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# Topological Graphs with no Self-Intersecting Cycle of Length 4

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Abstract. Let $G$ be a topological graph on $n$ vertices in the plane, i.e., a graph drawn in the plane with its vertices represented as points and its edges represented as Jordan arcs connecting pairs of points. It is shown that if no two edges of any cycle of length 4 in $G$ cross an odd number of times, then $|E(G)| = O(n^{8/5})$.

## 1. Introduction

A *geometric graph* is a simple graph drawn in the plane so that its vertices are represented by points in general position (i.e., no three are collinear) and its edges by straight-line segments connecting the corresponding points. *Topological graphs* are defined similarly, except that now each edge is represented by a simple (non-self-intersecting) Jordan arc passing through no vertices other than its endpoints. Clearly, every geometric graph is also a topological graph. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$, respectively. We will make no notational distinction between the vertices (resp. edges) of the underlying abstract graph, and the points (resp. arcs) representing them in the plane. Throughout this paper, we assume that if two edges of a topological graph $G$ share an interior point, then at this point they properly cross. We also assume, for simplicity, that no three edges cross at the same point and that any two edges cross only a finite number of times.

In the mid-sixties Avital and Hannani [**2**], Erdős, and Perles initiated, later Kupitz [**9**] and many others continued the systematic study of extremal problems for geometric graphs. In particular, they proposed the following general question for geometric graphs, which is then naturally extended to topological graphs. Let $H$ be a so-called *forbidden* geometric configuration or a *class* of forbidden configurations. *What is the maximum number of edges that a topological graph with n vertices can have without containing any forbidden configuration?* For example, $H$ may consist of $k$ pairwise crossing edges or may be the class of all configurations of $k+1$ edges, one of which crosses all the others, etc. For a survey of many results of this type, consult [**10**].

In the present paper, we consider the following related question.

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**Problem 1:** Let $K$ be a fixed abstract graph. What is the maximum number $ex_{\mathrm{cr}}(n, K)$ of edges that a topological graph on $n$ vertices can have if it contains no self-intersecting copy of $K$?

Pach, Pinchasi, Tardos, and Tóth ([**11**]) recently analyzed the case where $K = P_k$, i.e., a path with $k$ edges. They showed that a geometric graph[1] containing no self-intersecting $P_3$ has at most $O(n\log n)$ edges, and this bound is asymptotically tight. For any fixed $k \geq 3$. Tardos [**13**] constructed a sequence of geometric graphs on $n$ vertices containing no self-intersecting path of length $k$ and still having a *super-linear* number of edges (approximately $n$ times the $\lfloor k/2 \rfloor$ times iterated logarithm of $n$).

In this paper we analyze Problem 1 in the case where $K = C_4$, namely, a cycle of length 4. One can note some easy observations on this problem. Let $\mathrm{ex}(n, K)$ denote the maximum number of edges in a $K$-free abstract graph on $n$ vertices. It is well known that $\mathrm{ex}(n, C_4) = \Theta(n^{3/2})$, (see e.g. [**3**]). In other words there are graphs with roughly $n^{3/2}$ edges which don't contain any copy of $C_4$ as an abstract graph. Therefore, $ex_{\mathrm{cr}}(n, C_4) = \Omega(n^{3/2})$. On the other hand, if an abstract (simple) graph $G$ has $\Omega(n^{5/3})$ edges, then, by the theorem of Kővári, Sós and Turán [**8**], $G$ must contain a subgraph isomorphic to $K_{3,3}$, which in turn is not planar. Moreover, in any drawing of $K_{3,3}$ there exist two edges that cross an odd number of times and do not share an endpoint [**7, 14**]. Clearly, these two edges belong to a cycle of length 4 in $G$. Therefore, $ex_{\mathrm{cr}}(n, C_4) = O(n^{5/3})$.

Here we give the first nontrivial upper bound for $ex_{\mathrm{cr}}(n, C_4)$.

**Theorem 1** *$ex_{cr}(n, C_4) = O(n^{8/5})$.*

As was pointed out by Tutte [**14**] and also by Hanani ([**7**]), *parity* plays an important role when considering crossing of edges of a topological graph. They showed that every drawing of a non-planar graph contains two edges which do not share an endpoint and cross each other an odd number of times.

This suggest that it may be interesting to consider the following variant of Problem 1.

**Problem 2:** Let $K$ be a fixed abstract graph. What is the maximum number $ex_{\text{odd-cr}}(n, K)$ of edges of a topological graph with $n$ vertices if no two edges which belong to the same copy of $K$ in $G$, cross an odd number of times?

In [**11**] it is shown that $ex_{\text{odd-cr}}(n, P_3) = \Theta(n^{3/2})$. Here, we in fact prove an analogue theorem for $ex_{\text{odd-cr}}(n, C_4)$, which is a stronger version of Theorem 1, and implies it easily.

