Linear Algebra Approach to Geometric Graphs

Rom Pinchasi[1]
Israel Institute of Technology, Technion

October 29, 2006

Abstract

We introduce a new linear algebra approach for studying extremal problems in geometric graphs. We give alternative proofs to well-established facts on geometric graphs, as well as new results about triangulations.

1 Preliminaries

Definition 1.1. *Let G be a simple graph. We denote by $\hat{G}$ the* line-graph *of G, that is the graph $\hat{G}$ whose vertices are the edges of G. Two (different) vertices in $\hat{G}$ are connected by an edge, if the corresponding two edges in G share a common vertex.*

Notation 1.2. *For a graph G, we denote by $A(G)$ the adjacency matrix of G. The rank of $A(G)$ is then called the rank of G, over the appropriate field which we usually denote by $\mathbb{F}$.*

Theorem 1.3. *Let G be a connected graph with n vertices. Let $\mathbb{F}$ be a field of characteristic 2. The rank of $\hat{G}$ over $\mathbb{F}$ is $n-1$ if n is odd, and $n-2$ if n is even.*

Proof. Let T be a spanning tree of G and let K_n be the complete graph on n vertices. Since the edges in T are a subset the edges in G which in turn are a subset of the edges of K_n , the rank of $\hat{G}$ is greater than or equal to the rank of $\hat{T}$, and is less than or equal to the rank of $\hat{K_n}$. It is therefore, enough to show that the theorem holds for the complete graph on n vertices as well as for any tree on n vertices.

Assume first that G is a tree on n vertices. Therefore, G has exactly $n-1$ edges. A vector in the kernel of $A(\hat{G})$, is an assignment of weights (from the field $\mathbb{F}$) to the edges of G, such that the sum of the weights of all edges adjacent to any given edge is 0. Assume that there exists such an assignment of weights to the edges of G. Let e be any edge of G and

[1]e-mail: room@math.technion.ac.il,
This research was supported by a Grant from the G.I.F., the German-Israeli Foundation for Scientific Research and Development.

let u, v be its two vertices. As $\mathbb{F}$ has characteristic 2, it follows that the sum of the weights of all edges adjacent to v equals the sum of the weights of all edges adjacent to u. Since G is connected, we conclude that there exists $a \in \mathbb{F}$ such that the sum of the weights of all edges adjacent to any given vertex in G, is a. In particular, all the edges going out from a leaf of G are assigned the weight a. By removing these leaves and arguing similarly on the new leaves, we can recover exactly all the weight assigned to all the edges of G. This shows that the kernel of $A(\hat{G})$ has dimension at most 1. Therefore, the rank of $\hat{G}$ over $\mathbb{F}$ is at least $n-2$ (and at most $n-1$ as G is a tree).

more specifically, we now show that if n is odd, then necessarily the dimension of $ker(A(\hat{G})) = \{0\}$ thus implying that the rank of $\hat{G}$ over $\mathbb{F}$ is $n-1$. Indeed, assume there is a nontrivial assignment of weights to the edges of G as above. We already know that the sum of all weights of the edges adjacent to any given vertex is some constant a. If $a = 0$, then the weight of each edge of G adjacent to a leaf is 0 and by induction on the size of the tree all weights are 0. Thus we may assume that $a \neq 0$. In this case na is the sum over all vertices in G of the sum of all the weights of the edges adjacent to these vertices. On the other hand each edges contributes its weight twice to this sum and therefore this sum must be 0 ($char(F) = 2$). We get a contradiction when n is odd.

Assume next that G is the complete graph on n vertices. We will show that the dimension of the kernel of $A(\hat{G})$ is at least $\binom{n-1}{2}$. Indeed, fix a vertex x of G and for every two other vertices u, v assign weight 1 to the edges (x, u), (x, v), and (u, v). Assign weight 0 to all other edges. It is easy to see that this assignment gives a vector in the kernel of $A(\hat{G})$ (as a matrix over $\mathbb{F}$). As these vectors are linearly independent, this shows that the dimension of the kernel of $A(\hat{G})$ is at least $\binom{n-1}{2}$ and thus the rank of $A(\hat{G})$ is at most $\binom{n}{2} - \binom{n-1}{2} = n-1$. Therefore, it is equal to $n-1$ when n is odd, as it is greater than or equal to the rank of $\hat{T}$ where T is any spanning tree of G (this rank is $n-1$ by the first part of the proof) .

If n is even, then the assignment which gives every edge the weight 1 is also in the kernel and independent of the previous kernel vectors (as it is not their sum). Since we know already that every tree T on n the rank of $\hat{T}$ is greater than or equal to $n-2$, it follows that if n is even,

$$n - 2 \leq \text{rank of } \hat{T} \leq \text{rank of } \hat{K_n} \leq n - 2.$$

Therefore we have equality.∎

Given a graph G on n vertices, define $R(G)$ to be the $n \times n$ matrix which equals the adjacency matrix of G plus the diagonal matrix D where the entry d_{ii} is the degree of the i'th vertex of G minus 2.

