

# Gallai-Sylvester Theorem for Pairwise Intersecting Unit Circles

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## Abstract

For every finite family of (at least 5) pairwise intersecting unit circles in the plane, there is an intersection point that lies on exactly two circles. This proves a conjecture of A. Bezdek.

## 1 Introduction

The celebrated Gallai-Sylvester theorem ([S93],[G44]), in its dual form, asserts that for every finite collection of lines in the real projective plane that do not form a pencil (i.e, do not share a common point) there is a point that lies on exactly two lines. A possible analogue for circles could be: In any finite collection of (at least 2) unit circles that are pairwise intersecting (i.e, every two circles meet) there is an intersection point that belongs to exactly two circles.

Unfortunately, this statement is false. A simple counterexample consists of three distinct unit circles through a common point  $P$ , each two meeting in two points, and a fourth unit circle that passes through the remaining three intersection points (see Figure 1). A. Bezdek [B90] conjectured that this is the only counterexample to the statement above. He confirmed this under the additional condition that the distance between the centers of any two unit circles is less than  $\sqrt{3}$ .

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**Definition 1.1.** A family  $\mathcal{C}$  of unit circles in the plane is called exceptional if it consists of four circles  $C, C_1, C_2, C_3$  where  $C_1, C_2,$  and  $C_3$  pass through a point  $P_0$ , each two meeting in two points, and  $C$  passes through the three remaining intersection points of  $C_1, C_2, C_3$  (see Figure 1).

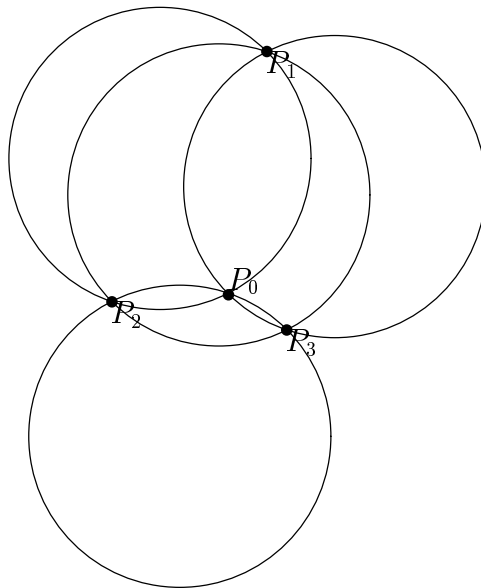


Figure 1: An exceptional configuration

The aim of this paper is to prove Bezdek's conjecture, namely, we prove the following theorem.

**Theorem 1.2.** Let  $\mathcal{C}$  be a finite family of (at least two) pairwise intersecting unit circles in the plane. Then there is a point through which exactly two circles pass, unless  $\mathcal{C}$  is exceptional.

## 2 Definitions and Notations

Throughout this paper  $\mathcal{C}$  denotes a finite family of at least three (clearly, Theorem 1.2 is valid when  $\mathcal{C}$  consists of just two circles) unit circles in the Euclidean plane. We assume that the diameter of the set of centers of those circles is  $\leq 2$ , or in other words, that every two (distinct) circles in  $\mathcal{C}$  either

intersect: they either *cross* (i.e., have two points in common) or *touch* (i.e., have a unique point in common, and a common tangent at this point). We denote by  $\mathcal{A}(\mathcal{C})$  the planar map induced by  $\mathcal{C}$ . Its vertices  $\mathcal{V}(\mathcal{C})$  are the intersection points of circles in  $\mathcal{C}$ , its edges are the closures of the components of  $\bigcup \mathcal{C} \setminus \mathcal{V}(\mathcal{C})$ , and its faces are the closures of the components of  $\mathbb{R}^2 \setminus \bigcup \mathcal{C}$ . We denote the (planar) graph of  $\mathcal{A}(\mathcal{C})$  by  $G_{\mathcal{C}}$ .  $G_{\mathcal{C}}$  may have multiple edges, but no loops (unless  $\mathcal{C}$  consists of just two touching circles, a case which we excluded).

We usually denote a unit circle by  $C$ , possibly with some modifier (subscript or superscript). The closed disk bounded by that circle is denoted by  $D$ , and its center by  $O$ , with the same modifier.

**Definition 2.1.** *The degree  $d(P)$  of a point  $P \in \mathcal{V}(\mathcal{C})$  is the number of circles in  $\mathcal{C}$  that pass through  $P$ . We call  $P$  a simple intersection point when  $d(P) = 2$ .*

For  $k \geq 2$  denote by  $t_k$  the number of intersection points of degree  $k$ . The number  $V$  of vertices is  $\sum_{k \geq 2} t_k$ . Denote by  $f_k$  the number of faces of  $G_{\mathcal{C}}$  which have  $k$  edges (we regard also the unbounded face). The number  $F$  of faces is  $\sum_{k \geq 2} f_k$ . Denote by  $E$  the number of edges of  $G_{\mathcal{C}}$ . We have

$$2E = \sum_{k \geq 2} k f_k \quad \text{as well as} \quad E = \sum_{k \geq 2} k t_k.$$

Therefore

$$2E = \sum_{k \geq 2} k f_k = 3F + \sum_{k \geq 2} (k - 3) f_k.$$

Clearly,  $G_{\mathcal{C}}$  is connected and has no loops. Hence, by Euler's formula,  $V - E + F = 2$ . Therefore,

$$\begin{aligned} -6 &= -3V + 3E - 3F = -3V + E + (2E - 3F) = \\ &= -3 \sum_{k \geq 2} t_k + \sum_{k \geq 2} k t_k + \sum_{k \geq 2} (k - 3) f_k = \\ &= \sum_{k \geq 2} (k - 3) t_k + \sum_{k \geq 2} (k - 3) f_k. \end{aligned}$$

Rearranging, we get

$$t_2 = 6 + \sum_{k \geq 3} (k - 3) t_k + \sum_{k \geq 3} (k - 3) f_k - f_2 \tag{1}$$

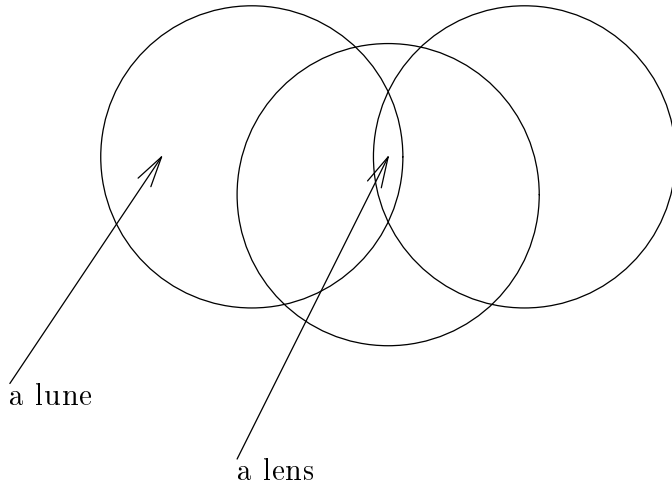


Figure 2: A lens and a lune

According to this notation Theorem 1.2 asserts that  $t_2 > 0$  unless  $\mathcal{C}$  is exceptional. We prove Theorem 1.2 by showing that the right hand side of 1 is positive unless  $\mathcal{C}$  is exceptional.

**Definition 2.2.** Let  $C_1, C_2$  be two circles in  $\mathcal{C}$ . In case  $C_1$  and  $C_2$  cross, we call  $L = D_1 \cap D_2$  a lens if it is a face with two edges in the planar graph  $G_{\mathcal{C}}$ . We then say that both  $C_1$  and  $C_2$  support  $L$  (see Figure 2). The two intersection points of  $C_1$  and  $C_2$  are called the vertices of  $L$ . If  $C_1$  and  $C_2$  touch at  $P$ , we call  $P$  a lens if at most one circle from  $\mathcal{C} \setminus \{C_1, C_2\}$  passes through  $P$ . Also in this case we say that  $C_1$  and  $C_2$  support  $P$ .

We call  $D_1 \setminus \text{int } D_2$  a lune if it is a face with two edges of the planar graph  $G_{\mathcal{C}}$ . We then call  $C_1 \setminus \text{int } D_2$  the longer arc of the lune and say that  $C_1$  supports the longer arc of that lune.

**Definition 2.3.** A lens that is not a singular point is called a proper lens.

**Observation 2.4.** The length of the longer arc of a lune is greater than  $\pi$ .

**Notation 2.5.** For a circle  $C$  and points  $A, B$  on  $C$  that are not antipodal, we denote by  $\text{arc}_C(AB)$  the closed shorter arc on  $C$  between  $A$  and  $B$ .

**Notation 2.6.** For two distinct points  $A$  and  $B$  in the plane we denote by  $\overline{AB}$  the line through  $A$  and  $B$ . We denote by  $\overrightarrow{AB}$  the closed ray which starts at  $A$  and includes  $B$ .  $[AB]$  denotes the closed interval between  $A$  and  $B$ .

**Notation 2.7.** Let  $P, Q, R$  be three different points in the plane, which are not collinear. We denote by  $\Delta(PQR)$  the closed triangle with vertices  $P, Q$ , and  $R$ . We denote by  $\angle PQR$  the closed convex region bounded by the rays  $\overrightarrow{QP}$  and  $\overrightarrow{QR}$ .  $\angle PQR$  denotes the angular measure of  $\angle PQR$ . Therefore  $0 < \angle PQR < \pi$ .

