Polygons: Odd Compression Ratio and Odd Plane Coverings

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Abstract

Let P be a polygon with rational vertices in the plane. We show that for any finite odd-sized collection of translates of P, the area of the set of points lying in an odd number of these translates is bounded away from 0 by a constant depending on P alone.

The key ingredient of the proof is a construction of an odd cover of the plane by translates of P. That is, we establish a family $\mathcal{F}$ of translates of P covering (almost) every point in the plane a uniformly bounded odd number of times.

1 Introduction

The starting point of this research is the following isoperimetric-type problem about translates of compact sets in $\mathbb{R}^d$:

Let $X \subset \mathbb{R}^d$ be a compact set, and let $Z \subset \mathbb{R}^d$ be a finite set of odd cardinality. Consider the finite odd-sized collection $\mathcal{F} = \{X + z\}_{z \in Z}$ of translates of X. Let $U \subset \mathbb{R}$ be the set of all points that belong to an odd number of the members of $\mathcal{F}$. How small can be the Lebesgue measure of U in terms of the Euclidean measure of X?

Denoting the infimum of this value by $Vol_{odd}(X)$, called the *odd volume* of X, we define the *odd compression ratio* of X as $\alpha^\circ(X) = Vol_{odd}(X) \,/\, Vol(X)$, where $Vol(X)$ is the Euclidean volume of X. Observe that $\alpha^\circ(X) \leq 1$, as $\mathcal{F}$ may consist of a single element X. Clearly, $\alpha^\circ(X)$ is an affine invariant.

It was observed by the second author about a decade ago that α° of a unit d-cube Q^d is 1. Indeed, informally, consider $\mathbb{R}^d$ under the action (i.e., translation) of $\mathbb{Z}^d$. The unit cube (with parts of its boundary removed) is a fundamental domain of $\mathbb{R}^d/\mathbb{Z}^d$. The quotient map $\phi : \mathbb{R}^d \to Q^d$ maps any translate of Q^d onto Q^d in an one-to-one manner. Moreover, the quotient map satisfies

$$\phi\left(\bigoplus_{\mathcal{T}\in\mathcal{F}} T\right) \;=\; \bigoplus_{\mathcal{T}\in\mathcal{F}} \phi(T)\,,$$

where $\bigoplus$ denotes the set-theoretic union modulo 2, i.e., the set of all points covered by an odd number of the members of $\mathcal{F}$. Since the quotient map is locally volume preserving, it is globally volume-nondecreasing, and so one concludes that the volume of $\bigoplus_{\mathcal{T}\in\mathcal{F}} T$ is at least that of $\phi(\bigoplus_{\mathcal{T}\in\mathcal{F}} T) \;=\; \bigoplus_{\mathcal{T}\in\mathcal{F}} \phi(T) \;=\; \bigoplus_{\mathcal{T}\in\mathcal{F}} Q^d \;=\; Q^d$.

A similar argument shows that α° of a centrally symmetric planar hexagon is 1 as well. But what about other sets, i.e., a triangle? The second author vividly remembers discussing this

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question with Jirka Matoušek in a pleasant cafe at Malà Strana, laughing that they are too old for Olympiad-type problems...[1]

The value of α^0 of the triangle (recall that any two triangles are affinely equivalent) was determined by the first author in [1]; it is $\frac{1}{2}$.

Next significant progress on the problem was obtained in [2]. It was shown there that for a union of two disjoint intervals of length 1 on a line with with a certain irrational distance between them, the odd compression ratio is 0. The proof uses some algebra of polynomials, and Diophantine Approximation. The construction easily extends to higher dimensions. In addition, [2] introduced a technique for obtaining lower bounds on $\alpha^\circ(X)$, and used it to show that for X's that are unions of finitely many cells of the 2-dimensional grid, $\alpha^\circ(X) > 0$.

In the present paper we further develop the technique of [2], and use it to prove that for any planar rational polygon P, the odd compression ratio $\alpha^\circ(P)$ is bounded away from 0 by some positive constant explicitly defined in terms of P. In fact, the statement applies to any compact planar figure with piecewise linear boundary, and (finitely many) rational vertices. In view of the above mentioned result from [2], the assumption of rationality cannot in general be dropped.

Perhaps more importantly, the value of $\alpha^\circ(X)$ is related here to the value of some other natural geometric invariant of X. The other invariant is $\theta^\circ(X)$, the smallest possible *average density* in an *odd cover* of $\mathbb{R}^2$ by a family $\mathcal{F}$ of translates of X. By odd cover we mean that every point $p \in \mathbb{R}^2$, with a possible exception of a measure 0 set, is covered by the members of $\mathcal{F}$ an odd and uniformly bounded number of times.

While [2] does not directly consider odd covers of $\mathbb{R}^2$, it still implies that $\alpha^\circ(X) \geq \theta^\circ(X)^{-1}$. We include here two complete proofs of this useful inequality.

The existence of odd covers of $\mathbb{R}^2$ by translates of a rational polygon is by no means obvious. Most of the present paper is dedicated to constructing such covers. We are aware of no related results in the literature.

While many of the results and constructions presented here can be easily extended to higher dimensions, some essential parts resist simple generalization, and more work is required in order to understand the situation there.

To conclude the Introduction, we hope that the present paper will somewhat elucidate the meaning of the odd compression ratio $\alpha^\circ(X)$, and that the odd covers introduced here will prove worthy of further study.

2 Preliminaries

2.1 Two Basic Operators

In what follows, we shall extensively use the following two operators on subsets of $\mathbb{R}^2$: $\oplus$ and $\mathring{+}$. Let us briefly discuss them here.

The first operator $\oplus$ is the set-theoretic union modulo 2. Given a family $\mathcal{F}$ of subsets of $\mathbb{R}^2$ so that any $p \in \mathbb{R}^2$ is covered at most finitely many times by $\mathcal{F}$, $\bigoplus_{X\in\mathcal{F}} X$ is the set of all points of $\mathbb{R}^2$ covered by an odd number of members in $\mathcal{F}$. Observe that $\oplus$ is commutative and associative.

The second operator, $\mathring{+}$, is less standard. It is the Minkowski sum modulo 2:

$$X \mathring{+} Z \;=\; \bigoplus_{x\in X,\, z\in Z} x + z \;=\; \bigoplus_{z\in Z} (X + z)\,,$$

[1] A discrete version of the problem about the translates of a square in $\mathbb{R}^2$ indeed eventually found its way to a Mathematical Olympiad [4].

where $X+z$ denotes the translate of X by z. Unlike the Minkowski sum, $X \mathring{+} Z$ is well defined only when every $a \in \mathbb{R}^2$ has at most finitely many representations of the form $x+z$ as above. This requirement is met, e.g., when Z is finite, or when Z is a discrete set of points at distance $\geq \epsilon > 0$ from each other, and X is bounded. Since the Minkowski sum extends to any finite number of sets, and it is commutative and associative, the same holds for $\mathring{+}$ (provided, as before, that every a has finitely many representations).