Before presenting the next theorem we will need a simple well known lemma usually referred to as an exercise in linear algebra:

Lemma 1.4. *Let B be an m by n matrix over a field $\mathbb{F}$. Then BB^t and B^tB have the same nonzero eigenvalues and each common nonzero eigenvalue has the same geometric multiplicity in both matrices.*

Theorem 1.5. *Let G be a graph on n vertices with E edges. Then the rank of $A(\hat{G})$ over a field $\mathbb{F}$ with $char(F) \neq 2$ is $E - n + rank(R(G))$.*

Proof. Let B be the $E \times n$ edge-vertex incidence matrix of the graph G. Then it is easy to see that $A(\hat{G}) = BB^t - 2I$ and $R(G) = B^tB - 2I$. Therefore, the rank of $\hat{G}$ is E minus the geometric multiplicity of the eigenvalue 2 of BB^t. But by Lemma 1.4, this equals the geometric multiplicity of the eigenvalue 2 in the matrix B^tB (here we use the fact that $2 \neq 0$ in $\mathbb{F}$). In other words, the rank of $\hat{G}$ equals E minus the dimension of the kernel of $R(G)$. This is equivalent to the statement in the theorem. ∎

Remark. Observe that the rank of $R(G)$ is at least 1, since $R(G)$ can never be the zero matrix. Therefore we conclude that the rank of $\hat{G}$ over a field $\mathbb{F}$ with $char(F) \neq 2$ is at least $E - n + 1$. The example of $G = K_4$ shows that this is best possible. (By K_n we denote the complete graph on n vertices.)

Corollary 1.6. *The rank of $\hat{K_n}$ over a field $\mathbb{F}$ with $char(F) \neq 2$ is $\binom{n}{2}$ (that is, largest possible), assuming that $n-2 \neq 0$ and $n-4 \neq 0$. If $n-4 \neq 0$ but $n-2=0$, then the rank of $\hat{K_n}$ is $\binom{n}{2} - 1$.*

Proof. In this case $G = K_n$, $E = \binom{n}{2}$, and the matrix $R(G)$ equals $(n-4)I + J$, where I is the identity matrix and J is the all 1's matrix. Therefore, the rank of $R(G)$ is n, if $n-2 \neq 0$ and $n-4 \neq 0$. It follows from Theorem 1.5 that in this case the rank of $\hat{G}$ is $\binom{n}{2}$.

If $n-4 \neq 0$ but $n-2=0$, then the rank of $R(G)$ is $n-1$, and it follows from Theorem 1.5 that rank of $\hat{G}$ equals $\binom{n}{2} - 1$. ∎

2 The matrix M of a geometric graph

A *geometric graph* is a graph drawn in the plane with its vertices drawn as points (usually in general position) and its edges drawn as straight line segments connecting corresponding points. A *convex* geometric graph is a geometric graph whose vertices are in strictly convex position.

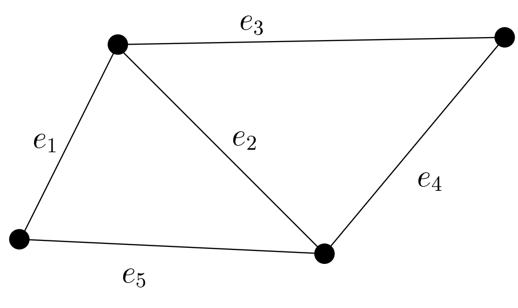

Figure 1: a geometric graph G

Assume that the plane is equipped with a Cartesian coordinate system. Let e be an edge of a geometric graph, and let a and b be the two vertices of e. If the x-coordinate of a is smaller than the x-coordinate of b, then a will be called the *left vertex* of e and b will be called the *right vertex* of e. Two edges e and f are said to share a common right (resp. left) vertex if they share a common vertex which is the right (resp. left) vertex for both edges.

Let G be a geometric graph with n vertices and E edges. For simplicity, we assume that no two edges of G are parallel. This can easily be achieved by a suitable projective transformation that does not change the combinatorial properties of G. We may also assume that no two vertices of G have the same x-coordinate. This can be done by a suitable rotation of the graph G in the plane. In what follows, by a *slope* of an edge e of a geometric graph, we mean to the slope of the line that includes e, with respect to the given Cartesian coordinate system. We will now define a matrix called $M = M(G)$ as follows: The rows and the columns of $M(G)$ will correspond to the edges of G. For e and e' two different edges of G we set $M_{ee'} = 1$ if e and e' share a common left vertex and the slope of e is greater than the slope of e'. We set $M_{ee'} = 1$ also if e and e' share a common right vertex and the slope of e is greater than the slope of e'.

We set $M_{ee'} = -1$ if e and e' share a common left vertex and the slope of e is smaller than the slope of e'. We set $M_{ee'} = -1$ also when e and e' share a common right vertex and the slope of e is smaller than the slope of e'.

