The minimum number of distinct areas of triangles determined by a set of n points in the plane

Rom Pinchasi
Israel Institute of Technology, Technion[1]

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Abstract

We prove a conjecture of Erdős, Purdy, and Straus on the number of distinct areas of triangles determined by a set of n points in the plane. We show that if P is a set of n points in the plane, not all on one line, then P determines at least $\lfloor\frac{n-1}{2}\rfloor$ triangles with pairwise distinct areas. Moreover, one can find such $\lfloor\frac{n-1}{2}\rfloor$ triangles all sharing a common edge.

1 Introduction

Let P be a set of n points in the plane. We consider all the triangles whose vertices are any three non-collinear points of P. We regard these triangles as triangles determined by the set P. Denote by $g(P)$ the number of distinct areas of triangles determined by P. Clearly, $g(P)$ can be as large as $\binom{n}{3}$ if P is in a general enough position. We are interested in the minimum possible value of $g(P)$. Denote by $g(n)$ the minimum possible value of $g(P)$ over all non-collinear sets P of n points in the plane.

The function $g(n)$ was first investigated by Erdős and Purdy ([EP77]). They showed that there exist positive absolute constants c_1, c_2 such that for every n we have

$$c_1 n^{3/4} \leq g(n) \leq c_2 n.$$

It is easy to observe an $\lfloor\frac{n-1}{2}\rfloor$ upper bound for $g(n)$ by taking P to be a set of equally spaced $\lfloor\frac{n}{2}\rfloor$ points on a line ℓ together with $\lceil\frac{n}{2}\rceil$ equally spaced points on a line ℓ' parallel to ℓ (see [BP79] and [S78]).

Erdős, Purdy, and Straus conjectured the following:

Conjecture 1.1. *For every n, $g(n) \geq \lfloor\frac{n-1}{2}\rfloor$.*

[1] e-mail: room@math.technion.ac.il

The problem of bounding the function $g(n)$ from below is listed for instance in the problem collection by Croft, Falconer, and Guy ([CFG91]) as well as in the problem collection by Brass, Moser, and Pach ([BMP05]).

A linear lower bound for $g(n)$ was obtained by Burton and Purdy in 1979 ([BP79]). They showed $g(n) \geq 0.32n$ by relating this problem to another problem on the minimum number of distinct directions determined by a non-collinear point set of size n. A complete solution of the latter problem was found only a little later in 1982 by Ungar ([U82]). Combining the result in [U82] with the technique of [BP79], one obtains the lower bound $g(n) \geq (\sqrt{2} - 1)n - \mathcal{O}(1)$.

Recently, Dumitrescu and Cs. Tóth ([DT]) improved the lower bound and showed $g(n) \geq \frac{17}{38}n - \mathcal{O}(1) \approx 0.4473n$.

In this paper we prove Conjecture 1.1.

Theorem 1.2. *The number of distinct areas of triangles determined by a non-collinear point set of size n is at least $\lfloor \frac{n-1}{2} \rfloor$.*

We note that all the previous mentioned results as well as the result in Theorem 1.2 are in fact a little stronger. The lower bounds refer even to triangles with distinct areas all sharing a common edge.

2 The Proof

We show the existence of a line γ determined by at least two points of the set P such that there are at least $\lfloor \frac{n-1}{2} \rfloor$ distinct positive distances of points of P from γ. This will clearly prove the theorem.

It will be more convenient for us to consider the dual problem in the plane. That is, we perform a standard duality transform which takes a point (a, b) to the line $y + ax + b = 0$ in the plane, and it takes a non-vertical line $y + a'x + b' = 0$ to the point (a', b').

It is easy to observe that if a point p lies on a line L, then the dual of p is a line containing the point which is the dual of L. It is also easy to see that two points with the same x-coordinate in the dual plane represent the dual of two parallel lines in the primal plane. Finally, observe that if a line $y + a'x + b' = 0$ lies above a point (a, b) (that is, if $b + a'a + b' < 0$), then the point (a', b'), dual to the line, lies below the line $y + ax + b = 0$, which is the dual of (a, b).

Therefore, we rotate the set P, if necessary, so that no two points of P have the same x-coordinate and then the dual of P is a collection $\mathcal{M}$ of n lines no two of which are parallel and not all passing through the same point (for otherwise the points of P are all collinear). Consider now an intersection point q of two or more lines in $\mathcal{M}$. q is the dual of a line γ containing two or more points of P. Draw a vertical line r through q and let $x_1, \ldots, x_k$ be all the intersection points of r with lines in $\mathcal{M}$, which lie above q on r. For each $i = 1, \ldots, k$ let ℓ_i be a line in $\mathcal{M}$ passing through x_i. Assume that for every $1 \leq i \leq k$, ℓ_i is the dual of the point $p_i \in P$. Then the points $p_1, \ldots, p_k$ lie strictly below γ, because the lines $\ell_1, \ldots, \ell_k$

lie strictly above q. Moreover, for every $1 \leq i < j \leq k$, the line through p_i and p_j cannot be parallel to γ, because ℓ_i and ℓ_j do not meet on r. It follows that the distances from $p_1, \ldots, p_k$ to γ are distinct. A similar scenario holds if we consider the intersection points of r with lines in $\mathcal{M}$, which are all strictly below q on r.

Theorem 1.2 therefore follows from the following theorem.