Moreover, the following distributive law holds. Let $\mathcal{G}$ be a family of sets in $\mathbb{R}^2$, and let $S \subset \mathbb{R}^2$. Assume that the family of sets $\{Y+s\}_{Y\in\mathcal{G},\, s\in S}$ covers any point of $\mathbb{R}^2$ at most finitely many times. Then:

$$\left(\bigoplus_{Y\in\mathcal{G}} Y\right) \mathring{+}\ S \ = \ \bigoplus_{Y\in\mathcal{G}} (Y \mathring{+} S)\,. \tag{1}$$

Indeed, the equality is trivial when S consists of a single element. Thus, by definition of $\mathring{+}$,

$$\left(\bigoplus_{Y\in\mathcal{G}} Y\right) \mathring{+}\ S \ = \ \bigoplus_{s\in S}\left(\left(\bigoplus_{Y\in\mathcal{G}} Y\right)+s\right) \ = \ \bigoplus_{s\in S}\bigoplus_{Y\in\mathcal{G}}(Y+s) \ \overset{*}{=} \ \bigoplus_{Y\in\mathcal{G}}\bigoplus_{s\in S}(Y+s) \ = \ \bigoplus_{Y\in\mathcal{G}}(Y \mathring{+} S)\,.$$

It remains to validate the change of order of summation in the starred equality. For $a \in \mathbb{R}^2$ consider the set $\{(Y,s)\,|\, a \in Y+s\} \subseteq \mathcal{G}\times S$. By our assumptions, this set is always finite. Therefore, for any a, $\mathbf{1}_a\big(\bigoplus_{s\in S}\bigoplus_{Y\in\mathcal{G}}(Y+s)\big) \ = \ \bigoplus_{s\in S}\bigoplus_{Y\in\mathcal{G}}\mathbf{1}_a(Y+s)$ has only finitely many nonzero terms. Hence, the order of summation in the double sum $\bigoplus_{s\in S}\bigoplus_{Y\in\mathcal{G}}(Y+s)$ is interchangeable.

Finally, notice that similarly to Minkowski sum, $X \mathring{+} \emptyset = \emptyset$, while $X \oplus \emptyset = X$.

2.2 Covers and Their Densities

It is important to stress that throughout this paper whenever we speak on covers or odd covers of the plane it always means covering **up to a set of measure** 0*, even if it is not explicitly said so. This convention helps to avoid discussing unnecessary technicalities related to the boundaries of the sets in the cover.*

For every compact measurable set $X \subset \mathbb{R}^2$, we denote by $A(X)$ the Euclidean area of X. Let $Z \subseteq \mathbb{R}^2$ be a discrete set. The family $\mathcal{F} = \{X+z\}_{z\in Z}$ is called a *cover* of $\mathbb{R}^2$ if $X+Z=\mathbb{R}^2$, up to a set of measure 0. We shall further require that the maximal degree of this cover is uniformly bounded, that is, there exists a constant $d_{\mathcal{F}}$ such that every $a \in \mathbb{R}^2$ belongs to at most $d_{\mathcal{F}}$ members of $\mathcal{F}$.

The (lower) *density* of $\mathcal{F}$, $\rho(\mathcal{F})$, is defined by

$$\rho(\mathcal{F}) \ = \ \liminf_{n\to\infty} \frac{\sum_{z\in Z} A(\,Q_n \cap (X+z))}{n^2} \ \geq \ 1\,,$$

where Q_n is the $n \times n$ square centered at the origin.

Since $\sum_{z\in Z} A(\,Q_n \cap X+z)/n^2$ is precisely the average of the cover degrees $d_{\mathcal{F}}(a)$ where a ranges over Q_n, the density $\rho(\mathcal{F})$ can be viewed as a kind of an average degree of the cover of $\mathbb{R}^2$ by $\mathcal{F}$.

Fixing Z and alternating X, the density of the family $\mathcal{F} = \{X+z\}_{z\in Z}$ is proportional to $A(X)$:

Claim 2.1. $\rho(\mathcal{F}) \ = \ c_Z \cdot A(X)$*, where c_Z is a constant depending solely on Z. (In particular, when Z is a lattice, c_Z is the reciprocal of the area of the fundamental domain of Z.)*

Proof. (Sketch) Assume w.l.o.g., that X contains the origin, and let $\delta = Diam(X)$. Consider $\Delta_n = \left| \sum_{z \in Z} A(Q_n \cap X + z) - |Z \cap Q_n| \cdot A(X) \right|$. How big can it be? On one hand,

$$|Z \cap Q_{n-2\delta}| \cdot A(X) \leq \sum_{z \in Z} A(Q_n \cap X + z) \leq |Z \cap Q_{n+2\delta}| \cdot A(X),$$

and therefore

$$\Delta_n \leq |Z \cap Q_{n+2\delta}| \cdot A(X) - |Z \cap Q_{n-2\delta}| \cdot A(X) = |Z \cap (Q_{n+2\delta} \setminus Q_{n-2\delta})| \cdot A(X).$$

On the other hand, since $Z \cap (Q_{n+2\delta} \setminus Q_{n-2\delta}) + X$ is contained in $Q_{n+4\delta} \setminus Q_{n-4\delta}$, covering no point there more than $d_{\mathcal{F}}$ times, it follows that $|Z \cap (Q_{n+2\delta} \setminus Q_{n-2\delta})| \cdot A(X)$ is at most $O(n) \cdot \delta \cdot d_{\mathcal{F}}$. Hence, $\Delta_n = O(n) \cdot \delta \cdot d_{\mathcal{F}}$, and so $\Delta_n / n^2 \to 0$. The conclusion follows:

$$\rho(\mathcal{F}) = \liminf_{n \to \infty} \frac{\sum_{z \in Z} A(Q_n \cap X + z)}{n^2} = \liminf_{n \to \infty} \frac{|Z \cap Q_n| \cdot A(X) \pm \Delta_n}{n^2} =$$

$$\liminf_{n \to \infty} \frac{|Z \cap Q_n|}{n^2} \cdot A(X) = c_Z \cdot A(X).$$

The fact that for a lattice Z, $\lim_{n \to \infty} |Z \cap Q_n| / n^2$ is the inverse of the the area of the fundamental domain of Z, is well known (see, e.g., [3]). □

The *covering density* of X, $\theta(X) \geq 1$, is defined as the infimum of $\rho(\mathcal{F})$ over all covers $\mathcal{F}$ as above. It is well known (see, e.g., [3]) that $\theta(X)$ is an affine invariant.

2.3 Odd Covers

Let $X \subset \mathbb{R}^2$ be a compact set of a positive area $A(X) > 0$. The family $\mathcal{F} = \{X + z\}_{z \in Z}$ for $Z \subseteq \mathbb{R}^2$ is called an *odd cover* of $\mathbb{R}^2$ if $X \mathring{+} Z$ is well defined, and equals to $\mathbb{R}^2$ up to a set of measure 0. Notice that if $\mathcal{F}$ is an odd cover of $\mathbb{R}^2$, then in particular it is a cover of $\mathbb{R}^2$. As before, we shall further require that the maximal degree of the cover of $\mathbb{R}^2$ by $\mathcal{F}$ is uniformly bounded.

The *odd covering density* of a compact X, $\theta^\circ(X) \geq 1$ is defined as the infimum of $\rho(\mathcal{F})$ over all odd covers $\mathcal{F}$ as above. If no such $\mathcal{F}$ exists, set $\theta^\circ(X) = \infty$. Notice that $\theta^\circ(X) \geq \theta(X)$. Similarly to the usual covering density $\theta(X)$, the odd covering density $\theta^\circ(X)$ is an affine invariant. This intuitively plausible statement can be proved formally along the same lines as the standard proof of the corresponding statement for the usual covers (see, e.g., [3]). [2]

2.4 Odd Compression Ratio: the Definition

Let $X \subset \mathbb{R}^2$ be a compact set of area $0 < A(X) < \infty$. Define $A_{odd}(X)$, the *odd area* of X, to be the maximum number such that for any finite and odd-sized collection $\mathcal{F}$ of translates of X, the set of all points in $\mathbb{R}^2$ belonging to an odd number of members of $\mathcal{F}$ has area $\geq A_{odd}(X)$. I.e., $A_{odd}(X)$ is the infimum of $A(X \mathring{+} K)$ over all finite odd-sized sets $K \subset \mathbb{R}^2$ (see [1, 2]).