### 3 Some Lemmata

**Lemma 3.1.** Assume  $C, C_1, C_2 \in \mathcal{C}$ ,  $C_1$  and  $C_2$  cross, and  $\text{int}(D \cap D_1 \cap D_2) = \emptyset$ . Then  $\angle O_1 O O_2 < \frac{\pi}{2}$ .

**Proof.** Denote by  $P$  the midpoint of  $[O_1 O_2]$ . Since  $C_1$  and  $C_2$  cross,  $P \in \text{int } D_1 \cap \text{int } D_2$ , and  $\angle O_1 O O_2 \geq \frac{\pi}{2}$  would imply that  $P \in \text{int } D$  as well. ■

**Lemma 3.2.** Let  $C, C_1, C_2 \in \mathcal{C}$ . If  $D \cap D_1$  and  $D \cap D_2$  are distinct lenses, then

1.  $(D \cap D_2) \cap \text{int } D_1 = \emptyset$ ,
2.  $(D_1 \cap D_2) \cap \text{int } D = \emptyset$ .

**Proof.** Denote  $l_1 = D \cap D_1$  and  $l_2 = D \cap D_2$ .

We first show 1. Since  $l_1$  and  $l_2$  are distinct lenses  $\text{int}(l_1 \cap l_2) = \emptyset$ . Now

$$\emptyset = \text{int}(l_1 \cap l_2) = \text{int}(D_1 \cap D \cap D_2) = \text{int } D_1 \cap \text{int}(D \cap D_2).$$

If  $l_2 = D \cap D_2$  is a proper lens, then it follows that also  $(D \cap D_2) \cap \text{int } D_1 = \emptyset$ .

Assume  $D \cap D_2 = \{P\}$  is not a proper lens. If 1 is not true then  $P \in \text{int } D_1$ . It follows that  $D \cap D_1$  is a proper lens. But one of its edges,  $C \cap D_1$ , contains the intersection point  $P$  in its relative interior. This means that  $D \cap D_1$  is not a lens (i.e., not a face with two edges in  $G_{\mathcal{C}}$ ).

To prove 2 we argue similarly

$$\emptyset = \text{int}(l_1 \cap l_2) = \text{int}(D \cap D_1 \cap D_2) = \text{int } D \cap \text{int}(D_1 \cap D_2).$$

If  $C_1$  and  $C_2$  cross then  $D_1 \cap D_2$  is two dimensional and it follows that  $(D_1 \cap D_2) \cap \text{int } D = \emptyset$ .

If  $C_1$  and  $C_2$  touch at  $P$  and 2 is not true, then  $P \in \text{int } D$ . It follows that  $l_1 = D \cap D_1$  is a proper lens. But  $P$  lies in the relative interior of the edge  $D \cap C_1$  of  $l_1$ , a contradiction. ■

**Lemma 3.3.** *Assume  $C, C_1, C_2 \in \mathcal{C}$ . If  $D \cap D_1$  and  $D \cap D_2$  are distinct lenses, then*

1.  $0 < \angle O_1 O O_2 \leq \frac{\pi}{2}$ . Equality holds only if  $C_1$  and  $C_2$  touch at a point  $P$ , and  $C$  passes through  $P$ .
2.  $0 < \angle O O_1 O_2 \leq \frac{\pi}{2}$ , and  $0 < \angle O_1 O_2 O \leq \frac{\pi}{2}$ .

**Proof.** We first prove 1. If  $\angle O_1 O O_2 = 0$  then  $O, O_1, O_2$  are colinear, and without loss of generality  $O_1$  is between  $O$  and  $O_2$ . It follows that  $C_1$  and  $C$  cross and  $C_2$  meets relint  $(C \cap D_1)$ . Therefore,  $D \cap D_1$  is not a face with two edges in  $\mathcal{A}(\mathcal{C})$  and we obtain a contradiction. By Lemma 3.2,  $D_1 \cap D_2 \cap \text{int } D = \emptyset$ . If  $C_1$  and  $C_2$  cross then by Lemma 3.1,  $\angle O_1 O O_2 < \frac{\pi}{2}$ .

If  $C_1$  and  $C_2$  touch at  $P$  and  $\angle O_1 O O_2 > \frac{\pi}{2}$  then  $P \in \text{int } D$ . This contradicts Lemma 3.2. If, on the other hand,  $\angle O_1 O O_2 = \frac{\pi}{2}$ , then the length of  $[OP]$  is 1, and  $C$  passes through  $P$ .

To prove 2, we restrict our attention to the subfamily  $\mathcal{C}' = \{C, C_1, C_2\}$ . 2 will follow from 1 if we show that within  $\mathcal{C}'$ ,  $D_1 \cap D_2$  is a lens.

If  $C_1$  and  $C_2$  touch, then  $D_1 \cap D_2$  is a lens within  $\mathcal{C}'$ . Assume that  $C_1$  and  $C_2$  cross. If  $D_1 \cap D_2$  is not a lens, then it is not a face with two edges in  $\mathcal{A}(\mathcal{C}')$ . It follows that  $C$  meets either relint  $(C_1 \cap D_2)$  or relint  $(C_2 \cap D_1)$ . Without loss of generality assume that the former case happens. Then  $C$  and  $C_2$  cross, and  $C_1$  meets relint  $(D \cap D_2)$ . This is a contradiction for we assume that  $D \cap D_2$  is a lens. ■

**Lemma 3.4.** *Assume  $C, C_1, C_2$  are distinct circles in  $\mathcal{C}$ , and  $D_1 \cap D_2$  is a lens with vertices  $A, B$  (possibly  $A = B$ , if  $D_1 \cap D_2$  is not a proper lens). Denote by  $Q$  the intersection point of the line  $\overline{O_1 O_2}$  with  $C_1$  which is not in  $D_2$ . Let  $\vec{r} \subset \overline{O_1 O_2}$  be the ray which starts at  $Q$  and does not include  $O_1$  (see Figure 3). Then*

1.  $D$  does not meet  $\vec{r}$ .
2. If  $\text{int } D$  is disjoint from  $D_1 \cap D_2$ , then the arc  $C_1 \cap D$  is included either in  $\text{arc}_{C_1}(QA)$  or in  $\text{arc}_{C_1}(QB)$ . (If  $A = B$  then by  $\text{arc}_{C_1}(QA)$  and  $\text{arc}_{C_1}(QB)$ , we mean to the two semicircles of  $C_1$  from both sides of  $[A, Q]$ .)

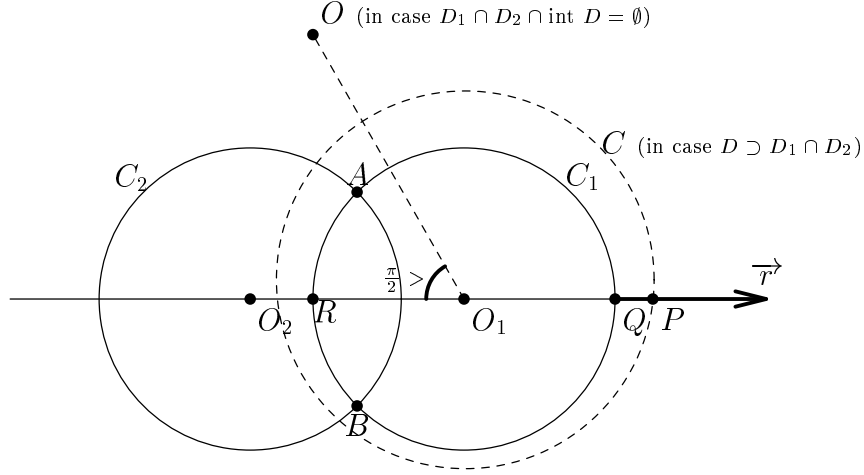


Figure 3: Lemma 3.4

**Proof.** 1. Assume  $C$  meets  $\vec{r}$  at a point  $P$  that is either equal to  $Q$  or lies beyond  $Q$  (on  $\vec{r}$ ).  $D_1 \cap D_2$  is a lens, and therefore either  $D \supset D_1 \cap D_2$  or  $(D_1 \cap D_2) \cap \text{int } D = \emptyset$ .

If  $D \supset D_1 \cap D_2$  then in particular  $D$  include the point  $R \in C_1$  that lies opposite to  $Q$  (see figure 3). It follows (since  $P, R \in C \neq C_1$ ) that The diameter of  $D$  is then greater than 2, which is impossible.

If, on the other hand,  $(D_1 \cap D_2) \cap \text{int } D = \emptyset$ , then, within the family  $\mathcal{C}' = \{C_1, C_2, C\}$ ,  $D \cap D_2$  is also a lens. By Lemma 3.3,  $\angle O_2 O_1 O \leq \frac{\pi}{2}$  (see Figure 3). This implies that the distance between  $P$  and  $O$  is greater than 1, i.e,  $P \notin C$ .