**Theorem 1'** *Let $G$ be a graph on $n$ vertices that can be drawn in the plane so that no two edges belonging to a cycle of length 4 cross an odd number of times. Then $|E(G)| \leq 32n^{8/5}$.*

We should mention here an important application of Theorem 1', which was in fact the motivation for this paper in the first place. Theorem 1' is one of the main engines through which Agarwal et al. [**1**], obtain an improved bound for the minimum number of cuts $\lambda(n)$ needed in order to cut an arrangement $C$ of $n$

[1] In fact it is shown for every topological graph which is *$x$-monotone*, that is, all of its edges are $x$-monotone curves.

pseudoparabolas[2] into an arrangement of pseudo-segments, i.e., so that any pair of resulting arcs intersect at most once. This problem was first considered by Tamaki and Tokuyama ([**12**]), who showed that $\lambda(n) = O(n^{5/3})$. This bound was recently improved in [**1**] to $O(n^{8/5})$. The important notion in understanding this problem is that of a *lens*. A lens $\lambda$ formed by $c, c' \in C$ is the union of two arcs, one of $c$ and one of $c'$, both delimited by the intersection points of $c$ and $c'$. A family of lenses formed by the curves in $C$ is called *pairwise non-overlapping* if the relative interiors of the arcs forming any two of them do not overlap. The crucial observation, made already in [**12**], is that the minimum number of cuts needed is roughly the maximum size of a family of pairwise non-overlapping lenses. Let $L(C)$ denote such family of the maximum size. Define a graph $G$ whose vertices are the curves in $C$, and put an edge between two vertices if the corresponding two curves create a lens in $L(C)$. A clever argument in [**1**] shows that $G$ can be drawn in the plane in such a way that it satisfies the conditions of Theorem 1' of this paper, namely, any two edges which belong to a cycle of length 4 in $G$ cross an even number of time. The conclusion on the size of $L(C)$ follows immediately. See [**1**] for interesting implications of this bound.

This paper is organized as follows. We prove Theorem 1' by reducing the problem to a purely combinatorial problem on certain circular sequences. The reduction is shown in Section 2, while the lemma on circular sequences, which forms the main core of our proof, is proved in Section 3. Finally, in Section 4 we discuss several open problems.

## 2. Reduction from geometry to a combinatorial problem

Let $G$ be a graph on $n$ vertices, drawn in the plane so that no two edges belonging to a cycle of length 4 cross an odd number of times. Label the vertices of $G$ arbitrarily from 1 to $n$. For $v \in V(G)$, let $N(v) \subseteq \{1, \ldots, n\}$ denote the set of its neighbors. In the sufficiently small[3] disk centered at $v$, the *counterclockwise* order of the edges from $v$ to the elements of $N(v)$ induces a *cyclic order* on $N(v)$. The *circular sequence* $C_v$ *of* $v$, is $N(v)$ with the underlying cyclic ordering of its elements. Sometimes, depending on the context, we will refer to $C_v$ just as a set of elements disregarding its ordering. Thus for example $C_v \cap C_u$ is in fact the set $N_v \cap N_u$.

A property of circular sequences, captured in the following Lemma, is the only geometric information we use in the rest of the proof of Theorem 1'.

LEMMA 1. *Let $u$, $v$ be two distinct vertices of $G$, and let $R = C_u \cap C_v$. If $|R| \geq 3$, then the order of the elements of $R$ in $C_u$ is reversed in $C_v$.*

**Proof** It suffices to show that the order in $C_u$ of any three elements $a$, $b$, $c \in R$ is reversed in $C_v$. Let $J_a$ be the *oriented* Jordan curve formed by edges $ua$ and $av$, and $J_b$ the oriented Jordan curve formed by edges $vb$ and $bu$. Let $J = J_a \cup J_b$ be the closed oriented Jordan curve formed by edges $ua$, $av$, $vb$ and $bu$ and let

[2]A finite family of Jordan curves in the plane, that are graphs of continuous functions everywhere defined on the set of real numbers, such that every two intersect at most twice, is called a family of pseudo-parabolas.

[3]No two edges adjacent to $v$ intersect in the interior of the disk.