Define $\alpha^\circ(X)$, the *odd compression ratio* of X, as $A_{odd}(X)/A(X)$. Clearly, $0 \leq \alpha^\circ(X) \leq 1$, and it is an affine invariant.

[2] The requirement that $\mathcal{F}$ has a uniformly bounded degree of cover does not appear in the standard definition of $\theta(X)$, despite the fact that it is used in the proof of the affine invariance of $\theta(X)$ and elsewhere. The reason is that for any $\epsilon > 0$, a cover $\mathcal{F}$ can be easily modified into a *periodic* cover $\mathcal{F}'$ with $\rho(\mathcal{F}') \leq \rho(\mathcal{F}) + \epsilon$, i.e., the corresponding Z' is of the form $\Lambda + K$, where Λ is a lattice, and K is finite (see, e.g., [3]). Thus, w.l.o.g., one may restrict the discussion of $\theta(X)$ to periodic covers, and those are always uniformly bounded for a compact X. In contrast, the odd covers apparently do not allow such a modification, and so the assumption about the uniformly bounded degree seems to be essential for them. This said, all odd covers occurring in this paper are periodic.

3 The Odd Cover Lemma

The following lemma, a variant, and in fact a special case, of Lemma 1 from [2], is a useful tool for obtaining lower bounds on the odd compression ratio of X. For completeness, we provide two different proofs for it. The first is shorter and simpler due to the preparation done in Section 2.2. It is a streamlined variant of the proof used in [2]. The second proof follows a somewhat different logic, and can be viewed as a generalization of the factor-space argument mentioned in the Introduction.

Lemma 3.1. *For any compact set X of a positive measure in $\mathbb{R}^2$, the odd compression ratio of X is at least the reciprocal of its odd covering density. That is,*

$$\alpha^\circ(X) \;\geq\; \theta^\circ(X)^{-1} \;.$$

Proof. (A) Let $\mathcal{F} = \{X + z\}_{z\in Z}$ be an odd cover of $\mathbb{R}^2$ of density $\rho(\mathcal{F})$, and maximal cover degree $d_{\mathcal{F}} < \infty$. (If no such $\mathcal{F}$ exists, the lemma is trivially true.) Let $K \subset \mathbb{R}^2$ be any finite set of odd cardinality. Set $Y = X \mathring{+} K$.

Consider the set $(X \mathring{+} Z) \mathring{+} K$. On one hand, it is equal to $\mathbb{R}^2$, up to a set of measure 0. This is because $(X \mathring{+} Z) = \mathbb{R}^2$, again up to a set of measure 0, and the cardinality of K is odd.

On the other hand, using the commutativity of $\mathring{+}$, one concludes that $(X \mathring{+} Z) \mathring{+} K = (X \mathring{+} K) \mathring{+} Z = Y \mathring{+} Z$. In other words, the family $\mathcal{G} = \{Y + z\}_{z\in Z}$ is an odd cover of $\mathbb{R}^2$ of a maximal covering degree at most $d_{\mathcal{F}} \cdot |K|$.

By Claim 2.1, there is a constant c_Z depending only on Z, such that for every measurable set $W \subset \mathbb{R}^2$ such that $\{W + z\}_{z\in Z}$ is a cover of $\mathbb{R}^2$, it holds that $\rho(\{W + z\}_{z\in Z}) = c_Z \cdot A(W)$. Therefore,

$$1 \;\leq\; \rho(\mathcal{G}) \;=\; c_Z \cdot A(Y) \;=\; \rho(\mathcal{F}) \cdot \frac{A(Y)}{A(X)} \qquad\Longrightarrow\qquad \rho(\mathcal{F})^{-1} \;\leq\; \frac{A(Y)}{A(X)} \;.$$

Taking the infimum over all odd-sized K's to minimize $A(Y)/A(X)$, and the infimum over all legal Z's to minimize $\rho(\mathcal{F})$, one concludes that $\theta^\circ(X)^{-1} \;\leq\; \alpha^\circ(X)$. □

Proof. (B) Let $\mathcal{F} = \{X + z\}_{z\in Z}$ be an odd cover of $\mathbb{R}^2$ as before, and let $S \subset \mathbb{R}^2$ be compact. Let $S(z) = S \cap (X + z)$. Obviously, $S(z) - z = (S - z) \cap X$ is a subset of X. Consider the following mapping ϕ of the compact sets S to the compact subsets of X:

$$\phi(S) \;=\; \bigoplus_{z\in Z}(S(z) - z) \;=\; \bigoplus_{z\in Z} (S - z) \cap X \;=\; (S \mathring{+} (-Z)) \cap X \,.$$

Claim 3.1.

1. *$\phi(-X + a) \;=\; X$;*
2. *$\phi\left(\bigoplus_{i=1}^k S_i\right) \;=\; \bigoplus_{i=1}^k \phi(S_i)$;*
3. *$A(\phi(S)) \leq A(S) \cdot \tilde{d}_{\mathcal{F}}(S)$, where $\tilde{d}_{\mathcal{F}}(S)$ is the average degree of cover of S by $\mathcal{F}$, i.e., the average of the cover degrees $d_{\mathcal{F}}(a)$, where a ranges over S.*

Proof. Indeed, for (1), keeping in mind that $X \mathring{+} Z = \mathbb{R}^2$, and that $a - \mathbb{R}^2 = \mathbb{R}^2$, one gets

$$\phi(-X+a) \;=\; \bigoplus_{z\in Z} (-X+a-z) \cap X \;=\; X \cap \bigoplus_{z\in Z} a+(-X-z) \;=\; X \cap\, a-(X \mathring{+} Z) \;=\; X \cap \mathbb{R}^2 \;=\; X \,.$$

For (2),

$$\phi\left(\bigoplus_{i=1}^{k} S_i\right) = \bigoplus_{z\in Z}\left(\left(\bigoplus_{i=1}^{k} S_i \; - z\right)\cap X\right) \; = \; \bigoplus_{z\in Z}\bigoplus_{i=1}^{k}(S_i - z)\cap X \; = \; \bigoplus_{i=1}^{k}\bigoplus_{z\in Z}(S_i - z)\cap X \; = \; \bigoplus_{i=1}^{k} \phi(S_i)\,.$$

For (3), since $\{X+z\}_{z\in Z}$ is a cover of $\mathbb{R}^2$,
$A(\phi(S)) \; = \; A\left(\sum_{z\in Z}(S\cap(X+z))-z)\right) \; \le \; \sum_{z\in Z} A(S\cap(X+z)) \; = \; A(S)\cdot\tilde{d}_{\mathcal{F}}(S)$. □

Instead of proving a lower bound on $\alpha^\circ(X)$, we shall prove one for $\alpha^\circ(-X+a)$, with a suitably chosen a. Since $\alpha^\circ(X)$ is invariant under affine transformations of $\mathbb{R}^2$, $\alpha^\circ(-X+a)=\alpha^\circ(X)$. For typographical reasons, set $X_a=-X+a$.