2. Let  $a$  denote the arc  $C_1 \cap D$  of the circle  $C_1$ . Since we assume  $\text{int } D \cap (D_1 \cap D_2) = \emptyset$ ,  $\text{arc}_{C_1}(AB) \cap \text{int } D = \emptyset$ . Moreover, by 1  $Q \notin D$ . It follows that  $a$  is included either in  $\text{arc}_{C_1}(QA)$  or in  $\text{arc}_{C_1}(QB)$ . ■

**Lemma 3.5.** *Assume  $C_1, C_2, C_3, C_4 \in \mathcal{C}$  are four distinct circles. If the four intersections  $D_1 \cap D_3, D_1 \cap D_4, D_2 \cap D_3$ , and  $D_2 \cap D_4$  are lenses, then  $C_1$  and  $C_2$  touch at a point  $P$ , and  $C_3$  and  $C_4$  touch at  $P$  as well.*

**Proof.** We restrict our attention to the subfamily  $\mathcal{C}' = \{C_1, C_2, C_3, C_4\}$ . Note that if the intersection  $D_i \cap D_j$  is not a lens in  $\mathcal{C}'$ , then it not a lens in  $\mathcal{C}$  as well. Assume that all four intersections are lenses. By Lemma 3.3, all twelve angles  $\angle O_i O_j O_k$  ( $i, j, k \in \{1, 2, 3, 4\}$ ) are less or equal to  $\frac{\pi}{2}$ . However, this is easily seen to be possible only if  $O_1, O_2, O_3, O_4$  are the vertices of a rectangle

$R$ . Let  $P$  the the center of  $R$ . The four intervals  $[P, O_i]$ ,  $i = 1, 2, 3, 4$ , all have the same length  $d$ . If  $d < 1$ , then  $P \in \text{int}(D_1 \cap D_2 \cap D_3 \cap D_4)$ . This is impossible since already  $\text{int}(D_1 \cap D_2 \cap D_3) = \emptyset$ , by Lemma 3.2. If  $d > 1$  then the circles centered at opposite vertices of  $R$  are disjoint. There remains the case  $d = 1$ . In this case all four circles pass through  $P$ , and the two circles centered at opposite vertices of  $R$  touch at  $P$ .  $P$  is not a lens, because it lies on four circles. It follows that the vertex of  $R$  opposite to  $O_1$  is  $O_2$ , and  $O_4$  lies opposite to  $O_3$ . ■

**Lemma 3.6.** *Let  $C, C_1, C_2, C_3 \in \mathcal{C}$  be four distinct circles. Assume that  $D \cap D_i$  ( $i = 1, 2, 3$ ) are lenses. Then for some  $j \in \{1, 2, 3\}$ ,  $C_j$  supports no lens other than  $D \cap D_j$ .*

**Proof.** By Lemma 3.3  $\angle O_i O O_j \leq \frac{\pi}{2}$  for  $1 \leq i < j \leq 3$ . Assume, without loss of generality, that  $O_2 \in \text{int} \angle O_1 O O_3$ . Rotate (and if necessary flip) the plane so that the line  $\overline{OO_2}$  is vertical, with  $O_2$  above  $O$ ,  $O_1$  is to the left of  $\overline{OO_2}$  and  $O_3$  is to the right of  $\overline{OO_2}$ . Denote by  $Q$  the intersection point of  $\overline{OO_2}$  with  $C_2$  which is outside  $D$ , and denote by  $P$  the intersection point of  $\overline{OO_2}$  with  $C$  which is outside  $D_2$ . Denote the intersection points of  $C$  and  $C_2$  by  $A, B$ . If  $D \cap D_2$  is not a proper lens, then  $A = B \in \overline{OO_2}$ . Otherwise, assume that  $A$  is to left of  $\overline{OO_2}$ . In the sequel, when  $A = B \text{ arc}_{C_2}(AQ)$  denotes the semicircle of  $C_2$  which is to the left of  $\overline{OO_2}$ , and  $\text{arc}_{C_2}(BQ)$  denotes the semicircle of  $C_2$  which is to the right of  $\overline{OO_2}$ .

We show that  $C_2$  supports no lens other than  $D \cap D_2$ . Assume to the contrary, that  $C_0 \in \mathcal{C}$ ,  $C_0 \neq C$  and  $D_0 \cap D_2$  is a lens.

By Lemma 3.3,  $\Delta(OO_0O_2)$  is a proper triangle, hence  $O_0 \notin \overline{OO_2}$ . Assume, without loss of generality, that  $O_0$  is to the right of  $\overline{OO_2}$ . We restrict our attention to the subfamily  $\mathcal{C}' = \{C, C_0, C_1, C_2\}$  and intend to show that  $D_0 \cap D_1$  is a lens in  $\mathcal{C}'$ , contradicting Lemma 3.5. (Indeed, by Lemma 3.5, this is possible only if  $C$  and  $C_0$  touch at a point  $V$ , and  $C_1, C_2$  touch at  $V$  as well. This is impossible, since  $O_1$  is to the left of  $\overline{OO_2}$  and  $O_0$  is to the right of  $\overline{OO_2}$ .)

We start with two applications of Lemma 3.2. Since  $D \cap D_2$  and  $D \cap D_2$  are lenses,

$$D \cap D_2 \cap \text{int} D_0 = \emptyset. \quad (2)$$

Since  $D \cap D_1$  and  $D \cap D_2$  are lenses,

$$D \cap D_2 \cap \text{int} D_1 = \emptyset. \quad (3)$$



Next come four applications of Lemma 3.4. With  $C_0, C_2, C$  standing for  $C, C_1, C_2$ , respectively,  $C_2 \cap D_0 \subseteq \text{arc}_{C_2}(QA) \setminus \{Q\}$ , or  $C_2 \cap D_0 \subseteq \text{arc}_{C_2}(QB) \setminus \{Q\}$ . Since  $O_0$  is to the right of  $\overline{OO_2}$ , we conclude that

$$C_2 \cap D_0 \subseteq \text{arc}_{C_2}(QB) \setminus \{Q\}. \quad (4)$$

With  $C_0, C, C_2$  standing for  $C, C_1, C_2$  respectively, in Lemma 3.4 (repeating the above argument), we find that

$$C \cap D_0 \subseteq \text{arc}_C(PB) \setminus \{P\}. \quad (5)$$

With  $C_1, C_2, C$  standing for  $C, C_1, C_2$ , respectively, in Lemma 3.4, we obtain

$$C_2 \cap D_1 \subseteq \text{arc}_{C_2}(QA) \setminus \{Q\}. \quad (6)$$

With  $C_1, C, C_2$  standing for  $C, C_1, C_2$ , respectively, in Lemma 3.4, we obtain

$$C \cap D_1 \subseteq \text{arc}_C(PA) \setminus \{P\}. \quad (7)$$

Now note that

$$\text{arc}_{C_2}(QA) \cap \text{arc}_{C_2}(QB) = \begin{cases} \{Q\} & \text{if } A \neq B \\ \{Q, A\} & \text{if } A = B, \end{cases} \quad (8)$$

and similarly,

$$\text{arc}_C(PA) \cap \text{arc}_C(PB) = \begin{cases} \{P\} & \text{if } A \neq B \\ \{P, A\} & \text{if } A = B. \end{cases} \quad (9)$$

If  $D \cap D_2$  is a proper lens, i.e., if  $A \neq B$ , then we conclude from 4,6, and 8 that

$$C_2 \cap D_0 \cap D_1 = \emptyset, \quad (10)$$

and from 5,7, and 9 that

$$C \cap D_0 \cap D_1 = \emptyset. \quad (11)$$

This means that  $D_0 \cap D_1$  is indeed a lens in  $\mathcal{C}'$ .

If  $D \cap D_2$  is an improper lens, i.e., if  $A = B$ , then the same arguments lead only to the inclusions

$$C_2 \cap D_0 \cap D_1 \subseteq \{A\}, \quad (12)$$

$$C \cap D_0 \cap D_1 \subseteq \{A\}. \quad (13)$$

$D_2 \cap D_0$  is a lens, and  $D_2 \cap D = \{A\}$  is a lens. Therefore, by Lemma 3.2,

$$A \notin \text{int } D_0. \quad (14)$$

Similarly,  $D \cap D_1$  is a lens, and  $D \cap D_2 = \{A\}$  is a lens. Therefore, by Lemma 3.2,

$$A \notin \text{int } D_1. \quad (15)$$

In order to show that  $D \cap D_0$  is a lens, it suffices to establish equalities 10 and 11. These, in turn, follow from the known inclusions 10 and 11, provided that we know that  $A \notin D_0 \cap D_1$ .

Assume, to the contrary, that

$$A \in D_0 \cap D_1. \quad (16)$$

From 14, 15, and 16 we conclude that  $A \in C_0 \cap C_1$ . But this is impossible since  $\{A\} = D \cap D_2 = C \cap C_2$  is an improper lens and therefore  $A$  can belong to at most one circle in  $\mathcal{C}' \setminus \{C, C_2\}$ . (See Definition 2.2.) ■

**Lemma 3.7.** *If  $\#(\mathcal{C}) = n$ , then there are at most  $n$  lenses in  $\mathcal{A}(\mathcal{C})$ .*

**Proof.** By induction on  $n$ . The statement is clearly true for  $n = 2, 3$ . If every circle in  $\mathcal{C}$  supports at most two lenses, then the number of lenses is at most  $n$ , since every lens is supported by exactly two circles. Assume  $n \geq 4$ . If a circle  $C \in \mathcal{C}$  supports three or more lenses, then, by Lemma 3.6, another circle  $C' \in \mathcal{C}$  supports just one lens. Define  $\mathcal{C}' = \mathcal{C} \setminus \{C'\}$ . By the induction hypothesis,  $\mathcal{C}'$  has at most  $n - 1$  lenses. Adding  $C'$  to  $\mathcal{C}'$ , will produce just one more lens (and could possibly kill a few lenses of  $\mathcal{A}(\mathcal{C}')$ ). ■

Our next aim is to study the lunes in  $\mathcal{A}(\mathcal{C})$ , and more specifically the relationships between lunes and lenses in  $\mathcal{A}(\mathcal{C})$ .

