{arc}_{C_1}(B_2B_3)$ with $\overline{B_1A_1}$ (indeed, this intersection point exists because $\overline{B_1A_1}$ seperates l_2 and l_3).

The only case which is left to check is where D includes only one of l_2 or l_3, and not l_1. Assume, without loss of generality, that D includes l_3 and not l_1 or l_2. By Claim 4.3, D includes a subarc of $\mathrm{arc}_{C_3}(B_1B_2)$. Since D is convex, D includes an interval connecting B_3 to a point on $\mathrm{arc}_{C_3}(B_1B_2)$. This interval intersects $\overline{B_1A_1}$ inside Δ_0. ∎

Claim 4.7. *For each of the lenses l_1, l_2, and l_3 there is a circle $C \in \mathcal{C}$ such that D includes that lens and not the two others.*

Proof. We prove the claim for l_1. Let Q be the intersection point of a circle from $\mathcal{C}$ and the ray $\overrightarrow{B_1A_1}$, which has maximum distance from B_1. One of the circles C through Q includes an edge of the unbounded face. By Claim 4.5, A_1 is included in the interior of the disk bounded by some circle from $\mathcal{C}$ and hence $Q \neq A_1$. Consequently, C is not one of C_1, C_2, C_3. We show that D includes l_1 and not any of l_2 or l_3.

By Lemma 4.6, $D \cap \Delta_0 \cap \overline{B_1A_1} \neq \emptyset$. It follows, from the convexity of D, that $D \supset [B_1A_1]$. Hence $D \supset l_1$.

Observe that if D includes l_2 or l_3, then, by Claim 4.4, C does not support a lens. This is a contradiction because C includes an edge of the unbounded face. ∎

For $i = 1, 2, 3$ let $C'_i \in \mathcal{C}$ be a circle which is garanteed by Claim 4.7, for l_i. Thus, D'_i includes l_i alone from l_1, l_2, and l_3.

Denote by $\alpha_1, \alpha_2, \alpha_3$ the lengthes of the arcs $C_1 \cap D_2, C_2 \cap D_3$, and $C_3 \cap D_1$ respectively. Denote by $\theta_1, \theta_2, \theta_3$ the lengthes of $\mathrm{arc}_{C_1}(B_2B_3), \mathrm{arc}_{C_2}(B_3B_1)$ and $\mathrm{arc}_{C_3}(B_1B_2)$ respectively.

Claim 4.8. *For every $1 \leq i \leq 3$, $\theta_i < \frac{\pi}{3}$.*

Proof. Reduce to the extrem case where C_1, C_2, and C_3 touch each other. ∎

Claim 4.9. *C'_1 meets C_2 at a point $F_{12} \in \mathrm{arc}_{C_2}(B_3B_1)$. C'_1 meets C_3 at a point $F_{13} \in \mathrm{arc}_{C_3}(B_1B_2)$. Moreover,*

$$|\mathrm{arc}_{C'_1}(F_{12}F_{13})| \leq \max(\theta_1, \theta_2, \theta_3).$$

Remark. By simetry Claim 4.9 is true if we replace C_1', C_2, C_3 respectively by C_2', C_1, C_3, or by C_3', C_1, C_2. In this way we analogously define the points $F_{21} \in \mathrm{arc}_{C_1}(B_2B_3)$ and $F_{23} \in \mathrm{arc}_{C_3}(B_1B_2)$ on C_2' as well as $F_{31} \in \mathrm{arc}_{C_1}(B_2B_3)$ and $F_{32} \in \mathrm{arc}_{C_2}(B_3B_1)$ on C_3'.

Proof. D_1' includes l_1 and is disjoint from the interiors of both l_2 and l_3. Therefore C_1' seperates the interior of l_1 from the interior of l_3 and thus meets $\mathrm{arc}_{C_2}(B_3B_1)$ at a point which we denote by F_{12}. Similarly, C_1' seperates the interior of l_1 from the interior of l_2 and thus meets $\mathrm{arc}_{C_3}(B_1B_2)$ at a point which we denote by F_{13} (see Figure 8).