Consider, as before, any finite set $K\subset\mathbb{R}^2$ of odd cardinality, and let $Y_a = X_a \mathring{+} K$. On one hand, by Claim 3.1(3), $A(\phi(Y_a)) \; \le \; \tilde{d}_{\mathcal{F}}(Y_a)\cdot A(Y_a)$. On the other hand, by Claim 3.1(2)&(1), $\phi(Y_a) \; = \; \phi(\bigoplus_{k\in K}(X_a+k)) \; = \; \bigoplus_{k\in K}\phi(X_a+k) \; = \; \bigoplus_{k\in K} X \; = \; X$. Thus,

$$\tilde{d}_{\mathcal{F}}(Y_a)\cdot A(Y_a) \; \ge \; A(\phi(Y_a)) \; = \; A(X) \qquad \Longrightarrow \qquad \frac{A(Y_a)}{A(X_a)} \; \ge \; \tilde{d}_{\mathcal{F}}(Y_a)^{-1}\ .$$

It remains to choose the translation vector a as to minimize $\tilde{d}_{\mathcal{F}}(Y_a)$. Getting back to the discussion of Section 2.2, a simple averaging argument shows that for a random uniform $a\in Q_n$, the expected value of $\tilde{d}_{\mathcal{F}}(Y_a)$ approaches $\tilde{d}_{\mathcal{F}}(Q_n)$ as n tends to infinity. Keeping in mind the definition of $\rho_{\mathcal{F}}$, this implies in turn that there is a sequence of a's such that $\tilde{d}_{\mathcal{F}}(Y_a)$ approaches $\rho_{\mathcal{F}}$. Minimizing over all legal odd covers $\mathcal{F}$, one concludes that the infimum of $\tilde{d}_{\mathcal{F}}(Y_a)$ over $a\in\mathbb{R}^2$ is at most $\theta^\circ(X)$. □

To demonstrate the usefulness of Lemma 3.1, assume that there is a tiling of $\mathbb{R}^2$ by translates of X. Then, $\theta^\circ(X)=1$, implying $\alpha^\circ(X)=1$. This yields the aforementioned result about the non-compressibility of the square and the centrally symmetric hexagon.

Further, assume that X is a triangle (a,b,c). Let Λ be the lattice spanned by $\{\frac{1}{2}(b-a),\ \frac{1}{2}(c-a)\}$. Then, $\mathcal{F}=\{X+z\}_{z\in\Lambda}$ is an odd cover of $\mathbb{R}^2$ covering each point in the plane either 1 or 3 times, with $\rho(\mathcal{F})=2$. This implies $\alpha^\circ(X)\ge\frac{1}{2}$, matching the optimal bound of [1].

4 Odd Covers by Stripe Patterns

A *stripe pattern* is a (non-singular) affine image of the set $\{(x,y)\in\mathbb{R}^2 \mid \lfloor y\rfloor \text{ is even}\}$. I.e., it is an infinite set of parallel stripes of equal width w, such that the distance between any two adjacent stripes is w as well (see Figure 1). The *direction* of a stripe pattern is, expectedly, the direction of a boundary line of any stripe in it. The *width* of the stripe pattern is the w as above.

Figure 1: A stripes pattern

We start with the following simple but useful observation about stripe patterns. The easy verification is left to the reader.

Observation 4.1. *Let S be a stripe pattern, and let ℓ and r be the two lines delimiting one of the stripes in S. Then, for every $a \in \ell$, $b \in r$, and $v = b - a$, it holds that:*

1. $S \mathring{+} \{0, v\} \; = \; S \oplus (S + v) \; = \; \mathbb{R}^2$.
2. $S \mathring{+} \{0, \frac{1}{2}v\} = S \oplus (S + \frac{1}{2}v)$ *is a stripe pattern with the same direction as S, whose width is equal to half of the width of S.*

The main result of this section is:

Lemma 4.1. *Let $S_1, \ldots, S_k$ be stripe patterns with pairwise distinct directions, and let $T = S_1 \oplus \cdots \oplus S_k$. Then, there exists a finite (and efficiently computable) set of vectors $U \subset \mathbb{R}^2$, $|U| \leq 2^{k-1}$, such that $T \mathring{+} U = \bigoplus_{u_i \in U} (T + u_i) = \mathbb{R}^2$, up to a set of measure 0.*

It will be technically more convenient to prove the following more general statement:

Lemma 4.2. *Let $S_1, \ldots, S_k$ be stripe patterns with pairwise distinct directions, and let $\{Z_i\}_{i=1}^k$ be a family of finite nonempty subsets of $\mathbb{R}^2$, with $Z_1 = \{0\}$. Let $T = \bigoplus_{i=1}^k (S_i \mathring{+} Z_i)$. Then, as before, there exists a finite (and efficiently computable) set of vectors $U \subset \mathbb{R}^2$, $|U| \leq 2^{k-1}$, such that $T \mathring{+} U = \bigoplus_{u_i \in U} (T + u_i) = \mathbb{R}^2$, up to a set of measure 0.*

Lemma 4.1 follows from Lemma 4.2 by setting $Z_i = \{0\}$ for all $i \geq 2$.

Notice the special role of S_1 in the statement of Lemma 4.2. In fact, the condition $Z_1 = \{0\}$ is essential even for $k = 1$. It is easy to verify that, using the notation of Observation 4.1, no finite set of translates of the set $T = S_1 \mathring{+} \{0, \frac{2}{3}v\}$ can oddly cover the plane[3].

Proof. (of Lemma 4.2) For every $i = 1, 2, \ldots, k$, let ℓ_i and r_i denote the two parallel lines delimiting some stripe in S_i. By the assumptions of the Lemma, for different i's these have different directions, and therefore intersect.

The proof is by induction on k.

For $k = 1$, the statement follows from Observation 4.1(1).

For $k = 2$, let a and b be the intersection points of ℓ_1 and r_1 with ℓ_2, respectively. Setting $v_2 = b - a$, we have $S_1 \mathring{+} \{0, v_2\} = \mathbb{R}^2$, by Observation 4.1(1). Moreover, since v_2 has the same direction as of S_2, we have $S_2 \mathring{+} \{0, v_2\} = \emptyset$. Keeping this in mind we have:

$$T \mathring{+} \{0, v_2\} \; = \; \big(S_1 \oplus (S_2 \mathring{+} Z_2)\big) \mathring{+} \{0, v_2\} \; = \; \big(S_1 \mathring{+} \{0, v_2\}\big) \oplus \big(S_2 \mathring{+} Z_2 \mathring{+} \{0, v_2\}\big) \; =$$
$$= \; \mathbb{R}^2 \oplus \big(S_2 \mathring{+} \{0, v_2\} \mathring{+} Z_2\big) \; = \; \mathbb{R}^2 \oplus (\emptyset \mathring{+} Z_2) \; = \; \mathbb{R}^2 \oplus \emptyset \; = \; \mathbb{R}^2$$

For $k > 2$, we proceed as follows. Let v_k be the (well-defined) vector such that, on one hand, $\ell_k + v_k = r_k$, and on the other hand, $\ell_1 + 2v_k = r_1$. Observation 4.1(1) implies that $S_k \mathring{+} \{0, v_k\} = \mathbb{R}^2$, and hence $(S_k \mathring{+} Z_k) \mathring{+} \{0, v_k\}$ equals $\mathbb{R}^2 \mathring{+} Z_k$, which is either $\emptyset$ or $\mathbb{R}^2$, depending on the parity of Z_k. Observation 4.1(2) implies that $S_1 \mathring{+} \{0, v_k\}$ is a stripe pattern with the same direction as S_1, and half its width. Consequently, $\big(S_1 \mathring{+} \{0, v_k\}\big) \oplus \big((S_k \mathring{+} Z_k) \mathring{+} \{0, v_k\}\big)$ is a stripe pattern with the same direction as S_1 and half its width as well.