Let us start with some simple observations. If  $D$  is a unit disk,  $A, B$  are distinct points in  $D$ , and  $C^*$  is any circle of radius  $\geq 1$  that passes through

$A$  and  $B$ , then  $\text{arc}_{C^*}(AB) \subset D$ . The same holds, of course, if we replace the disk  $D$  by an arbitrary intersection of unit disks in the plane. It follows that if  $C, C'$  are distinct circles in  $\mathcal{C}$ , then  $D \cap D'$  is the union of all short unit circle arcs with endpoints on  $C \cap D'$ . Therefore if  $C, C', C''$  are distinct circles in  $\mathcal{C}$ , then  $D \cap D' \subset D \cap D''$  iff  $C \cap D' \subset C \cap D''$ . Moreover,  $D' \cap D'' \subset D$  iff  $C \cap C' \subset D$ , i.e., iff the two arcs  $C' \cap D$  and  $C'' \cap D$  have two points in common.

Now consider the relative position of the two arcs  $C \cap D'$  and  $C \cap D''$  on  $C$ . There are essentially three possible cases:

1.  $C \cap D' \subset C \cap D''$  (or  $C \cap D'' \subset C \cap D'$ ).
2. The arcs  $C \cap D'$  and  $C \cap D''$  overlap, but no one includes the other.
3. The arcs  $C \cap D'$  and  $C \cap D''$  are disjoint, or have just one endpoint in common.

In case 1 the arcs  $D \cap C'$  and  $D \cap C''$  are either disjoint, or they have just one endpoint in common.

In case 2 the points of  $C \cap C'$  separate the points of  $C \cap C''$  on  $C$ , and the circles  $C'$  and  $C''$  cross once inside  $D$  and once outside  $D$ . In case 3 the intersection  $D' \cap D''$  maybe either entirely inside  $D$ , or entirely outside  $D$ , possibly with one vertex on  $C$ .

Thus we see that the inclusion  $D' \cap D'' \subset D$  is possible only in case 3. Next we show that this inclusion is impossible when both arcs  $C \cap D'$  and  $C \cap D''$  lie in a semicircle of  $C$ . But first we need one more technical lemma. We denote the length of a circular arc  $S$  by  $|S|$ .

**Lemma 3.8.** *Let  $C, C'$  be distinct circles in  $\mathcal{C}$ , with  $C \cap C' = \{A, B\}$  (possibly  $A = B$ , if  $C$  and  $C'$  tangent). Assume  $|\text{arc}_C(AB)| = |\text{arc}_{C'}(AB)| = \alpha$ . Denote by  $L, L'$  the rays tangent to  $\text{arc}_C(AB)$  and  $\text{arc}_{C'}(AB)$  at  $A$ . Then:*

1. *the angle between  $L$  and  $L'$  is  $\alpha$ .*
2. *the intersection of  $C$  (or  $C'$ ) with the convex angular region bounded by  $L$  and  $L'$  has length  $2\alpha$ .*

**Proof.** See Figure 4. Note that the ray  $\overrightarrow{AB}$  makes an angle  $\frac{1}{2}\alpha$  with each of the tangents  $L, L'$ . ■

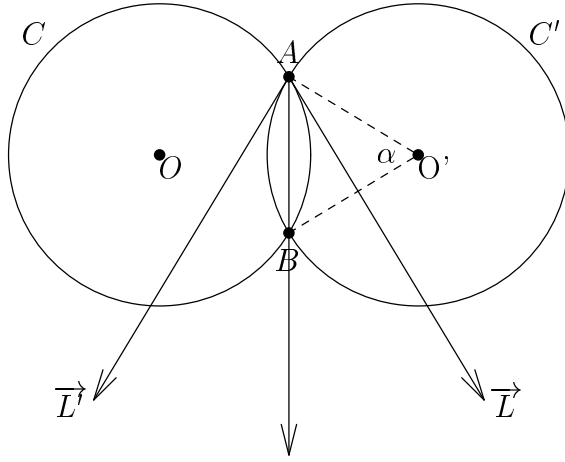


Figure 4: Lemma 3.8

**Lemma 3.9.** *Let  $C, C', C''$  be distinct circles in  $\mathcal{C}$ . Assume that the arcs  $C \cap D'$  and  $C \cap D''$  are disjoint, or have just one endpoint in common, and both lie on a open semicircle  $S \subset C$ . Then the intersections  $D \cap D'$  and  $D \cap D''$  are disjoint or share just one point on  $C$ . In the latter case  $D' \cap D''$  is not a point, i.e.,  $C'$  and  $C''$  are not tangent.*

**Proof.** Assume  $C \cap C' = \{A', B'\}$  (possibly  $A' = B'$ ) and  $C \cap C'' = \{A'', B''\}$  (possibly  $A'' = B''$ ), and that the points  $A'B', A''B''$  appear in this order on  $S$  (possibly  $B' = A''$ ), with  $|C \cap D'| = \alpha \geq 0$ ,  $|\text{arc}_C(B'A'')| = \beta \geq 0$ ,  $|C \cap D''| = \gamma \geq 0$ , and  $\alpha + \beta + \gamma < \pi$ .

Draw the ray  $L'$  tangent to  $C' \cap D$  at  $B'$ , and the ray  $L''$  tangent to  $C'' \cap D$  at  $A''$ . Since  $2\alpha + \beta + 2\gamma < 2\pi$ , by Lemma 3.8, these rays do not cross inside  $D$ , and they separate  $D \cap D'$  from  $D \cap D''$ . If the intersection of  $D \cap D'$  and  $D \cap D''$  share one point on  $C$  then this point must be  $B' = A''$ . If  $C'$  and  $C''$  tangent at  $B'$  then  $L$  and  $L'$  coincide. In this case  $\beta = 0$  and  $\alpha + \gamma = \pi$  contradicting the assumption that  $\alpha + \beta + \gamma < \pi$ . ■

**Lemma 3.10.** *Suppose  $C \in \mathcal{C}$  supports the longer arc of a lune. Then for any two distinct circles  $C_1, C_2 \in \mathcal{C} \setminus \{C\}$ .  $D_1 \cap D_2 \not\subseteq D$ .*

**Proof.** Suppose  $D \setminus \text{int } D'$  is a lune. Then the arc  $C \cap D'$  is included in a open semicircle  $S$ , and for any circle  $C_i \in \mathcal{C} \setminus \{C\}$  we have  $C \cap D_i \subset C \cap D' \subset S$ . We conclude that  $D_1 \cap D_2 \not\subseteq D$ . This is obvious if the arcs  $C \cap D_1$  and  $C \cap D_2$

do have interior points in common, and follows from Lemma 3.9 if they do not. ■

**Corollary 3.11.** *Suppose  $C \in \mathcal{C}$  supports the longer arc of a lune. Then every lens that is included in  $D$  must be supported by  $C$ . Moreover, if  $C_1$  and  $C_2$  are two circles which are tangent at a point  $P \in D$  then either  $C = C_1$  or  $C = C_2$ .*

**Proof.** If the lens  $D_1 \cap D_2$  is included in  $D$  but not supported by  $C$  (i.e, both  $C_1$  and  $C_2$  are different from  $C$ ), then  $D_1 \cap D_2 \subset D$  contradicting Lemma 3.10.

For the second part, assume that  $C_1, C_2 \in \mathcal{C}$  are two distinct circles which are tangent at a point  $P \in D$  and both  $C_1$  and  $C_2$  are distinct from  $D$ . Then  $D \supset D_1 \cap D_2$  which contradicts Lemma 3.10 ■

From this point on we assume that every intersection point of circles in  $\mathcal{C}$  has degree  $\geq 3$ .

**Lemma 3.12.** *Let  $C_1, C_2 \in \mathcal{C}$  be two different circles, then either  $D_1 \cap D_2$  includes a lens or there is a circle which is tangent to  $C_1$  or to  $C_2$  at a point on the boundary of  $D_1 \cap D_2$ .*

**Proof.** Clearly, if  $C_1$  and  $C_2$  touch, then there is nothing to prove. Assume then that  $C_1$  and  $C_2$  cross and  $C_1 \cap C_2 = \{A, B\}$ . We assume that no circle in  $\mathcal{C}$  is tangent to  $C_1$  or to  $C_2$  at a point on the boundary of  $D_1 \cap D_2$ .

Let  $G'_\mathcal{C}$  be the planar subgraph of  $G_\mathcal{C}$  which is induced by the vertices of  $G_\mathcal{C}$  which are inside  $D_1 \cap D_2$ . We regard the complement of  $D_1 \cap D_2$  as the unbounded face of  $G'_\mathcal{C}$ .

For every  $k \geq 2$  let  $u_k$  denote the number of vertices inside  $\text{int}(D_1 \cap D_2)$  through which exactly  $k$  circles pass. Let  $v_k$  denote the number of vertices, different from  $A$  and  $B$ , on the boundary of  $D_1 \cap D_2$  through which exactly  $k$  circles pass. Denote by  $w_k$  the number of faces in  $G'_\mathcal{C}$  which have exactly  $k$  edges.

Let  $V, E, F$  denote the number of vertices, edges, and faces of  $G'_\mathcal{C}$  respectively. Then  $V = 2 + \sum_{k \geq 2} u_k + v_k$  and  $F = \sum_{k \geq 2} w_k$ .