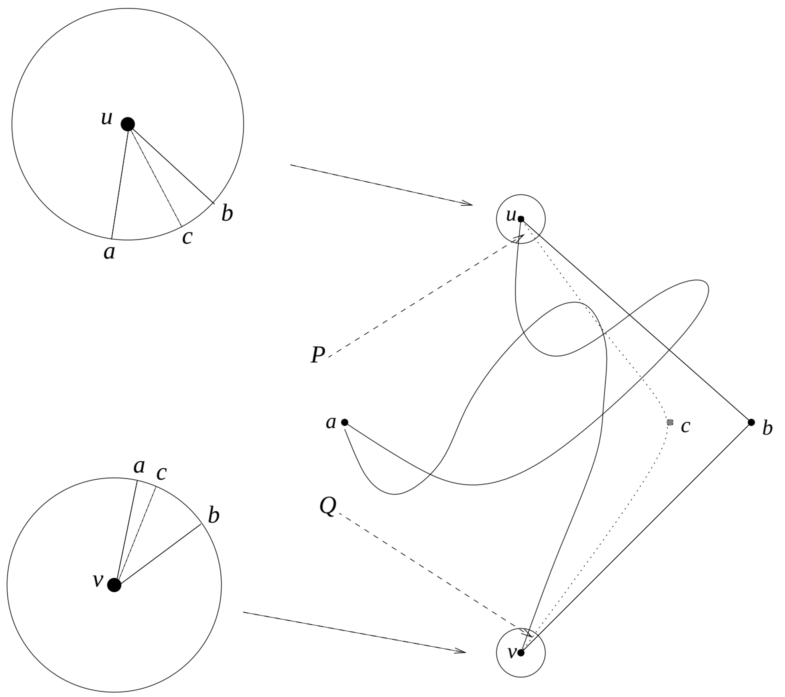


FIGURE 1. Lemma 1.

$\mathrm{Int}(J)$ denote the interior of $J$, i.e., the set of all points $x$ in the plane such that any ray with $x$ as its apex intersects $J$ an odd number of times. Let $P$ and $Q$ denote the intersections of $\mathrm{Int}(J)$ with sufficiently small disks centered at $u$ and $v$, respectively (see Figure 1). Without loss of generality, assume that $P$ is to the left of $ua$. If one takes a walk from $u$ to $v$, along $J_a$, then one crosses $J$ an even number of times. Indeed, every crossing with $J_a$ is met twice, while the number of crossings with $J_b$ is equal to the total number of crossings of any of the edges $ua$, $av$ with any of the edges $vb$, $bu$, and thus, is even due to our condition on $G$. At every crossing with $J$, $\mathrm{Int}(J)$ switches from the left of $J_a$ to the right of $J_a$ and vice versa. Since $\mathrm{Int}(J)$ is on the left of $J_a$ at the beginning of the walk and the number of crossings with $J$ is even, then $\mathrm{Int}(J)$ is on the left of $J_a$ at the end of the walk as well. Therefore, $Q$ is to the left of $av$. Let $J_c$ be the Jordan curve formed by edges $uc$ and $cv$. Then $J_c$ crosses $J$ an even number of times. Therefore, $J_c$ starts within $P$ if and only if it ends within $Q$ (see Figure 1), implying that the order of $a$, $b$, $c \in R$ is reversed in $C_v$. □

In Section 3, we prove the following general lemma on circular sequences.

LEMMA 2. *Let $n$ be a positive integer and let $C_u$, $u \in \{1, \ldots, n\}$ be $n$ circular sequences, each consists of $m$ different elements from $\{1, \ldots, n\}$. If for every $1 \leq u < v \leq n$ with $|C_u \cap C_v| \geq 3$, the order of elements of $C_u \cap C_v$ in $C_u$ is reversed in $C_v$, then $m \leq 16n^{3/5}$.*

Next, we finish the proof of Theorem 1', using Lemma 2.

**Proof of Theorem 1'** Let $G$ be a graph on $n$ vertices, drawn in the plane so that any two edges belonging to a cycle of length 4 in $G$ cross an even number of times. We prove the claim by induction on $n$. It is clearly true for $n = 1, 2$. Suppose that the degree of a vertex $v$ in graph $G$ is less than $32n^{3/5}$. Then the subgraph $G - \{v\}$, obtained from $G$ by removing the vertex $v$ and all the edges incident to it, satisfies the condition of Theorem 1', and therefore, by induction, has at most $32(n-1)^{8/5}$ edges. Then $|E(G)| \leq 32(n-1)^{8/5} + 32n^{3/5} \leq 32n^{8/5}$, the last inequality holding for every $n \geq 2$. Therefore, we may assume that every vertex is incident to at least $32n^{3/5}$ edges.

Label the vertices of $G$ arbitrarily from 1 to $n$, and for each vertex $v \in V(G)$ form the circular sequence $C'_v$ as described at the beginning of this section. For each $v \in V(G)$, let $C_v$ be an arbitrary subsequence of $C'_v$ of size $m = \lfloor 32n^{3/5} \rfloor$. By Lemma 1, $\{C_v\}_{v \in V(G)}$ satisfy the condition of Lemma 2. Therefore, by Lemma 2, $m \leq 16n^{3/5}$, which is a contradiction. □

## 3. Proof of Lemma 2

First, notice that any 3 circular sequences may share at most 2 elements. Otherwise, there would be 2 circular sequences with a triple of common elements in the same order. This observation alone implies that $n\binom{m}{3} \leq 2\binom{n}{3}$, i.e., $m \leq 2n^{2/3}$.