In all other cases we set $M_{ee'} = 0$. For example the matrix $M(G)$ of the graph G drawn in Figure 1 is:

$$M(G) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Lemma 2.1. *Let G be a geometric graph with E edges and n vertices. The rank of $M(G)$ over the field of real numbers is at least $E - 2n + 2$.*

Proof. Let the vertices of G be $v_1, \ldots, v_n$ ordered according to the value of their x-coordinate. A vector in the kernel of M is an assignment of weights to the edges of G in such a way that for every edge e the sum of the weights of all edges which share a common right vertex or left vertex with e and have a smaller slope than e equals the sum of the weights of all edges which share a common right vertex or left vertex with e and have a larger slope than e.

Such a vector can be reconstructed from the $2n$ numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, where a_i is the sum of the weights of the edges whose right vertex is v_i and b_i is the sum of the weights of the edges whose left vertex is v_i. Indeed, let e be the steepest edge in G. It follows that the sum of the weights of all edges sharing a common left vertex or right vertex with e is zero. Therefore, if v_j is the left vertex of e and v_k is the right vertex of e, then $b_j + a_k$ equals twice the weight of e. We can continue this way and find the weight of the second steepest edge and so on.

Since we always have $a_1 = 0$ and $b_n = 0$, this shows that the dimension of the kernel of M is at most $2n - 2$ and therefore the rank of M is at least $E - 2n + 2$. ∎

In the case where the set of vertices of G lies in convex position, we have the following improvement to Lemma 2.1. Before stating it we introduce a definition that will serve us also in the sequel.

Definition 2.2. *Let G be a geometric graph in the plane that is equipped with a Cartesian coordinate system.*

$\widetilde{G}$ will denote a geometric graph obtained by the following modification of G. We split every vertex v of G to v_R and v_L, except the rightmost and the leftmost vertices of G. v_R and v_L will be positioned very close to each other. The set of edges of $\widetilde{G}$ remains the same as that of G, where v_R inherits all the edges the left vertex of which is v. Similarly, v_L inherits those edges the right vertex of which is v. It is easy to see that $\widetilde{G}$ is a bipartite graph and that $M(G) = M(\widetilde{G})$. (See Figure 2.) Those vertices of $\widetilde{G}$ that serve as the right vertices of edges we call vertices of type R. The vertices of $\widetilde{G}$ that serve as left vertices of edges will be called vertices of type L.

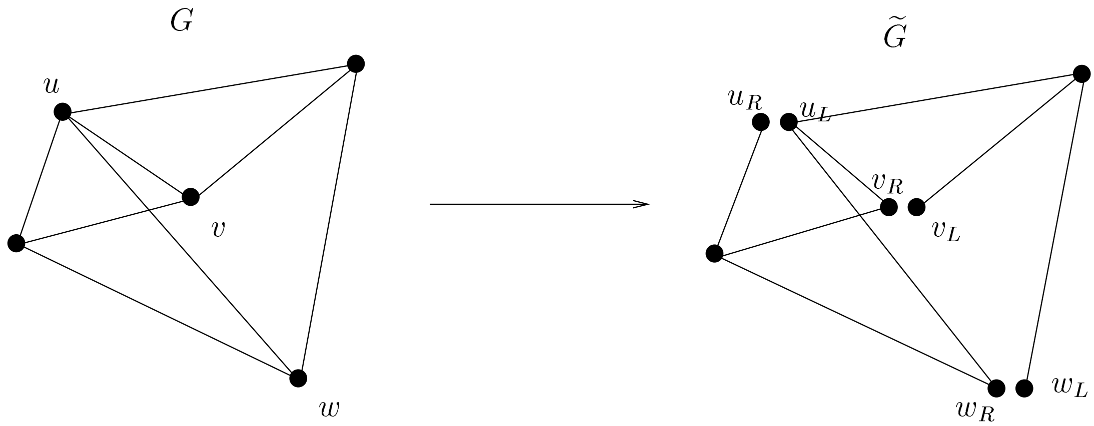

Figure 2: G and $\widetilde{G}$

Lemma 2.3. *Let G be a convex geometric graph with E edges and n vertices. The rank of $M(G)$ over the field of real numbers is at least $E - n + 1$.*

Proof. Since $M(\widetilde{G}) = M(G)$, it is enough to show the lemma for $H = \widetilde{G}$. A vector in the kernel of $M(H)$ is an assignment of weights to the edges of H in such a way that for every edge e with vertices a and b, the sum of the weight of those edges incident either to a or to b whose slope is strictly larger than that of e, minus the sum of the weights of those edges incident either to a or to b whose slope is strictly smaller than that of e, is equal to zero.

H is a bipartite graph on $n-1$ vertices of type R and $n-1$ vertices of type L. For every vertex v of H, let S_v denote the sum of the weights of all edges incident to v. We show that the weights assigned to the edges of H, can be recovered from the numbers $\{S_v | v \text{ is of type } L\}$. This will show that the dimension of the kernel of $M(H)$ is at most $n-1$.