Counting the number of edges in two different ways we get

$$2E = \sum_{k \geq 2} k w_k = 3F + \sum_{k \geq 2} (k - 3) w_k. \quad (17)$$

Since we assume that  $D_1 \cap D_2$  includes no lens, all faces of  $G'_C$  have at least 3 edges. Note that the unbounded face has exactly  $2 + \sum_{k \geq 2} v_k$  edges. Therefore,

$$\sum_{k \geq 2} (k-3)w_k \geq \sum_{k \geq 2} v_k - 1. \quad (18)$$

Observe that the degree in  $G'_C$  of a vertex counted by  $u_k$  is  $2k$  and the degree of a vertex counted by  $v_k$  is  $k+1$ . Let  $d(A)$  and  $d(B)$  denote the degrees in  $G'_C$  of  $A$  and  $B$  respectively. Counting the sum of the degrees of all vertices in two different ways we obtain

$$2E = d(A) + d(B) + \sum_{k \geq 2} 2ku_k + (k+1)v_k. \quad (19)$$

By Euler's formula  $V - E + F = 2$ , so that plugging in 17, 18, and 19 we obtain

$$\begin{aligned} -12 &= -6V + 6E - 6F = -6V + 2E + 2(2E - 3F) = \\ &= -6(2 + \sum_{k \geq 2} u_k + v_k) + d(A) + d(B) + \\ &+ \sum_{k \geq 2} (2ku_k + (k+1)v_k) + 2 \sum_{k \geq 2} (k-3)w_k = \\ &= -12 + d(A) + d(B) + \sum_{k \geq 2} ((2k-6)u_k + (k-5)v_k) + \\ &+ 2 \sum_{k \geq 2} (k-3)w_k \geq \\ &\geq -12 + d(A) + d(B) + \sum_{k \geq 2} ((2k-6)u_k + (k-5)v_k) + \\ &+ 2(\sum_{k \geq 2} v_k - 1) = \\ &= -14 + d(A) + d(B) + \sum_{k \geq 2} ((2k-6)u_k + (k-3)v_k). \end{aligned}$$

Rearranging, we get

$$\begin{aligned} 2u_2 + v_2 &\geq -2 + d(A) + d(B) + \sum_{k \geq 3} ((2k-6)u_k + (k-3)v_k) \geq \\ &\geq -2 + d(A) + d(B) \geq 2. \end{aligned} \quad (20)$$

The last inequality is because  $d(A), d(B) \geq 2$ . Every vertex counted by  $u_2$  is an intersection point of precisely two circles in  $\mathcal{C}$ . Moreover, since we assume that no circle in  $\mathcal{C}$  touches  $C_1$  or  $C_2$  at a point on the boundary of  $D_1 \cap D_2$ , every vertex counted by  $v_2$  is an intersection point of precisely two circles in  $\mathcal{C}$ . We assume that through every intersection point in  $\mathcal{A}(\mathcal{C})$  at least 3 circles pass and hence  $u_2 = v_2 = 0$ . In view of 20, this is a contradiction. ■

**Corollary 3.13.** *Assume that  $C_1, C_2 \in \mathcal{C}$  and each of  $C_1, C_2$  supports the longer arc of a lune, then either  $D_1 \cap D_2$  is a proper lens or  $C_1$  and  $C_2$  touch.*

**Proof.** By Lemma 3.12, either  $D_1 \cap D_2$  includes a lens  $l$  or it includes a tangency point  $P$  of two circles  $C_3, C_4 \in \mathcal{C}$ . In the first case, using Corollary 3.11,  $l$  is supported by  $C_1$  as well as by  $C_2$  and hence  $l = D_1 \cap D_2$ . In the second case since  $P \in D_1$ , by the second part of Corollary 3.11,  $C_1$  is one of  $C_3, C_4$ , and similarly  $C_2$  is one of  $C_3, C_4$ . Hence,  $C_1$  and  $C_2$  touch at  $P$ . ■

**Corollary 3.14.** *There are at most three lunes in  $\mathcal{A}(\mathcal{C})$ .*

**Proof.** Assume  $C_1, C_2, C_3, C_4 \in \mathcal{C}$  and each of them supports the longer arc of some lune. By Corollary 3.13, for every  $1 \leq i < j \leq 4$  either  $D_i \cap D_j$  is a proper lens or  $C_i$  and  $C_j$  touch.

We restrict our attention to the subfamily  $\mathcal{C}' = \{C_1, C_2, C_3, C_4\}$ . Every tangency point of two circles in  $\mathcal{C}'$  must be a lens (in  $\mathcal{C}'$ ). To see this assume, without loss of generality, that  $C_1$  and  $C_2$  touch at a point  $P$ . We show that no other circle from  $\mathcal{C}'$  passes through  $P$  (and hence  $P$  is a lens in  $\mathcal{C}'$ ). Assume to the contrary that  $C_3$  passes through  $P$ . By the second part of Corollary 3.11,  $C_3$  must coincide with  $C_1$  or  $C_2$ .

We conclude that within the family  $\mathcal{C}'$  all six intersection  $D_i \cap D_j$  are lenses. This contradicts Lemma 3.7, for we obtained six lenses in a family of just four circles. ■

**Lemma 3.15.** *Suppose that  $D_1 \cap D_2$  is a lens, then  $C_1$  includes an edge of the unbounded face.*

**Proof.** Denote by  $Q$  the intersection point of  $\overline{O_1 O_2}$  with  $C_1$  which is not in  $D_2$ . By Lemma 3.4, the ray  $\overrightarrow{r} \subset \overline{O_1 O_2}$  which start at  $Q$  and does not include  $O_1$ , meets no circle of  $\mathcal{C}$  except for  $C_1$  at  $Q$ . But that means that  $Q$  belongs to the interior of some edge of the unbounded face, which is included in  $C_1$ . ■

**Corollary 3.16.** *The number of edges of the unbounded face is at least the number of lenses.*

**Proof.** Let  $C_1, \dots, C_m$  be all the circles in  $\mathcal{C}$  that support a lens. By Lemma 3.15, the unbounded face has at least  $m$  edges. By Lemma 3.7, there are at most  $m$  lenses. ■

Denote by  $m$  the number of lenses and by  $l$  the number of lunes. The faces of  $G_{\mathcal{C}}$  counted by  $f_2$  are exactly the *proper* lenses and the lunes. Therefore  $f_2 \leq m + l$ .

Substituting this in 1 we get

$$t_2 = 6 + \sum_{k \geq 3} (k-3)t_k + \sum_{k \geq 3} (k-3)f_k - f_2 \geq 6 + (m-3) - (m+l) = 3-l$$

where the inequality is because  $f_2 \leq m + l$ ,  $\sum_{k \geq 3} (k-3)t_k \geq 0$ , and  $\sum_{k \geq 3} (k-3)f_k \geq m-3$  due to the unbounded face.

Since we assume  $t_2 = 0$  we must have  $l = 3$  (because of Corollary 3.14). Note also that all inequalities must be equalities. Hence we deduce the following equalities.  $f_2 = m + l$ , implying that all lenses are proper lenses.  $\sum_{k \geq 3} (k-3)t_k = 0$ , implying that every intersection point has degree 3.  $\sum_{k \geq 3} (k-3)f_k = m-3$ , implying that the unbounded face has exactly  $m$  edges and hence (by the proof of Corollary 3.16) every circle which includes an edge of the unbounded face must support a lens. Moreover, every bounded face which is and not a lens nor a lune must be a triangle (that is, a face with just three edges). An important corollary is that no two circles in  $\mathcal{C}$  touch. For if two circles in  $\mathcal{C}$  touch at a point  $P$ , then  $d(P) = 3$  implies that  $P$  is a lens which is not a proper lens and we obtain a contradiction.

From now on we assume that every two circles in  $\mathcal{C}$  cross.

**Lemma 3.17.** *Let  $C, C_1, C_2$  be three distinct unit circles. Assume that  $C_1$  and  $C_2$  cross, and  $D \supset D_1 \cap D_2$ . Denote the intersection points of  $C_1$  and  $C_2$  by  $M, N$ . Denote the intersection points of  $C$  and  $C_1$  by  $X_1, Y_1$  and those of  $C$  and  $C_2$  by  $X_2, Y_2$  (as indicated in Figure 5). Let  $\theta = |\text{arc}_C(X_1X_2)|$  and let  $\alpha = |\text{arc}_{C_1}(MN)|$ . Then*

$$|\text{arc}_{C_1}(Y_1M)| + |\text{arc}_{C_2}(Y_2M)| = \pi + \alpha - \theta.$$

**Proof.** Rotate (and if necessary flip) the plane so that  $\overline{O_1O_2}$  is horizontal,  $O_1$  is to the right of  $O_2$  and  $X_1, X_2$  are below  $\overline{O_1O_2}$ . We show how to deal with the case where  $O$  is on or above  $\overline{O_1O_2}$ . The other case can be treated



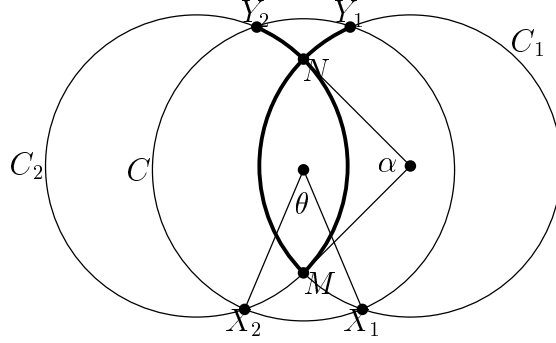


Figure 5: Lemma 3.17

similarly. Note that each of the quadrilaterals  $OX_1O_1Y_1$  and  $OX_2O_2Y_2$  is a rhombus.