The proof proceeds by induction on $n$. The claim is clearly true for $n = 1$. Suppose, on the contrary, that $m > 16n^{3/5}$. Let $k = \frac{1}{256}\frac{m^4}{n^2}$. For the sake of simplicity, we may assume that $m$ is divisible by $k$. Partition each circular sequence $C_u$ into $\frac{m}{k}$ *blocks* of $k$ consecutive elements. The order of elements within the blocks is inherited from the order of elements in $C_u$. Using $m \leq 2n^{2/3}$, it is easy to see that each circular sequence is partitioned into at least two blocks; moreover, since $m > 16n^{3/5}$, every block has at least two elements.

For every $u, v \in \{1, \ldots, n\}$, $u \neq v$, let $s_{uv}$ be the number of ordered pairs $(i, j)$ of elements which appear in the same order in a block of $C_u$ and in a block of $C_v$. Let $d_{uv}$ denote the number of ordered pairs $(i, j)$, such that the order of elements $i$ and $j$ in a block of $C_u$ is reversed in a block of $C_v$. Let $D = \sum_{u<v} d_{uv}$ and $S = \sum_{u<v} s_{uv}$. The main idea is to obtain good lower and upper bounds on $D - S$.

First, we obtain an upper bound by the following "double counting" argument. For every $i, j \in \{1, \ldots, n\}$, $i \neq j$, let $b_{ij}$ be the total number of blocks over all sequences $C_u$, $u \in \{1, \ldots, n\}$, in which $i$ appears before $j$.

Observe that $D = \sum_{i<j} b_{ij}b_{ji}$ and $S = \sum_{i<j}(\binom{b_{ij}}{2} + \binom{b_{ji}}{2})$. Then,

$$D - S \leq \frac{1}{2}\sum_{i<j}(b_{ij} + b_{ji}) \leq \frac{1}{4}nmk, \tag{1}$$

where we use the fact that $\sum_{i \neq j} b_{ij} = n\frac{m}{k}\binom{k}{2} \leq \frac{1}{2}nmk$ and $\binom{b_{ij}}{2} + \binom{b_{ji}}{2} - b_{ij}b_{ji} = \frac{1}{2}((b_{ij} - b_{ji})^2 - (b_{ij} + b_{ji}))$.

Next, we establish a lower bound on $D - S$, by separating the sum $D - S = \sum_{u<v}(d_{uv} - s_{uv})$ into three parts and finding a lower bound for each of the parts. Define a new parameter $M = 4\sqrt{k}$. Then $M = \frac{m^2}{4n}$ and $M \geq 4$. Let

$$D - S = \sum_1 (d_{uv} - s_{uv}) + \sum_2 (d_{uv} - s_{uv}) + \sum_3 (d_{uv} - s_{uv}), \tag{2}$$

where

- $\sum_1$ is over all $u < v$ such that $|C_u \cap C_v| < M$;

- $\sum_2$ is over all $u < v$ such that $|C_u \cap C_v| \geq M$ and there is a block $B_u$ of $C_u$ and a block $B_v$ of $C_v$ such that $B_u \cap B_v$ contains at least $|C_u \cap C_v|/2$ elements; and
- $\sum_3$ is over all $u < v$ such that $|C_u \cap C_v| \geq M$ and there are no blocks $B_u$ of $C_u$ and $B_v$ of $C_v$ such that $B_u \cap B_v$ contains at least $|C_u \cap C_v|/2$ elements.

The following claim, though a simple observation, is repeatedly used in the rest of the proof.

Claim 1. *Let $u, v \in \{1, \ldots, n\}$, $u \neq v$. Suppose that $|C_u \cap C_v| \geq 3$ and $s_{uv} > 0$. Then there is exactly one block $B_u$ of $C_u$ that contains a pair of elements appearing in the same order in $B_u$ as in a block of $C_v$.*

**Proof** Since $s_{uv} > 0$, there exists a block $B_u$ of $C_u$, containing a pair of elements $i, j$ that also appear in a block of $C_v$ in the same (counterclockwise) order. Without loss of generality, assume that $i$ appears before $j$ in both blocks. Suppose, on the contrary, that there exists another block $B'_u$ (different from $B_u$) of $C_u$, containing a pair of elements $i', j'$ that also appear in a block of $C_v$ in the same (counterclockwise) order as in $B'_u$. Without loss of generality, assume that $i'$ appears before $j'$ in both blocks. Then, $(i, j, i', j')$ is the cyclic order of these elements in $C_u$. Hence, the cyclic order of these elements in $C_v$ is the reversed one, namely $(i, j', i', j)$. Since $i$ appears before $j$ in a block of $C_v$, it follows that $j'$, $i'$ appear in that order in the *same* block of $C_v$, which is a contradiction, as $i'$ appears before $j'$ in that block. □