Theorem 2.1. *Let $\mathcal{A}$ be an arrangement of n blue non-vertical lines in the plane such that no two of the lines are parallel, and not all the lines pass through the same point. Draw vertical red lines such that each red line passes through an intersection point of (at least) two blue lines, and every intersection point of two (or more) blue lines lies on a red line. Then there is an intersection point p of at least two blue lines, and a red line ℓ through p such that either there are at least $\lfloor \frac{n-1}{2} \rfloor$ different intersection points of ℓ with blue lines such that all are above p on ℓ, or there are at least $\lfloor \frac{n-1}{2} \rfloor$ different intersection points of ℓ with blue lines, all lie below p on ℓ.*

Proof. Let $\mathcal{L}$ denote the collection of red lines, and let $m = |\mathcal{L}|$.

For every line $\ell \in \mathcal{L}$ let $b(\ell)$ denote the intersection point of (at least) two blue lines on ℓ with the least y-coordinate. Similarly, let $t(\ell)$ denote the intersection point of (at least) two blue lines on ℓ with the largest y-coordinate (it is possible that $b(\ell) = t(\ell)$). Denote by $S(\ell)$ the set of all intersection points of ℓ with blue lines that are either strictly below $b(\ell)$, or strictly above $t(\ell)$. Denote by $Q(\ell)$ the set of all intersection points of ℓ with a single blue line that are strictly above $b(\ell)$ and strictly below $t(\ell)$. Finally, denote by $R(\ell)$ the set of all intersection points of (at least) two blue lines on ℓ.

Observe that there are at least $S(\ell)/2 + Q(\ell) + R(\ell) - 1$ intersection points on ℓ that are either all above $b(\ell)$ or all below $t(\ell)$.

Let $S = \cup_{\ell \in \mathcal{L}} S(\ell)$, $Q = \cup_{\ell \in \mathcal{L}} Q(\ell)$, and $R = \cup_{\ell \in \mathcal{L}} R(\ell)$.

Let p and q be two intersection points in $R(\ell)$ for some red line ℓ. We say that p and q are adjacent if there is no point $s \in R(\ell)$ on the open line segment between p and q (on ℓ). Since there are exactly m lines and each includes at least one point from R, we deduce that there are exactly $|R| - m$ pairs of adjacent intersection points from R. Let p and q be a pair of adjacent intersection points on a red line ℓ. We say that the pair (p, q) is *nice* if there is no blue line separating between p and q. Consequently, if p and q are adjacent but (p, q) is not a nice pair, then there must be a point in Q on the open line segment of ℓ between p and q. Therefore, the number of nice pairs of adjacent intersection points is at least $|R| - m - |Q|$.

Consider now the arrangement $\mathcal{A}$ of the blue lines only. We denote, as usual, by f_k the number of faces in the arrangement with precisely k edges. It is important to note that we consider the arrangement $\mathcal{A}$ on the *projective* plane, and so we identify the two ends at infinity for each of the blue lines. For $k \geq 2$ denote by t_k the number of intersection points through which there are precisely k blue lines. Observe that since not all the blue lines pass through the same point, there is no face with two edges in $\mathcal{A}$. Recall (see for example [F04] p.73) that from Euler's formula it follows that

$$3 + \sum_{k \geq 2}(k-3)t_k + \sum_{k \geq 3}(k-3)f_k = 0. \tag{1}$$

One crucial observation is that if (p, q) is a nice pair, then p and q are vertices of the same face in $\mathcal{A}$. Moreover the line segment between p and q is a diagonal of that face. For a face F with k edges the maximum possible number of diagonals in the vertical direction is $\lfloor \frac{k-2}{2} \rfloor$. Observe that this number is always less than or equal to $k - 3$, as long as $k \geq 3$ which is true in our case. It follows that the maximum number of nice pairs (p, q) is at most $\sum_{k \geq 3}(k-3)f_k$. This together with the lower bound on the number of nice pairs gives:

$$|R| - m - |Q| \leq \sum_{k \geq 3}(k-3)f_k. \tag{2}$$

Observe also that by definition:

$$|R| = \sum_{k \geq 2} t_k. \tag{3}$$

(2) together with (3) give:

$$|Q| \geq \sum_{k \geq 2} t_k - m - \sum_{k \geq 3}(k-3)f_k. \tag{4}$$

We are now ready to complete the proof. Since every red line crosses each of the n blue lines we conclude that $|S| + |Q| = mn - \sum_{k \geq 2} kt_k$.

Therefore,

$$\begin{aligned}
\frac{|S|}{2} + |Q| + |R| &= \frac{|S| + |Q|}{2} + \frac{|Q|}{2} + |R| \geq \\
&\geq \frac{1}{2}(mn - \sum_{k \geq 2} kt_k + \sum_{k \geq 2} t_k - m - \sum_{k \geq 3}(k-3)f_k) + \sum_{k \geq 2} t_k = \\
&= \frac{1}{2}(m(n-1) - (\sum_{k \geq 2}(k-3)t_k + \sum_{k \geq 3}(k-3)f_k)) = \frac{m(n-1)+3}{2}
\end{aligned}$$

Since there are precisely m red lines, it follows by averaging that there exists a red line ℓ such that

$$\frac{|S(\ell)|}{2} + |Q(\ell)| + |R(\ell)| \geq \frac{n-1}{2} + \frac{3}{2m}.$$

It follows that either there are at least $\lceil \frac{n-1}{2} + \frac{3}{m} - 1 \rceil = \lfloor \frac{n-1}{2} \rfloor$ intersection points of ℓ with blue lines all lie below $t(\ell)$, or at least this many intersection points of blue lines with ℓ all lie above $b(\ell)$. ∎

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