$$\begin{aligned}
|\text{arc}_{C_2}(MY_2)| + |\text{arc}_{C_1}(MY_1)| &= \alpha + \angle O_1O_2Y_2 + \angle O_2O_1Y_1 = \\
&= \alpha + \angle O_1O_2O + \angle OO_2Y_2 + \angle O_2O_1O + \angle OO_1Y_1 = \\
&= \alpha + \angle O_1O_2O + \angle O_2OX_2 + \angle O_2O_1O + \angle O_1OX_1 = \\
&= \alpha + \angle O_1O_2O + \angle O_2O_1O + \angle O_2OO_1 - \theta = \\
&= \pi + \alpha - \theta.
\end{aligned}$$

■

**Lemma 3.18.** *Let  $C, C_1, C_2$  be three pairwise crossing unit circles. If  $D \supset D_1 \cap D_2$ , then  $\angle O_1OO_2 > \frac{\pi}{2}$ .*

**Proof.** Denote by  $A, B$  the intersection points of  $C_1$  and  $C_2$ . Let  $S_A, S_B$  be the unit disks centered at  $A, B$  respectively. Note that  $O \in S_A \cap S_B$ . Observe that the boundaries of  $S_A$  and  $S_B$  intersect at  $O_1$  and  $O_2$ .  $S_A \cap S_B$  is included in the disk whose diameter is  $[O_1O_2]$ . Hence  $\angle O_1OO_2 > \frac{\pi}{2}$ . ■

## 4 Proof of the Main Theorem

**Proof of Theorem 1.2.** Assume to the contrary that every intersection point  $P \in \mathcal{A}(C)$  has  $d(P) \geq 3$ . As we have shown already, it follows from here that there are exactly three lunes in  $\mathcal{A}(C)$  and every intersection point  $P$  has

$d(P) = 3$ . Moreover, every circle which includes an edge of the unbounded face must support some lens, and every bounded face which is not a lens or a lune must be a triangle. We also noted that no two circles in  $\mathcal{C}$  touch, and every lens in  $\mathcal{C}$  is a proper lens.

Let  $C_1, C_2, C_3$  be the three circles which support the longer arcs of the three lunes in  $\mathcal{A}(\mathcal{C})$ . By Corollary 3.13, for every  $1 \leq i < j \leq 3$   $D_i \cap D_j$  is a proper lens. Denote  $l_1 = D_2 \cap D_3, l_2 = D_1 \cap D_3, l_3 = D_1 \cap D_2$ .

**Claim 4.1.** *Let  $C \in \mathcal{C}$ . Then  $D$  includes at least one of  $l_1, l_2, l_3$ .*

**Proof.** If  $C$  is one of  $C_1, C_2, C_3$  then clearly  $D$  includes two of  $l_1, l_2, l_3$ . Let  $C \in \mathcal{C} \setminus \{C_1, C_2, C_3\}$  and assume that  $D$  does not include any of  $l_1, l_2, l_3$ . Hence,  $D$  is disjoint from the interior of each of  $l_1, l_2, l_3$ .

$D \cap \text{int } l_2 = \emptyset$  implies that  $\text{int } D \cap \text{int } D_1 \cap \text{int } D_3 = \emptyset$ . It follows that  $D_3 \cap \text{int } (D \cap D_1) = \emptyset$ . Similarly,  $D \cap \text{int } l_3 = \emptyset$  implies  $D_2 \cap \text{int } (D \cap D_1) = \emptyset$ . In other words, if we restrict our attention to the subfamily  $\mathcal{C}' = \{C, C_1, C_2, C_3\}$  then  $D \cap D_1$  is a lens.

In the same way we obtain that also  $D \cap D_2$  and  $D \cap D_3$  are lenses (within  $\mathcal{A}(\mathcal{C}')$ ). This is a contradiction to Lemma 3.5. ■

By Lemma 3.2,  $D_1 \cap D_2 \cap D_3$  has an empty interior so that it is either empty or a singular point.

**Case 1.**  $D_1 \cap D_2 \cap D_3$  is a point  $P$ . Let  $C \in \mathcal{C} \setminus \{C_1, C_2, C_3\}$ . By Claim 4.1,  $D$  must include at least one of  $l_1, l_2, l_3$ . Hence  $P \in D$ . We claim that  $P \in \text{int } D$ . Assume to the contrary that  $P \in C$ . Then  $D$  includes either one or two lenses of  $l_1, l_2, l_3$ . If  $D$  includes just one lens, say  $l_1$ , then it must be disjoint from the interiors of  $l_2$  and  $l_3$ . This is possible only if  $C$  touches  $C_1$  at  $P$ , which is impossible since every two circles in  $\mathcal{C}$  cross. If  $D$  includes two lenses, say  $l_2$  and  $l_3$ , and is disjoint from the interior of the third,  $l_1$ , then it is easily seen by inspection that  $C$  must coincide with  $C_1$ , contradicting our assumption on  $C$ .

Hence  $P \in \text{int } D$ . It follows now that  $D$  includes all lenses  $l_1, l_2, l_3$ . Denote the remaining intersection points of the circles  $C_1, C_2$ , and  $C_3$  by  $P_1, P_2, P_3$  (see figure 1).

**Lemma 4.2.**  $\Delta(P_1P_2P_3)$  overlaps  $\Delta(O_1O_2O_3)$ . Moreover, the circle which passes through  $P_1, P_2$ , and  $P_3$  is a unit circle.

**Proof.** We regard the affine plane as the two dimensional vector space  $\mathbf{R}^2$

with  $P$  as its origin. Then

$$\begin{aligned} P_1 &= O_2 + O_3, \\ P_2 &= O_1 + O_3, \\ P_3 &= O_1 + O_2. \end{aligned}$$

Therefore the points  $P_1, P_2$ , and  $P_3$  are at unit distance from the point  $O_1 + O_2 + O_3$ , which shows that the circle through  $P_1, P_2$ , and  $P_3$  is a unit circle.

To see that  $\Delta(O_1O_2O_3)$  and  $\Delta(P_1P_2P_3)$  overlap observe that

$$\Delta(P_1P_2P_3) = (O_1 + O_2 + O_3) - \Delta(O_1O_2O_3).$$

■

Let  $C \in \mathcal{C} \setminus \{C_1, C_2, C_3\}$ . We know that  $D$  includes  $l_1 \cup l_2 \cup l_3$  and hence  $D \supset \Delta(P_1P_2P_3)$ . By Lemma 3.1, the triangle  $\Delta(O_1O_2O_3)$  is acute. Therefore, by Lemma 4.2,  $\Delta(P_1P_2P_3)$  is also acute. Let  $C_0$  be the circle through  $P_1, P_2$ , and  $P_3$ .  $C_0$  is the smallest circle which includes  $\Delta(P_1P_2P_3)$ . By Lemma 4.2,  $C_0$  is a unit circle. Hence  $C$  must coincide with  $C_0$ . In other words  $\#\mathcal{C} \leq 4$ . If  $\mathcal{C} = \{C_0, C_1, C_2, C_3\}$  then  $\mathcal{C}$  is exceptional. Otherwise,  $\mathcal{C} = \{C_1, C_2, C_3\}$  and each of  $P_1, P_2$ , and  $P_3$  has degree 2.

**Case 2.**  $D_1 \cap D_2 \cap D_3 = \emptyset$ . Denote by  $B_1, B_2, B_3$  and  $A_1, A_2, A_3$  the intersection points of  $C_1, C_2$ , and  $C_3$  as indicated in figure 6.

**Claim 4.3.** *Assume  $C \in \mathcal{C}$  and  $D$  includes  $l_1$  but not  $l_2$  or  $l_3$ . Then  $C$  intersects  $C_1$  in two points on  $\text{arc}_{C_1}(B_2B_3)$ .*

**Remark.** By symmetry, Claim 4.3 is true if we change the role of  $l_1$  by  $l_2$  or  $l_3$ , and correspondently,  $C_1$  by  $C_2$  or  $C_3$ .

**Proof.** We show that  $O$  is inside  $\angle O_2O_1O_3$ .  $D$  is disjoint from  $\text{int } l_2 = \text{int}(D_1 \cap D_3)$  and from  $\text{int } l_3 = \text{int}(D_1 \cap D_2)$ . Therefore, by Lemma 3.1, the triangles  $\Delta(O_1O_2O)$  and  $\Delta(O_1O_3O)$  have all their angles acute. Moreover, by Lemma 3.2, all the angles of  $\Delta(O_1O_2O_3)$  are acute. If  $O \notin \angle O_2O_1O_3$ , then either  $O \in \angle O_1O_3O_2$  or  $O \in \angle O_1O_2O_3$ . We assume, without loss of generality, that  $O \in \angle O_1O_3O_2$ . Since  $O \notin \angle O_2O_1O_3$ , it follows that  $O_3 \in \angle O_1OO_2$ . By Lemma 3.18,  $\angle O_2OO_3 > \frac{\pi}{2}$ . We obtain a contradiction since now  $\angle O_2OO_3 \leq \angle O_2OO_1 < \frac{\pi}{2}$ .