Corollary 1. *Let $u, v \in \{1, \ldots, n\}$, $u \neq v$. Suppose that $|C_u \cap C_v| \geq 3$ and $s_{uv} > 0$. Then there is exactly one block $B_u$ of $C_u$ and exactly one block $B_v$ of $C_v$ that contain identically ordered pairs of elements.*

A corollary of the following claim immediately gives a lower bound for $\sum_1$.

Claim 2. *Let $u, v \in \{1, \ldots, n\}$, $u < v$, and let $r = |C_u \cap C_v|$. Then $d_{uv} - s_{uv} \geq -\frac{1}{2}r$.*

**Proof** The claim is clearly true if $r \leq 2$ or $s_{uv} = 0$. Thus, we may assume that $r \geq 3$ and $s_{uv} > 0$. By Corollary 1 there is exactly one block $B_u$ of $C_u$ and exactly one block $B_v$ of $C_v$ that contain identically ordered pairs of elements. Let $R' = B_u \cap B_v$ and let $r' = |R'|$. Clearly, $r' \leq r$ and the pairs counted by $s_{uv}$ are some pairs of elements of $R'$. Let $i_1, i_2, \ldots, i_{r'}$ be the elements of $R'$ in the order in which they appear in $B_u$. This order is reversed in $C_v$, so there is an index $j$, $1 \leq j \leq r'$, such that $i_j, i_{j-1}, \ldots, i_1, i_{r'}, i_{r'-1}, \ldots, i_{j+1}$ is the order of elements of $R$ in $B_v$. It follows that $s_{uv} = j(r'-j)$, while $d_{uv} \geq \binom{j}{2} + \binom{r'-j}{2}$. This is an inequality rather than equality, because the pairs of elements of blocks other than $B_u$ and $B_v$ may contribute to $d_{uv}$. Therefore $d_{uv} - s_{uv} \geq \frac{1}{2}((r'-2j)^2 - r') \geq -\frac{1}{2}r' \geq -\frac{1}{2}r$. □

Corollary 2. $\sum_1 \geq -\frac{1}{4}n^2 M$.

**Proof** This follows immediately from Claim 2, since there are at most $\binom{n}{2}$ pairs of circular sequences, each pair intersecting in at most $M$ elements. □

Now, we bound $\sum_2$. Let $a_i, 0 \leq i \leq n$, denote the number of pairs $(u, v)$, $u, v \in \{1, \ldots, n\}$, $u < v$, such that $|C_u \cap C_v| = i$. Clearly, $\sum_{i=0}^{n} a_i = \binom{n}{2}$. Let $a'_i$ denote the number of pairs $(u, v)$, $u, v \in \{1, \ldots, n\}$, $u < v$, such that $|C_u \cap C_v| = i$ and there is a block of $C_u$ and a block of $C_v$ whose intersection contains at least $i/2$ elements. Claim 2 implies

$$\sum_2 \geq -\frac{1}{2} \sum_{i \geq M} a'_i i. \tag{3}$$

The following claim provides an estimate for the right hand side of 3.

Claim 3. $\sum_{i \geq M} a'_i i \leq 2nm$

**Proof** Let $B$ be a block of $C_u$. Let $a_i^B$ be the number of circular sequences $C_v, v \neq u$, such that $|C_u \cap C_v| = i$ and there is a block in $C_v$ whose intersection with $B$ contains at least $i/2$ elements.

Let $t = \sum_{i \geq M} a_i^B$. Suppose $t \geq M/4$. Then, there exist $M/4$ circular sequences $C_{v_1}, \ldots, C_{v_{M/4}}$, such that $|B \cap C_{v_j}| \geq M/4$. Let $A_j = B \cap C_{v_j}$, $j = 1, \ldots, M/4$. Observe that for every $x \neq y$, $|A_x \cap A_y| \leq 2$, for otherwise $C_u \cap C_{v_x} \cap C_{v_y}$ contains at least 3 elements, which is a contradiction with the observation made at the beginning of this section. Hence,

$$\begin{aligned} k = |B| &\geq \frac{M}{2} + (\frac{M}{2} - 2) + \ldots + (\frac{M}{2} - 2(\frac{M}{4} - 1)) \\ &= \frac{M^2}{8} + \frac{M}{4} > \frac{M^2}{8} = 2k, \end{aligned}$$

which is a contradiction. Therefore, $t < M/4$.