Indeed, let x be the leftmost vertex of type R. We first recover the weights of all edges incident to x. Without loss of generality, assume that x belongs to the upper half of the boundary of the convex hull of the vertices of G. That is, the ray going vertically upwards from x does not intersect the convex hull of the vertices of G (the other case is treated similarly). Let $e_1, \ldots, e_k$ be the edges incident to x, ordered according to their increasing slopes. Let y_i denote the left vertex of e_i. Observe that from the condition on the weights of the edges and the convexity of G and because of the minimality of x, we have $S_{y_1} = S_x$. Moreover, we claim that for every $2 \leq i \leq k$ we have $S_{y_i} - S_{y_{i-1}} = -2w(e_{i-1})$. To see this observe that since x is the leftmost vertex of type R and it belongs to the upper half of the boundary of the convex hull of the vertices of G, then for every $1 \leq i \leq k$ the edge e_i is the steepest edge incident to y_i. Therefore, for every $1 \leq i \leq k$ we deduce from the condition on the weights of the edges, applied to $e_i = (y_i, x)$:

$$S_{y_i} - w(e_i) + (w(e_1) + \ldots + w(e_{i-1})) = w(e_{i+1}) + \ldots w(e_k).$$

It now follows that for every $2 \leq i \leq k$ we have $S_{y_i} - S_{y_{i-1}} = -2w(e_{i-1})$. Therefore, we can read the weights of $e_1, \ldots, e_{k-1}$ and eventually, also e_k as we know S_x.

After recovering the weights of all edges incident to x we move to the next leftmost vertex of type R. ∎

Conjecture 2.4. *The bound in Lemma 2.3 is valid for a general geometric graph G.*

As a first illustration of a geometric application of Lemma 2.1, we will now give an alternative proof for the linearity of the maximum number of edges in a geometric graph with n vertices and no crossings. (Clearly such graphs are planar and therefore have at most $3n - 6$ edges as follows from Euler's formula for instance.)

The question of whether one can conclude a linear bound to the number of edges of a planar graph on n vertices without using Euler's formula was raised by Székely in [3] (open problem 5). We now suggest a positive solution to Székely's problem.

Let G be a geometric graph with n vertices, E edges, and no crossings. Let $v_1, \ldots, v_n$ be the vertices of G, ordered according to the value of their x-coordinates (we assume, without loss of generality, that no two vertices have the same x-coordinate). To each edge e of G we assign a vector $W_e \in \mathbb{C}^n$ as follows. If v_j and v_k ($j < k$) are the vertices of e, we let the j'th coordinate of W_e to be the complex number i and the k'th coordinate of W_e to be $-i$. For l different from j and k, we let the l'th coordinate of W_e to be -1 if $j < l < k$ and v_l is above e, The l'th coordinate of W_e will be 1 if $j < l < k$ and v_l is below e. For every other vertex v_l we set the l'th coordinate of W_e to be 0.

The following simple observations are crucial. If e and e' are two edges of G that do not share a common vertex (and of course do not cross), then $< W_e, \overline{W_{e'}} >$ is a real number. If e and e' share a common vertex v, but for one edge v is the right vertex while for the other it is the left vertex, then $< W_e, \overline{W_{e'}} >= -1$ and in particular it is real. If e and e' have the same left vertex or the same right vertex v, then: If the slope of e is greater than the slope of e', then $< W_e, \overline{W_{e'}} >$ has imaginary part equals to 1. If the slope of e is smaller than the slope of e', then $< W_e, \overline{W_{e'}} >$ has imaginary part equals -1.

Let A be the matrix whose rows are W_e. Then $A\overline{A^t}$ is a complex matrix whose imaginary part equals $M(G)$. It is not hard to see that if S and T are real n by n matrices, then the rank of $Q = S + iT$ over $\mathbb{C}$ is at least half of the rank of T (also of S) over $\mathbb{R}$. Indeed, If the rank of Q over $\mathbb{C}$ is k, then the rows of Q lie in a linear subspace generated by k vectors in $\mathbb{C}^n$ over $\mathbb{C}$. But then the rows of T lie in a vector space generated by $2k$ vectors in $\mathbb{R}^n$ over $\mathbb{R}$, that is the $2k$ vectors that are the real and imaginary parts of the k complex vectors generating the rows of Q.

By Lemma 2.1, the rank over $\mathbb{R}$ of the imaginary part of $A\overline{A^t}$, which is nothing else but the matrix $M(G)$, is at least $E - 2n + 2$. Therefore the rank of $A\overline{A^t}$ over $\mathbb{C}$ is at least $\lceil \frac{E-2n+2}{2} \rceil$. However, the rank of A is at most n as it has n columns. We conclude that $E \leq 4n - 2$. ∎

3 A Theorem About Triangulations

Definition 3.1. *Let P be a finite set of points in the plane. A* triangulation *of P is a planar geometric graph G whose vertices are all the points in P such that every edge of the convex hull of P is in G and every face of the planar graph G, other than the unbounded face, is a triangle. (See for example Figure 1.)*

Let P be a set of n points in general position in the plane. Fix a coordinate system in the plane and assume that no two points of P have the same x-coordinate. Let G be a triangulation of P.