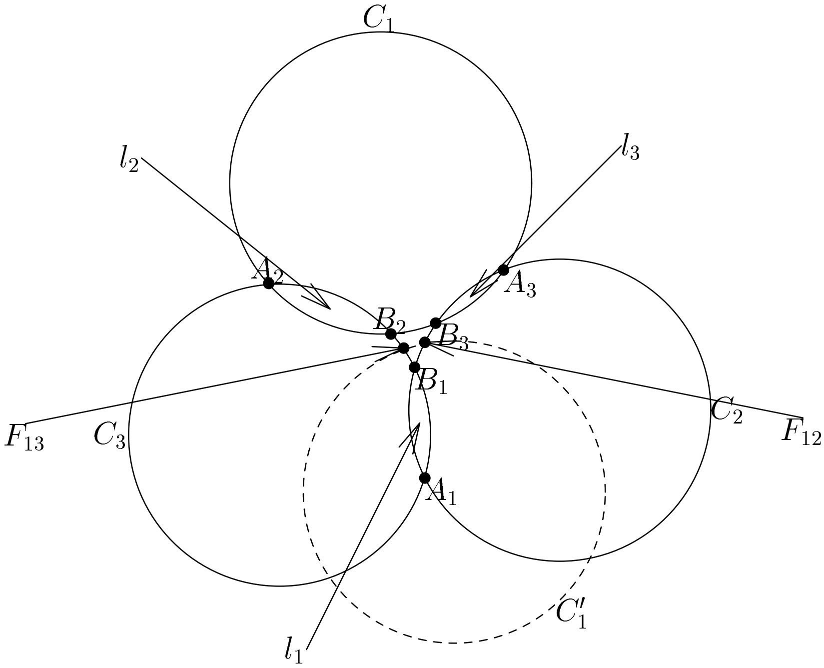

Figure 8: Claim 4.9

The length of $\mathrm{arc}_{C_1'}(F_{12}F_{13})$ depends monotoniously on the length of the interval $[F_{12}F_{13}]$. Observe that $[F_{12}F_{13}]$ is included in $\Delta(B_1B_2B_3)$ and therefore the length of $[F_{12}F_{13}]$ is less or equal to the maximum of the lengthes of the intervals $[B_3B_1]$,$[B_1B_2]$, and $[B_2B_3]$. Consequently, the length of $\mathrm{arc}_{C_1'}(F_{12}F_{13})$ is at most $\max(\theta_1, \theta_2, \theta_3)$. ∎

Let θ_1' denote the length of $\mathrm{arc}_{C_1'}(F_{12}F_{13})$ garanteed by Claim 4.9. Similarly, denote $\theta_2' = |\mathrm{arc}_{C_2'}(F_{21}F_{23})|$ and $\theta_3' = |\mathrm{arc}_{C_3'}(F_{31}F_{32})|$.

Let F_{12}' denote the second intersection point of C_1' and C_2 (other than

F_{12}). Let F'_{13} denote the second intersection point of C'_1 and C_3. Similarly, define F'_{21} and F'_{23} to be the additional intersections of C'_2 with C_1 and C_3 respectively. Finaly, let F'_{31} and F'_{32} be the additional intersections of C'_3 with C_1 and C_2 respectively.

$D'_1 \supset D_2 \cap D_3$. Therefore, using Lemma 3.17 with C'_1, C_2, and C_3, We obtain

$$|\mathrm{arc}_{C_2}(B_1 F'_{12})| + |\mathrm{arc}_{C_3}(B_1 F'_{13})| = \pi + \alpha_1 - \theta'_1. \tag{21}$$

Similarly, using Lemma 3.17 twice more, once with C'_2, C_1, C_3 and once with C'_3, C_1, C_2, we obtain

$$|\mathrm{arc}_{C_1}(B_2 F'_{21})| + |\mathrm{arc}_{C_3}(B_2 F'_{23})| = \pi + \alpha_2 - \theta'_2, \tag{22}$$

$$|\mathrm{arc}_{C_1}(B_3 F'_{31})| + |\mathrm{arc}_{C_2}(B_3 F'_{32})| = \pi + \alpha_3 - \theta'_3. \tag{23}$$

Since C_1 supports the longer arc of a lune, $\mathrm{arc}_{C_1}(B_2 F'_{21}), \mathrm{arc}_{C_1}(B_3 F'_{31})$, and $\mathrm{arc}_{C_1}(B_2 B_3)$ are included in an open semicircle on C_1. Therefore,

$$|\mathrm{arc}_{C_1}(B_2 F'_{21})| + |\mathrm{arc}_{C_1}(B_3 F'_{31})| + |\mathrm{arc}_{C_1}(B_2 B_3)| < \pi. \tag{24}$$

Similar argument for C_2 and C_3 gives

$$|\mathrm{arc}_{C_2}(B_3 F'_{32})| + |\mathrm{arc}_{C_2}(B_1 F'_{12})| + |\mathrm{arc}_{C_2}(B_3 B_1)| < \pi, \tag{25}$$

$$|\mathrm{arc}_{C_3}(B_1 F'_{13})| + |\mathrm{arc}_{C_3}(B_2 F'_{23})| + |\mathrm{arc}_{C_3}(B_1 B_2)| < \pi. \tag{26}$$

Summing up 24, 25, and 26, we get

$$\begin{aligned} 3\pi &> (|\mathrm{arc}_{C_2}(B_1 F'_{12})| + |\mathrm{arc}_{C_3}(B_1 F'_{13})|) + \\ &+ (|\mathrm{arc}_{C_1}(B_2 F'_{21})| + |\mathrm{arc}_{C_3}(B_2 F'_{23})|) + \\ &+ (|\mathrm{arc}_{C_1}(B_3 F'_{31})| + |\mathrm{arc}_{C_2}(B_3 F'_{32})|) + \\ &+ (\theta_1 + \theta_2 + \theta_3). \end{aligned} \tag{27}$$

In view of 21, 22, and 23, the right side of 27 equals

$$\begin{aligned} (\pi &+ \alpha_1 - \theta'_1) + (\pi + \alpha_2 - \theta'_2) + (\pi + \alpha_3 - \theta'_3) + (\theta_1 + \theta_2 + \theta_3) = \\ &= 3\pi + (\alpha_1 + \alpha_2 + \alpha_3 + \theta_1 + \theta_2 + \theta_3) - (\theta'_1 + \theta'_2 + \theta'_3). \end{aligned} \tag{28}$$

Combining 27 and 28, gives

$$\theta_1' + \theta_2' + \theta_3' \quad > \quad \alpha_1 + \alpha_2 + \alpha_3 + \theta_1 + \theta_2 + \theta_3. \tag{29}$$

Observe that the right side of 29 equals to the sum of the internal angles of $\Delta(O_1O_2O_3)$ which is π.

By Claim 4.9, the left side of 29 is less or equal to $3\max(\theta_1, \theta_2, \theta_3)$ which in turn is, by Claim 4.8, less or equal to π. We thus reached the desired contradiction. ∎

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