Let $C_{v_1}, \ldots, C_{v_t}$ be all the sequences $C_v$ (other than $C_u$), such that $|C_u \cap C_v| \geq M$ and there is a block of $C_v$ whose intersection with $B$ consists of at least $|C_u \cap C_v|/2$ elements. Then,

$$\sum_{i \geq M} a_i^B i \leq 2(|C_{v_1} \cap B| + \ldots + |C_{v_t} \cap B|). \tag{4}$$

As we observed earlier, for every $x \neq y$, $|C_{v_x} \cap C_{v_y} \cap B| \leq 2$, and therefore:

$$\begin{aligned} k &= |B| \geq \sum_{j=1}^{t} (|C_{v_j} \cap B| - 2(j-1)) \\ &= \sum_{j=1}^{t} |C_{v_j} \cap B| - t(t-1). \end{aligned} \tag{5}$$

Hence, using $t < M/4$,

$$\sum_{j=1}^{t} |C_{v_j} \cap B| \leq k + t(t-1) < k + (M/4)^2 = 2k. \tag{6}$$

From (4) and (6) it follows that $\sum_{i \geq M} a_i^B i \leq 4k$.

It is easy to see that

$$2\sum_{i\geq M} a_i' i \leq \sum_{u=1}^{n}\sum_{B}\sum_{i\geq M} a_i^B i,$$

where the middle sum on the right hand side is over all blocks $B$ in $C_u$. Therefore, $\sum_{i\geq M} a_i' i \leq \frac{1}{2}n\frac{m}{k}4k = 2nm.$ □

From (3), we have

COROLLARY 3. *$\sum_2 \geq -nm$.*

In order to obtain a bound for $\sum_3$, we will need the following simple claim which uses the overlay of block partitions of a pair of circular sequences.

CLAIM 4. *Let $u, v \in \{1, \ldots, n\}$, $u < v$, and $r = |C_u \cap C_v|$, where $r \geq 4\frac{m}{k}$. Then $d_{uv} \geq \frac{1}{8}r^2\frac{k}{m}$.*

**Proof** Let $R = C_u \cap C_v$. The cyclic order in which the elements of $R$ appear in $C_u$ is reversed in $C_v$. Hence, the block partitions of $C_u$ and $C_v$ induce a refined partition of the elements of $R$ into $N \leq 2\frac{m}{k}$ blocks. To be more precise, think of the elements of $R$ arranged in the cyclic order induced by $C_u$. We put 'bars' *between* two consecutive elements $a$ and $b$ (here *between* $a$ and $b$ means in the cyclic portion where they are consecutive) if there is such a 'bar' *between* $a$ and $b$ in the original partitioning of $C_u$ into blocks, or if there is such a bar *between* $a$ and $b$ in the partitioning induced by $C_v$ (observe that $a$ and $b$ are consecutive also in the order induced by $C_v$, which is just the reversed of that induced by $C_u$). Observe that every 'bar' of the original partitions $C_u$ and $C_v$ gives rise to at most one 'bar' in the refined partitioning. Therefore, the number of parts in the refined partition is at most $2\frac{m}{k}$.

Every block of the refined partition is included in a block of $C_u$ as well as in a block of $C_v$. Let $x_1, x_2, \ldots, x_N$ denote the number of elements in the blocks of the refined partition. We claim that every pair of elements which is in the same refined block is counted by $d_{uv}$, and thus $d_{uv} \geq \sum_{i=1}^{N}\binom{x_i}{2}$. Indeed, let $i$ and $j$ be two such elements. This means in particular that $i$ and $j$ belong to the same block in $C_u$ and to the same block in $C_v$. Without loss of generality assume that $i$ appears before $j$ in a block of $C_u$. All we have to show is that $j$ appears before $i$ in a block of $C_v$. Assume to the contrary that $i$ comes before $j$ also in a block of $C_v$. Then, in the refined partitioning induced by $C_u$ and $C_v$ there will be 'bars' both when going from $i$ to $j$ and when going from $j$ to $i$ in the counterclockwise order induced by $C_u$. This is a contradiction to that $i$ and $j$ belong to the same refined block.

We may therefore conclude that $d_{uv} \geq \sum_{i=1}^{N}\binom{x_i}{2} \geq N\binom{r/N}{2}$, where the last inequality follows from Jensen's inequality. Since $r - N \geq \frac{1}{2}r$, we have $d_{uv} \geq \frac{1}{8}r^2\frac{k}{m}$. □

**Note** Notice that the proof of Claim 4 above does not use the fact that all the blocks of the circular sequences have size $k$.