Consider now the set $T' = T \mathring{+} \{0, v_k\} = T \oplus (T + v_k)$. Using the properties of the operators $\oplus$ and $\mathring{+}$, one gets:

$$T' \; = \; T \mathring{+} \{0, v_k\} \; = \; \left(\bigoplus_{i=1}^{k} (S_i \mathring{+} Z_i)\right) \mathring{+} \{0, v_k\} \; = \; \bigoplus_{i=1}^{k} (\, S_i \mathring{+} Z_i \mathring{+} \{0, v_k\}\,) \qquad (2)$$

[3]Perhaps expectedly, the same T has also the complementary extremal property: $T \mathring{+} \{0, \frac{1}{3}v, \frac{2}{3}v\} = \emptyset$.

As we have just seen, the $\oplus$ of the first and the k'th terms of the latter sum is a stripe pattern S_1' with the same direction as S_1. Thus, setting $Z_i' = Z_i \mathring{+} \{0, v_k\}$, one arrives at

$$T' \;=\; S_1' \;\oplus\; \bigoplus_{i=2}^{k-1} (S_i \mathring{+} Z_i') \tag{3}$$

By the induction hypothesis applied to T', there exists a finite set $U' \subset \mathbb{R}^2$ such that $T' \mathring{+} U' = \mathbb{R}^2$ up to a set of measure 0. However,

$$T' \mathring{+} U' \;=\; T \mathring{+} \{0, v_k\} \mathring{+} U' \;=\; T \mathring{+} (U' \mathring{+} \{0, v_k\}) \tag{4}$$

Therefore, setting $U = U' \mathring{+} \{0, v_k\}$, one concludes that $T \mathring{+} U = T' \mathring{+} U' = \mathbb{R}^2$. This completes the construction of the desired set U.

It remains to estimate the size of U. The recursive definition $U = U' \mathring{+} \{0, v_k\}$ for $k > 2$, combined with the base cases $|U| = 2^{k-1}$ for $k = 1, 2$, implies the desired bound: $|U| \leq 2^{k-1}$. ☐

5 Odd Covers by Rational Polygons: A Special Case

In this section we prove our main theorem for the special case of a rational polygons with no two parallel edges.

Given a rational polygon P, let P_{INT} be the integer polygon of the minimal area affinely equivalent to P, and let $A_{\text{INT}}(P) = A(P_{\text{INT}})$ be its area.

Theorem 5.1. *Let P be a rational polygon with k vertices, and no parallel edges. Then, there exists a bounded degree odd cover $\mathcal{F}$ of $\mathbb{R}^2$ by translates of P with density $\rho(\mathcal{F}) \leq A_{\text{INT}}(P) \cdot 2^{k-1}$. Consequently, $\alpha^\circ(P) \geq A_{\text{INT}}(P)^{-1} \cdot 2^{-(k-1)}$.*

Before starting with the proof, we need one more observation about the structure of $\oplus$-sums of stripe patterns. For $i = 1, \ldots r$, let L_i be an affine image of the family of parallel lines $\{(x, y) \in \mathbb{R}^2 \,|\, y \in \mathbb{Z}\}$. Respectively, let S_i be a stripe pattern whose boundary is L_i. (Notice that there are exactly two such stripe patterns: S_i and its complement $\overline{S}_i = \mathbb{R}^2 \setminus S_i$.) Assume that $S_1, \ldots, S_r$ have pairwise distinct directions. The union of all these lines $\bigcup_{i=1}^r L_i$ partitions $\mathbb{R}^2$ into pairwise disjoint open cells, each cell being a convex polygon. Call two cells *adjacent* if they share a 1-dimensional edge.

It is a folklore to show that the cells of $\mathbb{R}^2 \setminus \bigcup_{i=1}^r L_i$ can be 2-colored in such a way that any two adjacent cells have different colors.

Claim 5.1. *Let T be the union of all cells of $\mathbb{R}^2 \setminus \bigcup_{i=1}^r L_i$ in one color class. Then, (up to the 0-measure boundary of T, i.e., $\bigcup_{i=1}^r L_i$) either $T = S_1 \oplus \cdots \oplus S_r$, or $T = \mathbb{R}^2 \setminus (S_1 \oplus \cdots \oplus S_r) = \overline{S}_1 \oplus S_2 \oplus \cdots \oplus S_r$.*

The claim is rather obvious, and can be formally verified, e.g., by induction on r. The full details are left to the reader (see Figure 2 for an illustration).

Proof. (of Theorem 5.1) Keeping in mind that both $\theta^\circ(P)$ and $\alpha^\circ(P)$ are affine invariants, one may assume without loss of generality that $P = P_{\text{INT}}$, and that the origin $O = (0, 0)$ is a vertex of P. Then, all the vertices of P belong to $\mathbb{Z}^2$. Observe also that some of the edges of P must contain an even number of integer lattice points. Otherwise, the coordinates of the vertices of P would all have the same parity, i.e., they would all be even. Scaling such an all-even P by a factor of $\frac{1}{2}$ would have yielded a smaller integer polygon affinely equivalent to P, contrary to the definition of P_{INT}.

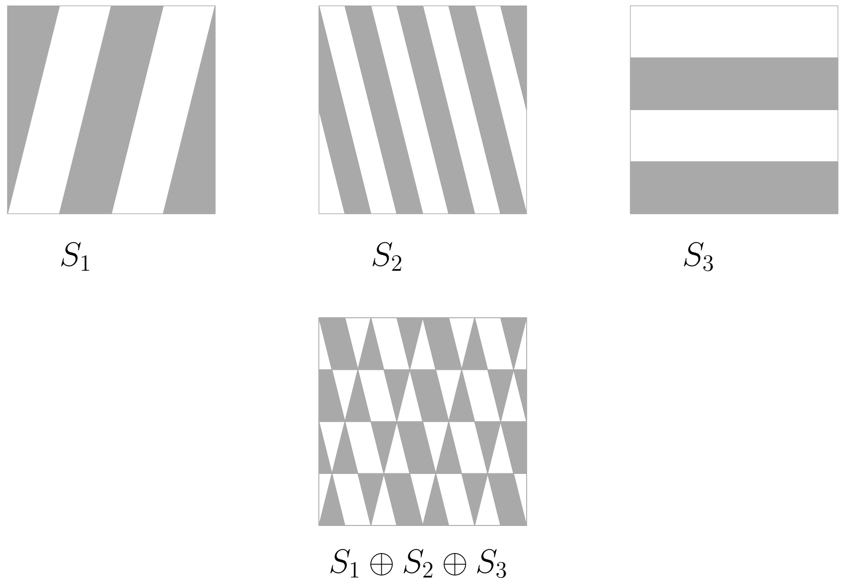

Figure 2: $\oplus$-sum of three stripes patterns

We claim that $P \mathring{+} \mathbb{Z}^2$ is equal to $S_1 \oplus \ldots \oplus S_r$, where $S_1, \ldots, S_1$ are stripe patterns with pairwise distinct directions, and r is at most the number of vertices (=edges) of P. Once this claim is established, the rest easily follows.

Indeed, assuming that the claim holds, by Lemma 4.1 there exists $U \subset \mathbb{R}^2$ with $|U| \leq 2^{r-1}$ such that $(P \mathring{+} \mathbb{Z}^2) \mathring{+} U = \mathbb{R}^2$. Equivalently, the (multi-) family of sets $\mathcal{F} = \{P + z + u\}_{z \in \mathbb{Z}^2,\, u \in U}$ is an odd cover of the plane. To employ the Odd Cover Lemma 3.1, one needs to estimate the density of this cover. Observe that $\{P + z\}_{z \in \mathbb{Z}^2}$ has a bounded maximal degree (being the maximal number of integer lattice points in any translate of P), while its average density is $A(P)$, as mentioned in Claim 2.1. Therefore, the maximal degree of $\mathcal{F}$ is at most $|U|$ times the maximal degree of the cover $\{P + z\}_{z \in \mathbb{Z}^2}$, while $\rho(\mathcal{F})$, the average degree of $\mathcal{F}$, is precisely $A(P) \cdot |U| \leq A(P) \cdot 2^{k-1}$. Hence, $\theta^\circ(P) \leq \rho(\mathcal{F}) \leq A(P) \cdot 2^{k-1}$. Applying the Odd Cover Lemma 3.1 one gets $\alpha^\circ(P) \geq \theta^\circ(P)^{-1} \geq A(P)^{-1} \cdot 2^{-(k-1)}$, as needed.