Theorem 3.2. *The rank of $M(G)$ (over any field $\mathbb{F}$) is $2n - 4$ regardless of the set P and the triangulation G.*

Proof. We will prove the theorem in two steps. First we assume that the points of P are in convex position. This case is rather easy. To prove the general statement we will use the technique of continuous motion and deduce the general case from the convex case.

A vector in the kernel of M is an assignment of a weight $w_e \in \mathbb{F}$ to every edge e of G in such a way that for every edge e the sum of the weights of all edges which share a common right vertex or left vertex with e and have a smaller slope than e equals the sum of the weights of all edges which share a common right vertex or left vertex with e and have a larger slope than e.

We will prove the theorem by calculating explicitly the dimension of the kernel of M.

Let us assume that we are given a vector in the kernel of M represented by weights assigned to the edges of the graph G in the manner described above. Let v be any vertex which is not the rightmost nor the leftmost. v is a vertex in a triangle Δvxy in the triangulation G such that the x-coordinate of v is between those of x and y. Without loss of generality assume that x is to the left of y and v is above the edge xy. Observe that since G is a triangulation, there are no edges in G whose right endpoint is v and whose slope is greater than that of vx. Similarly, there are no edges in G whose left endpoint is v and whose slope is smaller than that of vy (see Figure 3).

Let W_x^+ denote the sum of the weights of all edges whose left vertex is x and whose slope is greater than the slope of vx. Let W_x^- denote the sum of the weights of all edges whose left vertex is x and whose slope is smaller than the slope of xy. Let W_y^+ denote the sum of the weights of all edges whose right vertex is y and whose slope is smaller than the slope of vy. Let W_y^- denote the sum of the weights of all edges whose right vertex is y and whose slope is greater than the slope of xy.

Let W_v^L denote the sum of the weights of all edges but vx whose right endpoint is v. Let W_v^R denote the sum of the weights of all edges but vy whose left endpoint is v. Let a, b, c denote the weights of vx, xy, yv, respectively (see Figure 3).

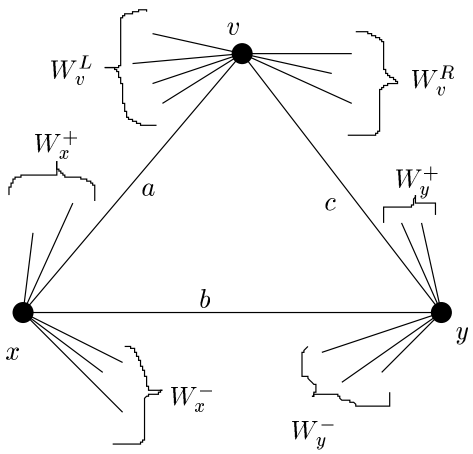

Figure 3: a kernel vector of the matrix M of a triangulation

By the condition on the weights of a vector in the kernel of $M(G)$, applied to the edges vx, xy, and yv, we have:

$$\begin{aligned} W_x^+ - W_x^- - b &= W_v^L \\ W_y^+ + c - W_y^- &= W_x^+ + a - W_x^- \\ W_v^R &= W_y^+ - b - W_y^- . \end{aligned}$$

Summing these three equations we get $W_v^R + c = W_v^L + a$. In other words, the sum of the weights of all edges whose left vertex is v equals the sum of the weights of all edges whose right vertex is v.

For every vertex v, let W_v denote the sum of the weights of all edges whose right (or equivalently left) vertex is v. If v is the rightmost vertex or the leftmost vertex, then W_v just denotes the sum of all weights assigned to edges incident to v. It is easy to see that if u and v are adjacent vertices on the boundary of the convex hull of P, then the condition on the weights, applied to the edge uv implies that $W_u = W_v$.

Assume first that the points of P are in convex position. Then $W_v = c$ for every vertex v of G where $c \in \mathbb{F}$ is an absolute constant independent of v. It is an easy observation, proved by induction for instance, that G (as an outer-planar graph) has at least two vertices with degree 2. We will now prove our theorem in the convex case by induction on n. For $n = 2$ the theorem is clear.

Observe that since P is in convex position, the number of edges of G is exactly $2n-3$. It is thus enough to show that the dimension of the kernel of M equals 1.

We now show that the weights w_e can be reconstructed from c. This will clearly imply that the dimension of the kernel of M is at most 1.

More generally, let w_e $e \in G$ be any assignment of weights to the edges of G. For every vertex v of G, let R_v denote the sum of the weights of the edges whose left endpoint is v. Similarly, let L_v denote the sum of the weight of all edges whose right endpoint is v. We show that the numbers w_e can always be uniquely reconstructed from the numbers R_v, L_v. The only thing we assume is that $\sum_v L_v = \sum_v R_v$.

We prove this by induction on n. This is clearly true when $n = 2$. In the general case, let v be a vertex of G with degree 2. If v is not the rightmost vertex or the leftmost vertex, then the two edges going from v go to opposite x-directions. Therefore, the weights of these edges are exactly R_v and L_v. We can now remove the vertex v and its two adjacent edges and get a new triangulation G' of $P \setminus \{v\}$ and conclude by induction.