CLAIM 5. *Let $u, v \in \{1, \ldots, n\}$, $u < v$, and $r = |C_u \cap C_v|$, where $r \geq 16\frac{m}{k}$. If $d_{uv} - s_{uv} < \frac{1}{64}r^2\frac{k}{m}$, then there is a block $B_u$ of $C_u$ and a block $B_v$ of $C_v$ such that $|B_u \cap B_v| \geq r/2$.*

**Proof** It follows from Claim 4 that $s_{uv} > 0$. By Corollary 1 there is a (unique) block $B_u$ of $C_u$ and a (unique) block $B_v$ of $C_v$ that contain identically ordered pairs of elements. Let $R = B_u \cap B_v$ and let $R' = (C_u \cap C_v) \setminus R$. Assume, by contradiction, that $|R| < r/2$.

Let $s^R_{uv}$ be the number of pairs of elements of $R$ that appear in the same order in $B_u$ and $B_v$. Similarly, let $d^R_{uv}$ be the number of pairs of elements of $R$ whose order in $B_u$ is reversed in $B_v$. From the proof of Claim 2, it follows that $s^R_{uv} = s_{uv}$ and $d^R_{xy} - s^R_{xy} \geq -\frac{1}{2}|R| > -r/4$.

Let $C_u^{R'}$ and $C_v^{R'}$ be the circular sequences obtained from $C_u$ and $C_v$, respectively, by removing the elements of $R$ from both. These new circular sequences inherit the block partition from $C_u$ and $C_v$, respectively. Let $s^{R'}_{uv}$ be the number of pairs of elements of $R'$ that appear in the same order in a block of $C_u^{R'}$ and a block of $C_v^{R'}$. Then, $s^{R'}_{uv} = 0$. Let $d^{R'}_{uv}$ be the number of pairs of elements of $R'$ whose order in a block of $C_u^{R'}$ is reversed in a block of $C_v^{R'}$. Clearly, $d_{uv} \geq d^R_{uv} + d^{R'}_{uv}$. Since $|R'| \geq \frac{1}{2}r \geq 8\frac{m}{k}$, we can apply Claim 4 for sequences $C_u^{R'}$ and $C_v^{R'}$ (see the Note above). We obtain $d^{R'}_{uv} \geq \frac{1}{8}|R'|^2\frac{k}{m} \geq \frac{1}{32}r^2\frac{k}{m}$. Therefore,

$$d_{uv} - s_{uv} \geq (d^{R'}_{uv} - s^{R'}_{uv}) + (d^R_{uv} - s^R_{uv}) > \frac{1}{32}r^2\frac{k}{m} - \frac{1}{4}r. \tag{7}$$

Since $r \geq 16\frac{m}{k}$, we have $-\frac{1}{4}r \geq -\frac{1}{64}r^2\frac{k}{m}$. From (7) it follows that $d_{uv} - s_{uv} > \frac{1}{64}r^2\frac{k}{m}$, which is a contradiction. □

Corollary 4. $\sum_3 \geq \sum_{i\geq M}(a_i - a'_i)\frac{1}{64}i^2\frac{k}{m}$.

**Proof** Since $m > 16n^{3/5}$, we have $M > 16\frac{m}{k}$. Therefore, Claim 5 can be applied. □

Using Corollaries 2, 3, 4, and recalling our upper bound (2) on $D - S$, we get

$$\sum_{i\geq M}(a_i - a'_i)\frac{1}{64}i^2\frac{k}{m} - nm - \frac{1}{4}n^2M \leq D - S \leq \frac{1}{4}nmk.$$

Therefore,

$$\sum_{i\geq M}(a_i - a'_i)\frac{1}{64}i^2\frac{k}{m} \leq \frac{1}{4}nmk + nm + \frac{1}{4}n^2M. \tag{8}$$

Next, we find an upper bound on $\sum_{i\geq M} a'_i\frac{1}{64}i^2\frac{k}{m}$. If $i > 2k$, then $a'_i = 0$, since the intersection of two blocks has at most $k$ elements. Therefore, using Claim 3,

$$\sum_{i\geq M} a'_i\frac{1}{64}i^2\frac{k}{m} \leq \frac{k^2}{32m}\sum_{i\geq M} a'_i i \leq \frac{k^2}{32m}nm \leq \frac{1}{32}nk^2. \tag{9}$$

From (8) and (9), it follows that

$$\sum_{i\geq M} a_i\frac{1}{64}i^2\frac{k}{m} \leq \frac{1}{32}nk^2 + nm + \frac{1}{4}nmk + \frac{1}{4}n^2M. \tag{10}$$

Now, we find an appropriate lower bound on $\sum_{i\geq M} a_i\frac{1}{64}i^2\frac{k}{m}$.