Thus, it is sufficient to show that $P \mathring{+} \mathbb{Z}^2$ is equal to $S_1 \oplus \ldots \oplus S_r$ as above. In the remainder of this section, we shall focus on proving this claim. The **argument** goes as follows.

Let $E(P)$ denote the set of all edges of P. For $e \in E(P)$, let L_e be the set of all lines parallel to e that contain points of $\mathbb{Z}^2$. Clearly, L_e is discrete set of lines as in Claim 5.1. Consider a point $x \in \mathbb{R}^2$. It belongs to $P \mathring{+} \mathbb{Z}^2$ exactly when $|P \cap (x - \mathbb{Z}^2)|$ is odd. Unless $x \in \bigcup_{e \in E(P)} L_e$, every point x' in a sufficiently small neighborhood of x will satisfy $|P \cap (x - \mathbb{Z}^2)| = |P \cap (x' - \mathbb{Z}^2)|$. Therefore, $P \mathring{+} \mathbb{Z}^2$ is a union of cells of $\mathbb{R}^2 \setminus \bigcup_{e \in E(P)} L_e$.

Call an edge e of P *active* if it contains an even number of integer lattice points, and *passive* otherwise. Respectively, if e is active, all the lines in L_e are called active, and if it is passive, the lines in L_e are called passive.

Let C_1 and C_2 be two adjacent cells in $\mathbb{R}^2 \setminus \bigcup_{e \in E(P)} L_e$ separated by a line $\ell \in L_e$ for some edge e of P. We claim that if e is active, then exactly one of C_1 and C_2 is contained in $P \mathring{+} \mathbb{Z}^2$, and if e is passive, then either both are contained in $P \mathring{+} \mathbb{Z}^2$, or none of them is.

Indeed, let $I \subset \ell$ denote the common 1-dimensional edge of C_1 and C_2. Observe that the only members in the family $\mathcal{F} = \{P + z\}_{z \in \mathbb{Z}}$ that distinguish between C_1 and C_2, that is, contain exactly one of the two, are those that contain I in their boundary. To get a clearer picture of this subfamily, let $J = [p, q] \subset \ell$ be the smallest interval with integer endpoints containing I. Notice that I has no integer points in its interior. Let us view e as a 1-dimensional interval $[s_e, t_e) \subset \mathbb{R}^2$, parallel to, and having the same orientation as, J . Then, $P + z$ contains I if and only if $p \in e + z$. Or, equivalently, $p - z \in e$.

This means that when e is active (i.e., it contains an even number of points in $\mathbb{Z}^2$), I is covered by an odd number of $(P + z)$'s, and when e is passive, it is covered by an even number of $(P + z)$'s. Consequently, in the former case the the degrees of cover of the cells C_1 and C_2 by $\mathcal{F}$ have a different parity, whereas in the latter case the parities are equal. Thus, when e is active, $P \mathring{+} \mathbb{Z}^2$ distinguishes between C_1 and C_2, and when it is passive, it does not. As claimed.

Let $AE(P)$ the (nonempty!) set of active edges of P. The conclusion is that $P \mathring{+} \mathbb{Z}^2$ is a union of cells of $\mathbb{R}^2 \setminus \bigcup_{e \in AE(P)} L_e$, satisfying the assumptions of Claim 5.1. Hence, $P \mathring{+} \mathbb{Z}^2$ is a $\oplus$-sum of stripe patterns, as desired. This completes the proof of Theorem 5.1. □

The assumption that P has no parallel edges was needed to justify the (tacit) assumption that for every line $\ell \in \bigcup_{e \in E(P)} L_e$, there is a *unique* edge e such that any translate of P may have contained in ℓ. When there are parallel edges, most of the argument still applies, however, it may fail at one fine point. The contributions of parallel edges may cancel out, leaving no active lines, and resulting in $P \mathring{+} \mathbb{Z}^2 = \emptyset$. Unfortunately, this situation indeed does occur for some rational polygons P, for example, for the centrally symmetric ones. To overcome this problem, a more refined family of translates will be constructed.

6 A Theorem About $\mathbb{Z}_2$-valued Functions on Integer Lattices

We shall need the following result of an independent interest. It will be proven here for any dimension d, but used in Section 7 only with $d = 2$.

Let $\mathcal{A}$ be a family of finite subsets of $\mathbb{Z}^d$. A function, or, rather, a weighting, $\mathfrak{F} : \mathbb{Z}^d \to \mathbb{Z}_2$, will be called *stable* with respect to $\mathcal{A}$, if for any $A \in \mathcal{A}$, all integer translates of A have the same $\mathfrak{F}$-weight. That is, the value of $\mathfrak{F}(A + p) = \bigoplus_{x \in A+p} \mathfrak{F}(x)$, does not depend on the choice of $p \in \mathbb{Z}^d$, but solely on A.[4] Further, call $\mathfrak{F}$ *0-stable* with respect to $\mathcal{A}$, if it is stable, and moreover, for every $A \in \mathcal{A}$, $\mathfrak{F}(A) = 0$. For example, if the function $\mathfrak{F}$ is everywhere 0, then it is 0-stable with respect to any family $\mathcal{A}$. If it is everywhere 1, it is stable with respect to any family $\mathcal{A}$, and 0-stable if $\mathcal{A}$ consists only of sets of even cardinality.

Theorem 6.1. *Let $\mathcal{A}$ be a (possibly infinite) family of non-empty finite subsets of $\mathbb{Z}^d$, and $\mathcal{A} \neq \emptyset$. There exists a function $\mathfrak{F} : \mathbb{Z}^d \to \mathbb{Z}_2$ that is stable, but not 0-stable, with respect to $\mathcal{A}$.*

Proof. We start with the 1-dimensional case, introducing the key construction to be used in all dimensions.

<u>Case $d = 1$</u>

We define a family $\{\mathfrak{f}_k\}_{k=0}^{\infty}$ of functions from $\mathbb{Z}$ to $\mathbb{Z}_2$ in the following recursive manner. We will show that one of this functions is the desired function $\mathfrak{F}$:

$\mathfrak{f}_0$ **is identically** 1**;**

[4] In this section, the operator $\oplus$ that was originally defined on sets, will be sometime applied to points. For consistency, regard points as single-element sets.