If v is the leftmost vertex of G (the case where it is the rightmost vertex is treated similarly), let a and b be its two neighbors in G. Δvab is a triangle in the triangulation G. Without loss of generality assume that the x-coordinate of a is smaller than that of b. Then the edge va is the only edge whose right vertex is a. Therefore, the weight of this edge is L_a and it follows that the weight of the edge vb is $R_v - L_a$. And again we can remove v and its two adjacent edges and conclude by induction.

Therefore, in order to settle the case where the points of P are in convex position, it remains to show that the kernel of M in this case is not trivial. This is however clear from the argument above as we can take $c = 1$ and thus find a nontrivial solution for a kernel vector of weights.

This settles the case where P is in convex position. We now move to discuss the general case. We show by induction on the number of vertices of P that are not extreme in the convex hull, that the dimension of the kernel of M equals 1 plus the number of points of P that are not extreme. This will be enough to prove the theorem as the number of edges of G equals

$$2n - 4 + (1 + \text{the number of points of } P \text{ that are not extreme}).$$

The convex case, that we just settled, forms the basis of induction.

Let x be a vertex which is not extreme. We will consider a continuous motion of the vertex x along the vertical direction downwards, until the first time that x becomes extreme. Throughout the motion we will try to maintain all the edges in the graph. We will have to make some adjustments when an edge is crossed by a vertex, and we may have to change the triangulation, but we will show that as long as the vertex x is not extreme, the dimension of the kernel of M does not change and when x becomes extreme the dimension of the kernel of M reduces by 1.

As long as no vertex crosses an edge through the continuous motion, G is still a triangulation and the matrix M does not change. The only event that might cause G to stop being a triangulation, or change the matrix M is when some edge is crossed by some vertex through the motion of x. Since x is the only vertex that actually moves, it must be that either x crosses some edge of G, or there is an edge, one of whose vertices is x, that is crossed by another vertex.

Assume for instance that when this kind of an event first happens, an edge ab is crossed by a vertex v, and without loss of generality assume that right before the collinearity of a, b, and v, the vertex v is above the edge ab. We claim that in this case the vertices a, b, and v must be the vertices of a triangle in the triangulation G, right before they are collinear. Indeed, the line through a and b does not lie above all other vertices of G (because v is above ab). Therefore ab must be an edge of a triangle Δ whose third vertex c lies above the line through a and b. If $c \neq v$, we get a contradiction because right before a, b, and v are collinear v is inside Δ.

Observe that since initially x is not an extreme vertex on the convex hull of P, then no edge incident to x is an edge of the convex hull of P. This remains true as long as x is not an extreme point of the convex hull of P. We will distinguish between two possible kinds of events that may happen through the continuous motion of x. One is when an internal edge ab is crossed by a vertex v. Observe that in this kind of an event v must be an internal vertex. Indeed, it is clear if $v = x$, and if v is different from x, then x is one of the vertices a and b. If v was extreme, then there is a line that separates v from the other points of P. Right before a, b, and v become collinear, we see that both a and b must be extreme vertices, a contradiction because x is one of the vertices a and b and we stop the motion in the first time that x becomes extreme.

The other event that may happen is when an external edge ab is crossed by a vertex v. Observe that in this case v must be the vertex x.

Consider the first kind of an event. Let v be an internal vertex that moves vertically, say downwards, and crosses an internal edge ab (we assume that a is to the left of b). As we observed earlier, v must be connected by edges to both a and b. Let u be the vertex below the line $\overline{ab}$ which forms a triangle in G together with a and b. As soon as v crosses ab we will delete the edge ab and add the edge uv to obtain a new triangulation G'. G' inherits all other edges of G. We now show a one to one correspondence between the kernel of M and that of $M' = M(G')$. Let $\{w_f\}_{f \in G}$ be the weights of the edges of G which form a vector in the kernel of M. Let $e' = uv$ be the new edge. Assume that the x-coordinate of u is smaller than that of v (see Figure 4). Define $w_{e'} = -w_e$ and add the weight w_e to w_{av} and also to w_{ub}. It is easy to see that the new weights form a vector in the kernel of M'. Indeed, we will verify that the conditions on the weights as forming a vector in the kernel of M still hold. The only edges whose status with respect to these conditions might have changed are those whose left vertex is a or u, or whose right vertex is v or b. We will analyze those edges whose left vertex is a, while the other cases are similar and left to the reader. Let ac be an edge whose left vertex is a. We have to show that with the new weights on the edges, the sum of the weights of all edges steeper than ac whose left vertex is a or whose right vertex is c, equals the sum of the weights of all edges whose slope is smaller than that of ac and whose left vertex is a or whose right vertex is c. This is clearly the case if ac is steeper than

av, or if the slope of ac is smaller than that of ab because the missing weight of e in this calculation is now gained on av whose weight is increased by w_e. The remaining case is when $c = v$. Then the sum of the weights of all edges whose slope is smaller than that of av that we take into account is indeed decreased by w_e but also the weight of all edges steeper than av is decreased because the weight of the new edge uv is $-w_e$. Hence we verified that every vector in the kernel of M gives rise to a vector in the kernel of M'.