Clearly, $\sum_{i=0}^n a_i i$ is the number of triples $(u, v, x)$, where $u, v \in \{1, \ldots, n\}$, $u < v$, and $x \in C_u \cap C_v$. Let $H$ be a bipartite graph on $V(H) = A \cup B$, where the vertices of $A$, $|A| = n$, correspond to the circular sequences $C_u$, $u \in \{1, \ldots, n\}$, and the vertices of $B$, $|B| = n$, correspond to the elements $\{1, \ldots, n\}$. Edge $e = ab$, $a \in A$, $b \in B$, is in $E(H)$, if the circular sequence corresponding to $a$ contains the element corresponding to $b$.

Let $d_1, \ldots, d_n$ denote the degrees in $H$ of the vertices in $B$. We may assume that $d_i \geq 2$ for every $1 \leq i \leq n$, for otherwise there is an element $i$ which appears in at most one circular sequence. Removing this sequence we obtain $n-1$ circular sequences of elements of $\{1, \ldots, n\} \setminus \{i\}$, and $m \leq 16(n-1)^{3/5}$ by induction hypothesis. This is contradicting our assumption that $m > 16n^{3/5}$. Therefore, using $\sum_{i=0}^{n} a_i i = \sum_{i=1}^{n} \binom{d_i}{2}$ and the mean inequality, we obtain

$$\sum_{i=0}^{n} a_i i \geq \sum_{i=1}^{n} \frac{1}{4} d_i^2 \geq \frac{1}{4n} (\sum_{i=1}^{n} d_i)^2 = \frac{|E(H)|^2}{4n} \tag{11}$$

Since $|E(H)| = nm$, then $\sum_{i=0}^{n} a_i i \geq \frac{1}{4} nm^2$. Using $\sum_{i=0}^{n} a_i = \binom{n}{2}$, we obtain

$$\sum_{i \geq M} a_i i \geq \frac{1}{4} nm^2 - \frac{1}{2} n^2 M = \frac{1}{8} m^2 n, \tag{12}$$

where the last equality follows from $M = \frac{m^2}{4n}$. Thus,

$$\sum_{i \geq M} a_i \frac{1}{64} i^2 \frac{k}{m} \geq M \frac{k}{64m} \sum_{i \geq M} a_i i \geq \frac{1}{2^9} Mkmn. \tag{13}$$

Finally, from (10) and (13), we have

$$\frac{1}{2^9} Mknm \leq \frac{1}{2^5} nk^2 + nm + \frac{1}{4} nmk + \frac{1}{4} n^2 M. \tag{14}$$

Using $M = \frac{m^2}{4n}$ and $k = \frac{1}{256} \frac{m^4}{n^2}$, (14) is equivalent to

$$2^2 m^7 n \leq m^8 + 2^{21} n^4 m + 2^{11} m^5 n^2 + 2^{17} m^2 n^4. \tag{15}$$

Since $m \leq 2n^{2/3}$, (15) implies

$$m^7 n \leq 2^{20} n^4 m^2,$$

which is a contradiction with $m > 16n^{3/5}$. □

## 4. Concluding remarks

In this paper we obtained the first nontrivial upper bound on $ex_{\mathrm{cr}}(n, C_4)$, i.e., the maximum number of edges that a topological graph with $n$ vertices can have if it contains no self-intersecting copy of $C_4$. We showed that $ex_{\mathrm{cr}}(n, C_4) = O(n^{8/5})$. It is an interesting open problem to either reduce the exponent in this bound or construct a topological graph on $n$ vertices that does not contain a self-intersecting copy of $C_4$, and the number of its edges is asymptotically larger than $n^{3/2}$. In this direction, it would also be exciting to improve Lemma 2, the main lemma in our proof, which is a purely combinatorial statement with no geometry involved.

It would be interesting to find nontrivial upper bounds for $ex_{\mathrm{cr}}(n, C_{2k})$, for any $k > 2$. It is proved in [**11**] that a $C_4$-free $x$-monotone topological graph on vertices and no self-intersecting $C_6$ has $O(n^{4/3} \log^{2/3} n)$ edges.

We cannot prove an $o(n^{8/5})$ bound even in the case of *geometric* graphs. For *convex* geometric graphs (geometric graphs with the set of vertices in convex position in the plane), however, it is very easy to observe that $ex_{\mathrm{cr}}(n, C_4) = \Theta(n^{3/2})$ [**4**].

Finally, let us remark that we do not know of a single forbidden subgraph $H$ such that $ex(n, H) = o(n^{5/3})$, $ex(n, H) = \Omega(n^{3/2})$, and any drawing of $H$ in the plane contains a self-intersecting copy of $C_4$. It is easy to see that $Q_8$, the graph of the 3-dimensional cube with the main diagonal, which satisfies $ex(n, Q_8) = O(n^{8/5})$ [**5, 6**], can be drawn in the plane with no self-intersecting $C_4$.

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