For $k > 0$, $\mathfrak{f}_k(0) = 1$, and $\mathfrak{f}_k(t) \;=\; \mathfrak{f}_k(t-1) \oplus \mathfrak{f}_{k-1}(t-1)$. [5]

For example, $\mathfrak{f}_1(t)$ is 1 if t is even, and 0 otherwise. The next one, $\mathfrak{f}_2(t)$, is 1 if $t \equiv 0, 3 \pmod 4$, and 0 otherwise. Observe that the repeated application of the recursive formula yields for any $c \in \mathbb{N}$,

$$\mathfrak{f}_k(t+c) \;=\; \mathfrak{f}_k(t) \bigoplus_{i=0}^{c-1} \mathfrak{f}_{k-1}(t+i)\,. \tag{5}$$

Claim 6.1. *If $\mathfrak{f}_{k-1}$ is 0-stable with respect to a finite $A \subseteq \mathbb{Z}$, then $\mathfrak{f}_k$ is stable with respect to A.*

Indeed, it suffices to show that for any $p \in \mathbb{Z}$, $\mathfrak{f}_k(A+p+1) = \mathfrak{f}_k(A+p)$. By definition of $\mathfrak{f}_k$,

$$\mathfrak{f}_k(A{+}p{+}1) \;=\; \bigoplus_{t\in A+p+1} \mathfrak{f}_k(t) \;=\; \bigoplus_{t\in A+p+1} \mathfrak{f}_k(t{-}1) \,\oplus \bigoplus_{t\in A+p+1} \mathfrak{f}_{k-1}(t{-}1) \;=\; \mathfrak{f}_k(A{+}p) \,\oplus\, \mathfrak{f}_{k-1}(A{+}p)\,.$$

Since $\mathfrak{f}_{k-1}(A+p) = 0$ by assumptions of the claim, one concludes that $\mathfrak{f}_k(A+p+1) = \mathfrak{f}_k(A+p)$.

Claim 6.2. *For any $k \geq 1$, $\mathfrak{f}_k(t) = 0$ for $1 \leq t \leq k$.*

Indeed, apply induction on k. For $k = 1$, $\mathfrak{f}_1(1) = \mathfrak{f}_1(0) \oplus \mathfrak{f}_0(0) = 1 \oplus 1 = 0$. For $k > 1$, using (5), one concludes that for any t in the range,

$$\mathfrak{f}_k(t) \;=\; \mathfrak{f}_k(0) \oplus \mathfrak{f}_{k-1}(0) \oplus \mathfrak{f}_{k-1}(1) \oplus \ldots \oplus \mathfrak{f}_{k-1}(t-1) \;=\; 1 \oplus 1 \oplus 0 \oplus \ldots \oplus 0 \;=\; 0.$$

We proceed to show that one of $\mathfrak{f}_k$'s satisfies the requirements of the theorem. Observe that $\mathfrak{f}_0$ is stable with respect to $\mathcal{A}$. By Claim 6.1, either there exists $k \geq 0$ such that $\mathfrak{f}_k$ is stable, but not 0-stable (precisely as desired), or all $\mathfrak{f}_k$'s are 0-stable. However, the latter situation does not occur. Consider any nonempty $A \in \mathcal{A}$, and let $a = \min(A)$, $b = \max(A)$, $r = b - a$. Then, by Claim 6.2, $\mathfrak{f}_r(A - a) = 1$, and thus $\mathfrak{f}_r$ is not 0-stable. This completes the case $d = 1$.

General Case

Let L be a linear function from $\mathbb{Z}^d$ to $\mathbb{Z}$ satisfying two requirements. The first requirement is that the coefficient of x_1 in L is 1. The second requirement is that for some $A \in \mathcal{A}$, L attains a minimum on A at a unique point. Such L's exist. E.g., assuming that A can be translated to a subset of a of the cube $[0, r-1]^d$, the function $\sum_{i=1}^{d} r^{i-1} x_i$ is one-to-one on A by the uniqueness of the base-r representation, and so its minimum on A is attained exactly once.

For $k \geq 0$ and $a \in \mathbb{Z}^d$, define $\mathfrak{F}_k(a) \;=\; \mathfrak{f}_k(L(a))$. Respectively, for a finite subset $A \subset \mathbb{Z}^d$, define $\mathfrak{F}_k(A) \;=\; \bigoplus_{a\in A} \mathfrak{F}_k(a)\,.$

The proof proceeds along the same lines as in the 1-dimensional case.

Claim 6.3. *If $\mathfrak{F}_{k-1}$ is 0-stable with respect to a finite $A \subseteq \mathbb{Z}^d$, then $\mathfrak{F}_k$ is stable with respect to it.*

It suffices to show that that for any $p \in \mathbb{Z}^d$, and any unit vector $e \in \mathbb{Z}^d$, $\mathfrak{F}_k(A{+}p{+}e) = \mathfrak{F}_k(A{+}p)$. Let $L(e) = c$. If $c = 0$, the statement is trivial. If $c < 0$ the statement reduces to the case $c > 0$ by considering $-e$ instead of e. Thus, without loss of generality, $c > 0$. By (5), the linearity of L, and the first requirement on it,

$$\mathfrak{F}_k(A{+}p{+}e) \;=\; \bigoplus_{t\in A+p+e} \mathfrak{f}_k(L(t)) \;=\; \bigoplus_{t\in A+p} \mathfrak{f}_k(L(t){+}c) \;=\; \bigoplus_{t\in A+p} \mathfrak{f}_k(L(t)) \,\oplus\, \bigoplus_{i=0}^{c-1} \bigoplus_{t\in A+p} \mathfrak{f}_{k-1}(L(t){+}i) \;=$$

[5] Observe that this recursive formula defines $\mathfrak{f}_k(t)$ for both positive and negative values of t. More explicitly, for $t < 0$ it becomes $\mathfrak{f}_k(t) \;=\; \mathfrak{f}_k(t+1) \oplus \mathfrak{f}_{k-1}(t)$, reducing either k or $|t|$ just as for $t > 0$.

$$= \mathfrak{F}_k(A+p) \oplus \bigoplus_{i=0}^{c-1} \bigoplus_{t\in A+p+i\cdot e_1} \mathfrak{f}_{k-1}(L(t)) = \mathfrak{F}_k(A+p) \oplus \bigoplus_{i=0}^{c-1} \mathfrak{F}_{k-1}(A+p+i\cdot e_1)\,.$$

Since $\mathfrak{F}_{k-1}$ is 0-stable with respect to A, the second summand is 0, and thus $\mathfrak{F}_k(A+p+e) = \mathfrak{F}_k(A+p)$. This concludes the proof of Claim 6.3.

To conclude the proof of the theorem, observe that $\mathfrak{F}_0$ is stable with respect to $\mathcal{A}$, and thus, by Claim 6.3, either there exists $k \geq 0$ such that $\mathfrak{F}_k$ is stable, but not 0-stable, precisely as desired, or all $\mathfrak{F}_k$'s are 0-stable.

As before, the second possibility does not occur. Indeed, by the second requirement on L, there exists $A \in \mathcal{A}$ on which L attains its minimum $a \in \mathbb{Z}$ at a unique point $p \in A$. Then, $L(A - a\cdot e_1) = L(A) - a$ is a subset of $[0,k]$ for some $k \in \mathbb{N}$, and 0 has a unique pre-image $p' = p - a\cdot e_1$. By Claim 6.2, $\mathfrak{F}_k(A - a\cdot e_1) = \mathfrak{f}_k(0) \oplus \bigoplus_{t\in A\setminus p'} \mathfrak{f}_k(L(t)) = 1 \oplus 0 = 1.$ ☐

7 The Main Theorem

We can now prove the main theorem in full generality, making no assumption about parallel edges.

Theorem 7.1. *Let P be a rational polygon with k distinct classes of parallel edges. Then, there exists a bounded degree odd cover $\mathcal{F}$ of $\mathbb{R}^2$ by translates of P with density $\rho(\mathcal{F}) \leq A_{\mathrm{INT}}(P)\cdot 2^{k-1}$. Consequently, $\alpha^\circ(P) \geq A_{\mathrm{INT}}(P)^{-1}\cdot 2^{-(k-1)}$.*

Proof. While the family of translates will be generally different from the one used in the proof of Theorem 5.1, the logical structure of the proof will be essentially identical. Let us re-examine this structure.