In the other direction, given a vector in the kernel of M', we obtain a vector in the kernel of M by giving the edge e the weight of $-w_{e'}$ and deduct the same amount from the weights of the edges av and ub.

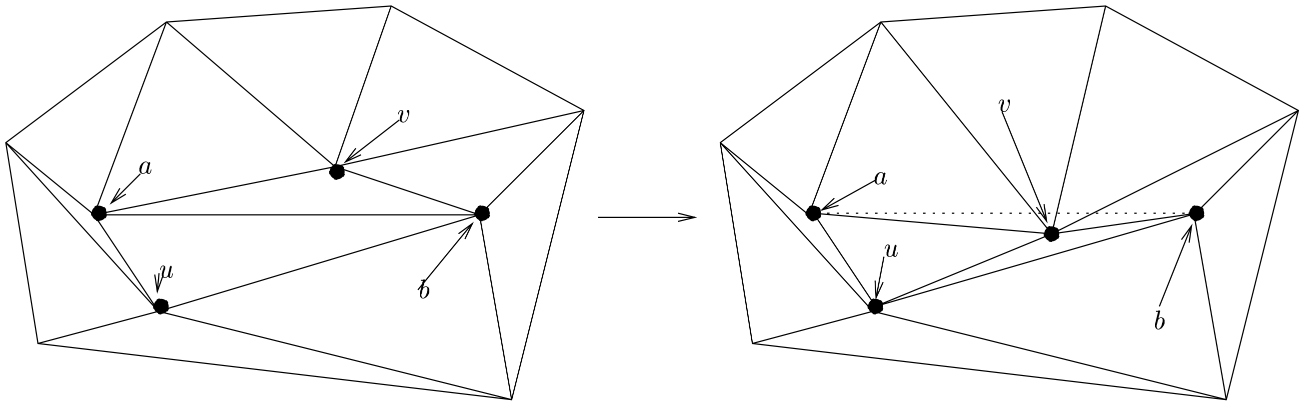

Figure 4: the vertex v crosses the edge ab

The case where the x-coordinate of u is bigger than that of v is treated similarly.

We now move to the second kind of event that may happen and assume that a vertex v crosses an edge ab of the convex hull of P, and thus becomes extreme (we already observed that $v = x$ in this case). In this case we consider a new triangulation G' which is obtained from G by removing the edge ab. Clearly every vector in the kernel of M' is also a vector in the kernel of M by letting $w_{ab} = 0$. Vice versa, every vector in the kernel of M in which $w_{ab} = 0$ corresponds to a vector in the kernel of M'.

In order to show that the dimension of the kernel of M is greater by 1 than the dimension of the kernel of M', it is enough to show that there is a vector in the kernel of M in which $w_{ab} = 1$.

It is in fact very easy to observe such a vector in the kernel. Since v is an internal vertex, there is a triangle Δvcd one of whose vertices is v such that the edge cd is above v (that is, v lies below the line $\overline{cd}$ and the x-coordinate of v is strictly between those of c and d). Assume first that $\{a, b\} \cap \{c, d\} = \emptyset$. In this case assigning weight 1 to ab and cd and weight -1 to av, bv, cv, dv while all other edges get the weight 0, gives a vector in the kernel.

If it happens that say $a = c$, then modify the above assignment by giving the edge va the weight -2. ∎

Corollary 3.3. *Let G be any triangulation of a convex polygon with the possibility of convex* *polygonal holes. Then the rank of M over $\mathbb{F}_2$ as well as over $\mathbb{R}$ is $2n-4$, where n is the number of vertices of G.*

Proof. We first show that the rank of M over $\mathbb{F}_2$ is $2n-4$. Consider the graph $\widetilde{G}$. It is easy to see that this graph is connected. The reason is that all the holes in the polygonal regions are convex and hence there are no isolated vertices in $\widetilde{G}$. Moreover, every triangle in G is transformed into a connected path of length 3 in $\widetilde{G}$. Since G is a triangulation of a connected region, it follows that from every edge one can get to any other edge by means of a path.

Now consider the matrix $M(G)$ over the field $\mathbb{F}_2$, and observe that it equals the matrix $\hat{\widetilde{G}}$. The number of vertices in $\widetilde{G}$ is even $(2n-2)$, and therefore, by Theorem 1.3, the rank of M over $\mathbb{F}_2$ is $2n-4$. This is a lower bound to the rank of M over the field of real numbers.

On the other hand, by adding edges to G we can obtain a triangulation H of the original polygonal regions, this time with no holes. The graph H is a triangulation of the convex hull of the vertices of G. By Theorem 3.2, the rank of the matrix $M(H)$ is $2n-4$. The matrix $M(G)$ is a sub-matrix of $M(H)$ and therefore $2n-4$ is also an upper-bound to the rank of $M(G)$ over $\mathbb{R}$. The Corollary follows. ∎

4 More Applications

Two edges in a geometric graph G are called *convergent*, if they are opposite edges of a convex quadrilateral in the plane. A conjecture of Kupitz ([2]), solved by Katchalski, Last, and Valtr ([1, 4]), asserts that a geometric graph with no pair of convergent edges has at most $2n-2$ edges. Their proof is quite involved, and based on simple bounds on Davenport-Schinzel sequences.