We conclude that  $\overrightarrow{O_1O}$  must be between  $\overrightarrow{O_1O_2}$  and  $\overrightarrow{O_1O_3}$ . Since  $D$  is disjoint from  $\text{int } l_2$  and  $\text{int } l_3$ ,  $D \cap C_1$  must be disjoint from the relative

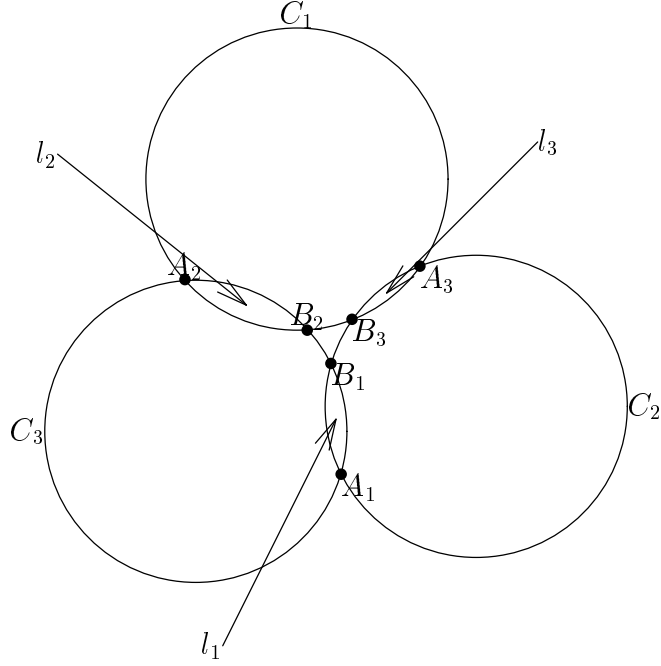


Figure 6:

interiors of both arcs  $D_2 \cap C_1$  and  $D_3 \cap C_1$ . The center of the arc  $C_1 \cap D$  is between the centers of  $C_1 \cap D_2$  and  $C_1 \cap D_3$ . It follows now that  $D \cap C_1$  must be included in  $\text{arc}_{C_1}(B_2B_3)$ . ■

**Claim 4.4.** *Assume that  $C \in \mathcal{C} \setminus \{C_1, C_2, C_3\}$ . If  $C$  supports a lens then  $D$  includes exactly one of the lenses  $l_1, l_2$ , or  $l_3$ .*

**Proof.** By Claim 4.1,  $D$  includes at least one of  $l_1, l_2$ , or  $l_3$ . Assume, without loss of generality, that  $D \supset l_1 \cup l_2$  and that  $D \cap D'$  is a lens.  $D$  includes also  $\text{arc}_{C_3}(B_1B_2)$ .  $D'$  must include only  $l_3$  (and not any of  $l_1, l_2$ ) for otherwise  $D \cap D'$  includes either  $l_1$  or  $l_2$  and therefore is not a lens.

By Claim 4.3,  $\text{int } D'$  meets the relative interior of  $\text{arc}_{C_3}(B_1B_2)$ , implying that also  $\text{int } (D \cap D')$  meets it. This is a contradicting to the assumption that  $D \cap D'$  is a lens. ■

**Claim 4.5.** *Each of  $A_1, A_2$ , and  $A_3$  is included in the interior of some unit disk bounded by a circle from  $\mathcal{C}$ .*

**Proof.** We prove the claim for  $A_1$ . Assume that the claim is not true for  $A_1$ . Let  $C'_2 \in \mathcal{C}$  be the circle so that  $D_2 \setminus \text{int } D'_2$  is a lune. Since  $A_1 \in D'_2$ ,

$A_1 \notin \text{int } D'_2$  implies that  $C'_2$  passes through  $A_1$ . Similarly, let  $C'_3 \in \mathcal{C}$  be the circle so that  $D_3 \setminus \text{int } D'_3$  is a lune, then  $C'_3$  passes through  $A_1$ . However,  $d(A_1) = 3$  and  $C_2, C_3$  pass through  $A_1$ . It follows that  $C'_2 = C'_3$ . Denote  $C' = C'_2 = C'_3$ .  $D'$  includes all lenses  $l_1, l_2, l_3$ . Denote the intersection points of  $C'$  with  $C_2$  and  $C_3$  other than  $A_1$  by  $Y, Z$  respectively (see figure 7).

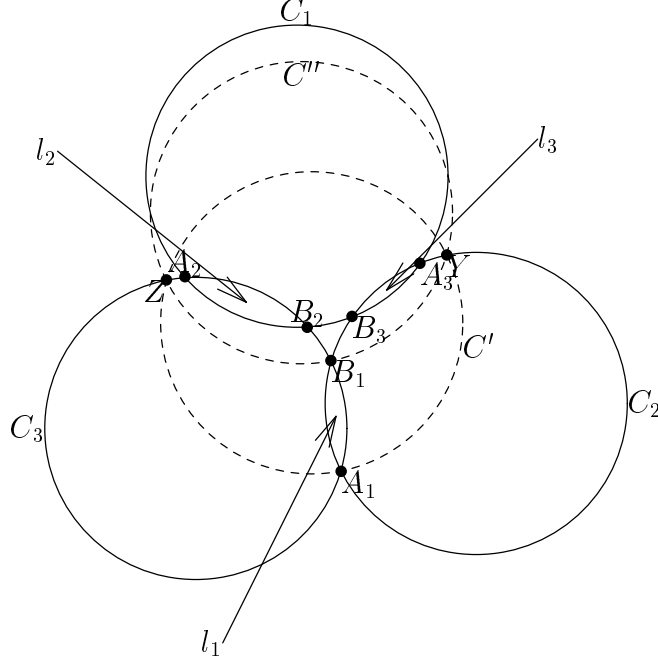


Figure 7:

The face whose two edges are  $\text{arc}_{C_3}(A_1B_1)$  and  $\text{arc}_{C'}(A_1Y)$  must have exactly three edges so there must be a circle (other than  $C_2$ ) in  $\mathcal{C}$  which passes through  $Y$  and  $B_1$ . Similarly, there must be a circle (other than  $C_3$ ) in  $\mathcal{C}$  which passes through  $Z$  and  $B_1$ . Since  $d(B_1) = 3$  and  $C_2$  and  $C_3$  pass through  $B_1$ , it follows that the same circle  $C'' \in \mathcal{C}$  passes through  $Y, B_1$ , and  $Z$ .

Observe that  $Y \neq A_3$  for otherwise there would be four circles through this point, namely  $C_1, C_2, C'$ , and  $C''$ . It is easily seen now that the face whose edge is  $\text{arc}_{C''}(YB_1)$  which is adjacent to the face  $A_1YB_1$  has more than three edges (as no other circles pass through  $Y$  nor  $B_1$ ), a contradiction. ■

Let  $\Delta_0$  denote the closed region which is bounded by the arcs  $\text{arc}_{C_1}(B_2B_3)$ ,  $\text{arc}_{C_2}(B_3B_1)$ , and  $\text{arc}_{C_3}(B_1B_2)$ .

**Lemma 4.6.** *Let  $C \in \mathcal{C}$ . Then  $D \cap \Delta_0 \cap \overline{B_1A_1} \neq \emptyset$ . (Similarly for  $\overline{B_2A_2}$  and  $\overline{B_3A_3}$ .)*

**Proof.** If  $D \supset l_1$  then clearly  $B_1 \in D \cap \Delta_0 \cap \overline{B_1A_1}$ . If  $D \supset (l_2 \cup l_3)$  then  $D$  includes  $\text{arc}_{C_1}(B_2B_3)$ . Therefore,  $D \cap \Delta_0 \cap \overline{B_1A_1}$  includes the intersection point of  $\text{arc}_{C_1}(B_2B_3)$  with  $\overline{B_1A_1}$  (indeed, this intersection point exists because  $\overline{B_1A_1}$  separates  $l_2$  and  $l_3$ ).

The only case which is left to check is where  $D$  includes only one of  $l_2$  or  $l_3$ , and not  $l_1$ . Assume, without loss of generality, that  $D$  includes  $l_3$  and not  $l_1$  or  $l_2$ . By Claim 4.3,  $D$  includes a subarc of  $\text{arc}_{C_3}(B_1B_2)$ . Since  $D$  is convex,  $D$  includes an interval connecting  $B_3$  to a point on  $\text{arc}_{C_3}(B_1B_2)$ . This interval intersects  $\overline{B_1A_1}$  inside  $\Delta_0$ . ■

**Claim 4.7.** *For each of the lenses  $l_1, l_2$ , and  $l_3$  there is a circle  $C \in \mathcal{C}$  such that  $D$  includes that lens and not the two others.*

**Proof.** We prove the claim for  $l_1$ . Let  $Q$  be the intersection point of a circle from  $\mathcal{C}$  and the ray  $\overrightarrow{B_1A_1}$ , which has maximum distance from  $B_1$ . One of the circles  $C$  through  $Q$  includes an edge of the unbounded face. By Claim 4.5,  $A_1$  is included in the interior of the disk bounded by some circle from  $\mathcal{C}$  and hence  $Q \neq A_1$ . Consequently,  $C$  is not one of  $C_1, C_2, C_3$ . We show that  $D$  includes  $l_1$  and not any of  $l_2$  or  $l_3$ .

By Lemma 4.6,  $D \cap \Delta_0 \cap \overline{B_1A_1} \neq \emptyset$ . It follows, from the convexity of  $D$ , that  $D \supset [B_1A_1]$ . Hence  $D \supset l_1$ .

Observe that if  $D$  includes  $l_2$  or  $l_3$ , then, by Claim 4.4,  $C$  does not support a lens. This is a contradiction because  $C$  includes an edge of the unbounded face. ■

For  $i = 1, 2, 3$  let  $C'_i \in \mathcal{C}$  be a circle which is guaranteed by Claim 4.7, for  $l_i$ . Thus,  $D'_i$  includes  $l_i$  alone from  $l_1, l_2$ , and  $l_3$ .