Assuming that $P = P_{INT}$, the first and main goal is to construct a family $\mathcal{F} = \{P+z\}_{z\in Z}$, $Z \subseteq \mathbb{Z}^2$, such that $P \mathring{+} Z$ is a $\oplus$-sum of at most k stripe patterns. (In the former proof, Z was just $\mathbb{Z}^2$.) A close reading of the proof of Theorem 5.1 reveals that in order to prove this fact about $\mathcal{F}$, it is sufficient to show that $\mathcal{F}$ has a certain property. To formulate it, let $D(P)$ be the set of all the classes of parallel edges of P, or simply the *directions* of P. For every $d \in D(P)$, let L_d be set of all lines in $\mathbb{R}^2$ in the direction of d that contain integer lattice points. As before, each L_d is a discrete set of parallel lines with equal distances between any two consecutive ones.

Consider the arrangement of lines $\bigcup_d L_d$. Observe that since $Z \subseteq \mathbb{Z}^2$, for any $(P+z) \in \mathcal{F}$, and any open cell C of $\mathbb{R}^2 \setminus \bigcup_d L_d$, either $C \subseteq (P+z)$, or $C \cap (P+z) = \emptyset$.

Property 7.1. *Let I be an edge of the arrangement $\bigcup_d L_d$ (i.e., a common 1-dimensional boundary of two adjacent cells) that lies on a line $\ell \in L_d$, $d \in D(P)$. Then, the parity of the number sets in $\mathcal{F} = \{P+z\}_{z\in Z}$ whose boundary contains I depends only on d, and not on I.*

Moreover, there exist $d \in D(P)$ such that this parity is odd. We call such a direction active, *and denote the set of all active directions by $AD(P)$.*

Once Property 7.1 is established for $\mathcal{F} = \{P+z\}_{z\in Z}$, the **argument** from the proof of Theorem 5.1 implies that $P \mathring{+} Z$ is a (nonempty) union of cells of $\mathbb{R}^2 \setminus \bigcup_{d\in AD(P)} L_d$ that satisfy the assumptions of Claim 5.1. Applying Claim 5.1, one concludes that $P \mathring{+} Z$ is equal to $S_1 \oplus \ldots \oplus S_r$ for some stripe patterns $S_1, \ldots, S_r$, and $r = |AD(P)|$, the number of active directions, is at most $|D(P)| = k$.

Once the main goal is achieved, the rest is easy. Lemma 4.1 is used to conclude that there exists a finite set $U \subset \mathbb{R}^2$ with $|U| \leq 2^{r-1}$, such that $(P \mathring{+} \mathbb{Z}^2) \mathring{+} U = \mathbb{R}^2$. Equivalently, the (multi-) family of sets $\mathcal{F} = \{P+z+u\}_{z\in Z,\, u\in U}$ is an odd cover of the plane. Since Z is a subset of $\mathbb{Z}^2$, the density of this odd cover is at most $A_{\mathrm{INT}}(P) \cdot |U| \le A_{\mathrm{INT}}(P) \cdot 2^{k-1}$. Finally, by the Odd Cover Lemma 3.1, one concludes that $\alpha^\circ(P) \ge A_{\mathrm{INT}}(P)^{-1} \cdot 2^{-(k-1)}$, establishing the theorem.

In view of the above, in order to prove Theorem 7.1, it suffices to construct $Z \subseteq \mathbb{Z}^2$ such that $\mathcal{F} = \{P + z\}_{z \in Z}$ has Property 7.1. The remaining part of this section is dedicated to constructing such Z, and proving that $\mathcal{F}$ has the required property.

The set of translates Z is constructed as follows. Assume that $P = P_{INT}$. In particular, the vertices of P are in $\mathbb{Z}^2$. For every direction $d \in D(P)$, define the vector $v_d \in \mathbb{Z}^2$ as the difference between (*any*) pair of two *consecutive* integer lattice points on (*any*) line in L_d, the set of all lines in direction d through an integer lattice point.

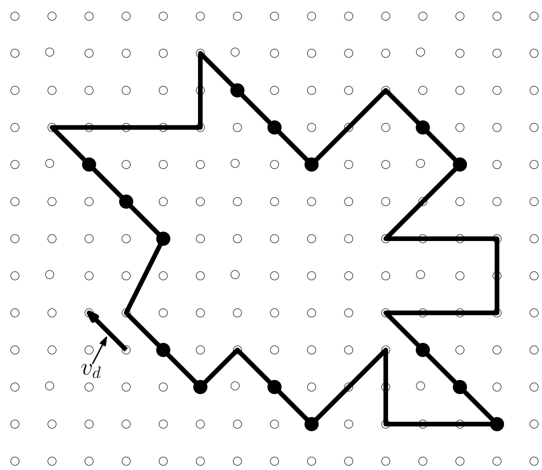

Figure 3: The points of A_d are the filled discs in the picture

Let A_d be the set of all integer lattice points z on the boundary of P such that both z and $z + v_d$ lie on an edge of P in the direction d (see Figure 3). Let $\mathcal{A} = \{-A_d\}_{d \in D(P)}$. By Theorem 6.1, there exists a $\mathbb{Z}_2$-weighting $\mathfrak{F}$ of $\mathbb{Z}^2$ that is stable, but not 0-stable, with respect to $\mathcal{A}$. Define Z as the support of $\mathfrak{F}$, i.e., $Z = \{z \mid \mathfrak{F}(z) = 1\}$. Finally, define $\mathcal{F} = \{P + z\}_{z \in Z}$. Our goal is to show that the family $\mathcal{F} = \{P + z\}_{z \in Z}$ has Property 7.1.

Call a direction d of an edge of P *active* if $\mathfrak{F}(-A_d) = 1$, and *passive* if $\mathfrak{F}(-A_i) = 0$.
We claim that a point $p \in \mathbb{Z}^2$ belongs to an odd number sets in $\{A_d + z\}_{z \in Z}$ if d is active, and to an even number of those sets if d is passive. Indeed, the number of solutions of the equation $a + z = p$, where $a \in A_d$, $z \in Z$, is precisely the size of $(-A_d + p) \cap Z$, and hence its parity is $\mathfrak{F}(-A_d + p) = \mathfrak{F}(-A_d)$, as desired.

Let $I \subset \ell \in L_d$, for some d, be an edge in the arrangement of lines $\bigcup_{d \in D(P)}$. Notice that I cannot contain integer lattice points in its (relative) interior. There exists two consecutive integer lattice points a and b on ℓ such that I is contained in the line segment $J = [p, q] \subset \ell$. Observe that $q - p$ is either v_d or $-v_d$; assume w.l.o.g., that $q - p = v_d$.

We claim that the parity of the number of sets from $\mathcal{F} = \{P + z\}_{z \in Z}$ whose boundary contains J is odd if d is active, and even if it is passive. Indeed, J is contained in the boundary of $(P + z)$, $z \in Z$, if and only if $(A_d + z)$ contains p. As we have already seen, the parity of the number of such sets is odd if and only if d is active. In particular, it depends only on d, and not on I, as desired. Moreover, by Theorem 6.1, there exists at least one active direction d.

This concludes the verification of Property 7.1 for the constructed family $\mathcal{F} = \{P + z\}_{z \in Z}$, which in turn concludes the proof of Theorem 7.1. $\square$

Notice that the above proof makes no use of the connectivity of P nor of the connectivity of its boundary. Thus, as it has been already mentioned in the Introduction, Theorem 7.1 applies equally well to any compact figure in $\mathbb{R}^2$ with non-empty interior, piecewise linear boundary, and finite number of vertices, all of which are rational.

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