We now show how to obtain a linear bound of $4n$ for the same problem using the method presented in this paper.

Let G be a geometric graph with n vertices, E edges, and no pair of convergent edges. Let $v_1, \ldots, v_n$ denote the vertices of G ordered in increasing order of their x-coordinates. (we assume that no two of these vertices have the same x-coordinate). To each edge e of G we assign a vector $W_e \in \mathbb{C}^n$ as follows. If v_j and v_k $(j < k)$ are the vertices of e, we let the j'th coordinate of W_e to be i and the k'th coordinate of W_e to be $-i$. For l smaller than j, or larger than k, we let the l'th coordinate of W_e to be 1 if v_l is below the line through e, and -1 if v_l is above the line through e. In all other cases we set the l'th coordinate of W_e to be equal to 0.

The simple yet crucial observation is that if e and f are two edges of G that do not share a common vertex (and of course are not convergent), then $< W_e, \overline{W_f} >$ has imaginary part which is -2 if the slope of e is smaller than that of f, and equals 2 if the slope of e is larger than the slope of f.

Moreover, if e and f are two edges of G that share a common left vertex or a common right vertex, then $< W_e, \overline{W_f} >$ has imaginary part which is -1 if the slope of e is smaller than that of f, and equals 1 if the slope of e is larger than the slope of f.

Claim 4.1. *Let A be the $|E|$ by n matrix whose rows are W_e. Then the rank over $\mathbb{R}$ of the imaginary part M of $A\overline{A^t}$ is greater than or equal to $E-(2n-2)$.*

Proof. We show that the kernel of the matrix M has dimension less than or equal to $2n-2$. Let V be a vector in the kernel. The coordinates of V can be regarded as weights assigned to the edges of G.

For every vertex v, let L_v denote the sum of the weights of the edges whose right vertex is v. Similarly let R_v denote the sum of the weights of all edges whose left vertex is v. Clearly, $L_v = 0$ for the leftmost vertex and $R_v = 0$ for the rightmost vertex. We claim that the vector V in the kernel of M can be reconstructed from the $2n-2$ numbers L_v and R_v (where v varies over the vertices of G, excluding L_s and R_t where s is the leftmost vertex and t is the rightmost vertex).

Indeed, let e be the edge which is the second steepest, and let f be the steepest edge. considering e, let a denote the left vertex of e and b denote its right vertex. There are two cases: **Case 1.** a is a left vertex of f or b is a right vertex of f. In this case we know that the sum of the weights of all edges but e whose left vertex is a or whose right vertex is b, plus twice the sum of all other edges, minus twice the weight of f is 0.

However, this sum can be expressed as $\sum_{v\neq a} R_v + \sum_{v\neq b} L_v - 2W_f$. Equating this to zero gives the weight of f in terms of L_v and R_v.

Case 2. a is not a left vertex of f and b is not a right vertex of f. In this case we know that the sum of the weights of all edges but e whose left vertex is a or whose right vertex is b, plus twice the sum of all other edges, minus four times the weight of f is 0.

However, this sum can be expressed as $\sum_{v\neq a} R_v + \sum_{v\neq b} L_v - 4W_f$. Equating this to zero gives the weight of f in terms of L_v and R_v.

After recovering f we can move to the third steepest edge and recover e (the second steepest). Continuing this way we will eventually recover all weights but that of the edge with the smallest slope. But then its weight too can be easily recovered from say R_v where v is the left vertex of that edge, and the weights of all other edges that we already know. ∎

It follows from Claim 4.1 that the rank over the *complex field* of the matrix $A\overline{A^t}$ (where A is the $|E|$ by n matrix whose rows are W_e) is at least $\lceil \frac{E-(2n-2)}{2} \rceil$. On the other hand this rank is at most n as A has n columns. We conclude that $E \leq 4n-2$. ∎

References

[1] M. Katchalski and H. Last, On geometric graphs with no two edges in convex position, *Discrete Comput. Geom.* **19** (1998), no. 3, Special Issue, 399–404.

[2] Y. Kupitz, On pairs of disjoint segments in convex position in the plane, *Ann. Discrete Math.* **20** (1984), 203–208.

[3] L. A. Székely, Progress on crossing number problems, *SOFSEM 2005: Theory and Practice of Computer Science: 31st Conference on Current Trends in Theory and Practice of Computer Science Liptovský Ján, Slovakia, January 22-28, 2005.* Editors: P. Vojtás, M. Bieliková, B. Charron-Bost, et al. Lecture Notes in Computer Science Vol. 3381, 2005, Springer-Verlag, 53–61.

[4] P. Valtr, On geometric graphs with no k pairwise parallel edges, *Discrete Comput. Geom.* **19** (1998), no. 3, Special Issue, 461–469.