Denote by  $\alpha_1, \alpha_2, \alpha_3$  the lengths of the arcs  $C_1 \cap D_2, C_2 \cap D_3$ , and  $C_3 \cap D_1$  respectively. Denote by  $\theta_1, \theta_2, \theta_3$  the lengths of  $\text{arc}_{C_1}(B_2B_3), \text{arc}_{C_2}(B_3B_1)$  and  $\text{arc}_{C_3}(B_1B_2)$  respectively.

**Claim 4.8.** *For every  $1 \leq i \leq 3$ ,  $\theta_i < \frac{\pi}{3}$ .*

**Proof.** Reduce to the extrem case where  $C_1, C_2$ , and  $C_3$  touch each other. ■

**Claim 4.9.**  *$C'_1$  meets  $C_2$  at a point  $F_{12} \in \text{arc}_{C_2}(B_3B_1)$ .  $C'_1$  meets  $C_3$  at a point  $F_{13} \in \text{arc}_{C_3}(B_1B_2)$ . Moreover,*

$$|\text{arc}_{C'_1}(F_{12}F_{13})| \leq \max(\theta_1, \theta_2, \theta_3).$$

**Remark.** By simetry Claim 4.9 is true if we replace  $C'_1, C_2, C_3$  respectively by  $C'_2, C_1, C_3$ , or by  $C'_3, C_1, C_2$ . In this way we analogously define the points  $F_{21} \in \text{arc}_{C'_1}(B_2B_3)$  and  $F_{23} \in \text{arc}_{C'_3}(B_1B_2)$  on  $C'_2$  as well as  $F_{31} \in \text{arc}_{C_1}(B_2B_3)$  and  $F_{32} \in \text{arc}_{C_2}(B_3B_1)$  on  $C'_3$ .

**Proof.**  $D'_1$  includes  $l_1$  and is disjoint from the interiors of both  $l_2$  and  $l_3$ . Therefore  $C'_1$  separates the interior of  $l_1$  from the interior of  $l_3$  and thus meets  $\text{arc}_{C_2}(B_3B_1)$  at a point which we denote by  $F_{12}$ . Similarly,  $C'_1$  separates the interior of  $l_1$  from the interior of  $l_2$  and thus meets  $\text{arc}_{C_3}(B_1B_2)$  at a point which we denote by  $F_{13}$  (see Figure 8).

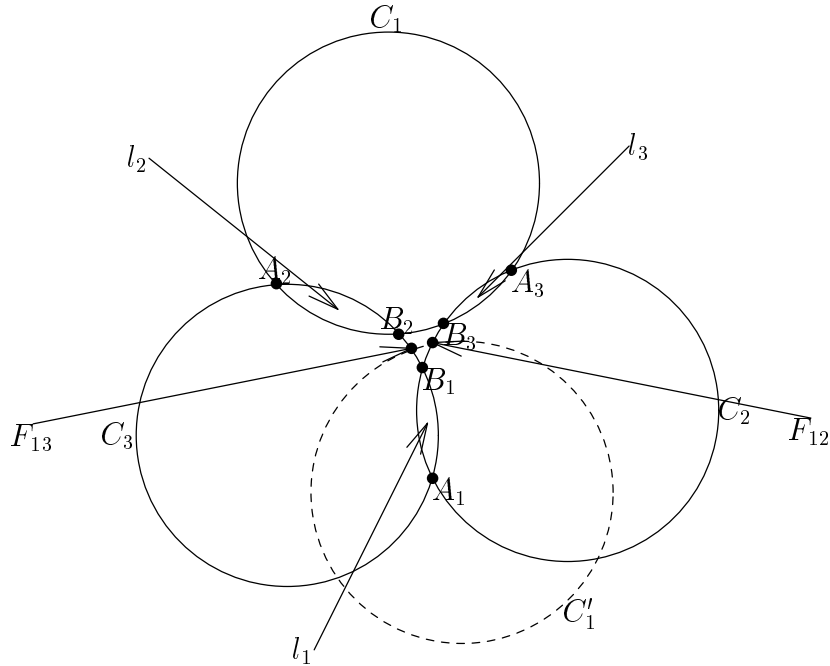


Figure 8: Claim 4.9

The length of  $\text{arc}_{C'_1}(F_{12}F_{13})$  depends monotoniously on the length of the interval  $[F_{12}F_{13}]$ . Observe that  $[F_{12}F_{13}]$  is included in  $\Delta(B_1B_2B_3)$  and therefore the length of  $[F_{12}F_{13}]$  is less or equal to the maximum of the lengthes of the intervals  $[B_3B_1], [B_1B_2]$ , and  $[B_2B_3]$ . Consequently, the length of  $\text{arc}_{C'_1}(F_{12}F_{13})$  is at most  $\max(\theta_1, \theta_2, \theta_3)$ . ■

Let  $\theta'_1$  denote the length of  $\text{arc}_{C'_1}(F_{12}F_{13})$  guaranteed by Claim 4.9. Similarly, denote  $\theta'_2 = |\text{arc}_{C'_2}(F_{21}F_{23})|$  and  $\theta'_3 = |\text{arc}_{C'_3}(F_{31}F_{32})|$ .

Let  $F'_{12}$  denote the second intersection point of  $C'_1$  and  $C_2$  (other than

$F_{12}$ ). Let  $F'_{13}$  denote the second intersection point of  $C'_1$  and  $C_3$ . Similarly, define  $F'_{21}$  and  $F'_{23}$  to be the additional intersections of  $C'_2$  with  $C_1$  and  $C_3$  respectively. Finally, let  $F'_{31}$  and  $F'_{32}$  be the additional intersections of  $C'_3$  with  $C_1$  and  $C_2$  respectively.

$D'_1 \supset D_2 \cap D_3$ . Therefore, using Lemma 3.17 with  $C'_1, C_2$ , and  $C_3$ , We obtain

$$|\text{arc}_{C_2}(B_1F'_{12})| + |\text{arc}_{C_3}(B_1F'_{13})| = \pi + \alpha_1 - \theta'_1. \quad (21)$$

Similarly, using Lemma 3.17 twice more, once with  $C'_2, C_1, C_3$  and once with  $C'_3, C_1, C_2$ , we obtain

$$|\text{arc}_{C_1}(B_2F'_{21})| + |\text{arc}_{C_3}(B_2F'_{23})| = \pi + \alpha_2 - \theta'_2, \quad (22)$$

$$|\text{arc}_{C_1}(B_3F'_{31})| + |\text{arc}_{C_2}(B_3F'_{32})| = \pi + \alpha_3 - \theta'_3. \quad (23)$$

Since  $C_1$  supports the longer arc of a lune,  $\text{arc}_{C_1}(B_2F'_{21}), \text{arc}_{C_1}(B_3F'_{31})$ , and  $\text{arc}_{C_1}(B_2B_3)$  are included in an open semicircle on  $C_1$ . Therefore,

$$|\text{arc}_{C_1}(B_2F'_{21})| + |\text{arc}_{C_1}(B_3F'_{31})| + |\text{arc}_{C_1}(B_2B_3)| < \pi. \quad (24)$$

Similar argument for  $C_2$  and  $C_3$  gives

$$|\text{arc}_{C_2}(B_3F'_{32})| + |\text{arc}_{C_2}(B_1F'_{12})| + |\text{arc}_{C_2}(B_3B_1)| < \pi, \quad (25)$$

$$|\text{arc}_{C_3}(B_1F'_{13})| + |\text{arc}_{C_3}(B_2F'_{23})| + |\text{arc}_{C_3}(B_1B_2)| < \pi. \quad (26)$$

Summing up 24, 25, and 26, we get

$$\begin{aligned} 3\pi &> (|\text{arc}_{C_2}(B_1F'_{12})| + |\text{arc}_{C_3}(B_1F'_{13})|) + \\ &+ (|\text{arc}_{C_1}(B_2F'_{21})| + |\text{arc}_{C_3}(B_2F'_{23})|) + \\ &+ (|\text{arc}_{C_1}(B_3F'_{31})| + |\text{arc}_{C_2}(B_3F'_{32})|) + \\ &+ (\theta_1 + \theta_2 + \theta_3). \end{aligned} \quad (27)$$

In view of 21, 22, and 23, the right side of 27 equals

$$\begin{aligned} (\pi + \alpha_1 - \theta'_1) + (\pi + \alpha_2 - \theta'_2) + (\pi + \alpha_3 - \theta'_3) + (\theta_1 + \theta_2 + \theta_3) = \\ = 3\pi + (\alpha_1 + \alpha_2 + \alpha_3 + \theta_1 + \theta_2 + \theta_3) - (\theta'_1 + \theta'_2 + \theta'_3). \end{aligned} \quad (28)$$

Combining 27 and 28, gives



$$\theta'_1 + \theta'_2 + \theta'_3 > \alpha_1 + \alpha_2 + \alpha_3 + \theta_1 + \theta_2 + \theta_3. \quad (29)$$

Observe that the right side of 29 equals to the sum of the internal angles of  $\Delta(O_1O_2O_3)$  which is  $\pi$ .

By Claim 4.9, the left side of 29 is less or equal to  $3 \max(\theta_1, \theta_2, \theta_3)$  which in turn is, by Claim 4.8, less or equal to  $\pi$ . We thus reached the desired contradiction. ■

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