Forbidden k-Sets in the Plane

Micha A. Perles[1]
Hebrew University of Jerusalem
and
Rom Pinchasi[2]
Israel Institute of Technology, Technion

September 27, 2006

Abstract

Let A be a set of nonnegative integers. We say that A is *skippable* if there are arbitrary large finite sets of points in the plane, not contained in a line, that determine no k-edge for any $k \in A$. In this paper we show, by construction, that there are arbitrary large skippable sets. We also characterize precisely the skippable sets with at most two elements.

1 Introduction

Let G be a finite set of points in the plane. We say that a line l is *determined* by G, if l passes through at least two points of G. A line l, determined by G, is called a k-edge, if in one of the two open half-planes bounded by l there are precisely k points of G. We say that G *skips* k, if G has no k-edge.

Definition 1.1. *Let A be a set of nonnegative integers. We say that A is* skippable*, if there are arbitrary large finite sets in the plane, that are not collinear, that skip k for every $k \in A$.*

This notion of a skippable set was defined in [PP1]. It is shown there that for any $k \geq 0$ the set $\{k, k+2\}$ is not skippable. This means that if a non-collinear set of points G is large enough, then it has either a k-edge or a $(k+2)$-edge. In this paper we show that skippable sets do exist. In particular, in Theorem 3.1 we show that $\{k\}$

[1]e-mail: perles@math.huji.ac.il
[2]e-mail: room@math.technion.ac.il, This research was conducted when the second author was an M.Sc student in the Hebrew University of Jerusalem

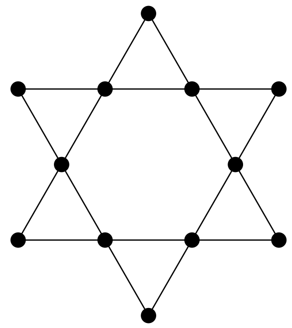

Figure 1: A set of points that skips $k = 2$

is skippable if and only if $k \geq 2$. Moreover, in Theorem 3.2 we show that one can find arbitrary large such sets of positive integers.

We also complete the picture from [PP1], and characterize precisely the skippable sets that consist of two elements (Theorem 4.5).

In the sequel, by referring to a *set of points* we mean a finite set of points in the two dimensional Euclidean plane.

If G is a set of points in general position, namely no three points of G lie on one line, then clearly G has a k-edge for every $0 \leq k \leq |G| - 2$. Therefore in our study of skippable sets we will mainly be concerned with sets that are not in general position. Figure 1 shows an example of a set that skips $k = 2$.

In fact, by adding an arbitrary large number of points very close to the center of the shape in Figure 1, we will remain with a set that skips $k = 2$, thus showing that $\{2\}$ is a skippable set. Similarly, the example in Figure 5 illustrates that $\{4, 5\}$ is a skippable set. The study of skippable sets was initiated by Kupitz and Perles. In [KP], Kupitz and Perles construct arbitrary large sets G, not contained in a line, that skip every k for $k = |G|, |G| - 1, \ldots, |G| - \log_2 |G|$. The question of whether one can fix some skippable values and find arbitrary large non-collinear sets that skip each of them was suggested by Perles. In this paper we give an affirmative answer to this question.

2 Tangency Paths

In this section we define the notion of a *tangency path* and learn its properties in connection with skippable sets. The notion of a tangency path will be crucial for most of the constructions in this paper.

Definition 2.1. *Let P be a convex polygon in the plane. A tangency path for P is a closed polygonal path with vertices $x_0, x_1, \ldots, x_{m-1}, x_m = x_0$ (m can be arbitrary) with the property that if l is the directed line $\overrightarrow{x_i x_{i+1}}$, then l is a tangent of P, $l \cap P$ is contained in the interior of the edge $[x_i, x_{i+1}]$, and the polygon P is in the closed* *half-plane to the left of l. In addition we require that the vertices of the path (namely, $x_0, x_1, \ldots, x_{m-1}$) are pairwise different. (See Figure 2 for an example.)*

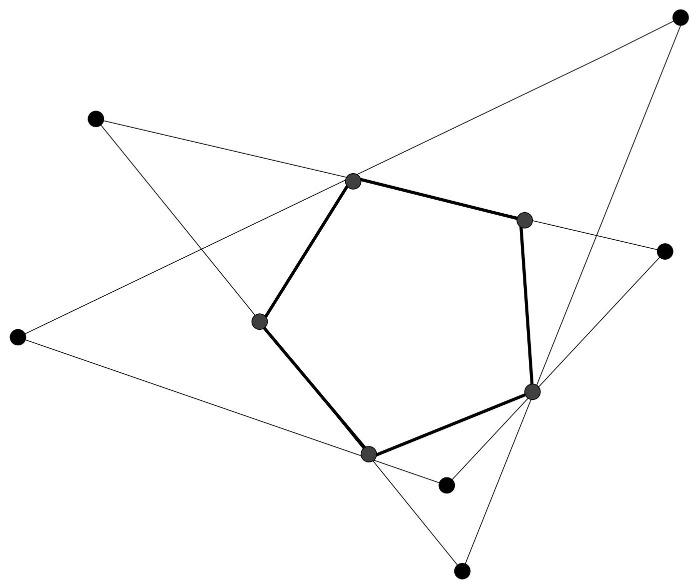

Figure 2: A tangency path to a pentagon

Notation 2.2. *Let G be a set of points in the plane and let l be any directed line. We denote by $A_G(l)$ the number of points of G that are inside the open half-plane to the right of l. We denote by $B_G(l)$ the number of points of G on l. When there is no ambiguity and the set G is known and fixed, we simply write $A(l)$ for $A_G(l)$, and $B(l)$ for $B_G(l)$.*

The following very simple lemma is the key observation regarding tangency paths. We recall that if γ is a closed oriented path in the plane, then the *index* of γ with respect to a point Q, not on γ, is the (counterclockwise) winding number of the path γ around Q. It is well known that if $\overrightarrow{r}$ is any ray emanating from Q, then the number of times γ crosses $\overrightarrow{r}$ from right to left minus the number of times γ crosses $\overrightarrow{r}$ from left to right, equals the index of γ with respect to Q.

Lemma 2.3. *Let P be a convex polygon, let $\mathcal{C}$ be a tangency path for P, and let G denote the set of vertices of $\mathcal{C}$. Then for every directed line l that touches P so that P is contained in the closed half-plane to the left of l, we have*

$$A_G(l) + B_G(l)/2 = I(\mathcal{C}),$$

where $I(\mathcal{C})$ is the index of the closed path $\mathcal{C}$ with respect to any point in the interior of P.

Proof of Lemma 2.3: Let $x_0, x_1, \ldots, x_{n-1}$ be the set of vertices of $\mathcal{C}$ cyclicly ordered as they appear along the path $\mathcal{C}$. Let m be the midpoint of $l \cap P$. Let l' be a directed line parallel to l that is strictly to the left of l, intersects the interior of P, but still to the right of all vertices of $\mathcal{C}$ which are to the left of l. Let $p \in l' \cap P$ be such that the ray $\overrightarrow{r}$ emanating from p and passing through m does not include any vertex of $\mathcal{C}$ (see Figure 3).

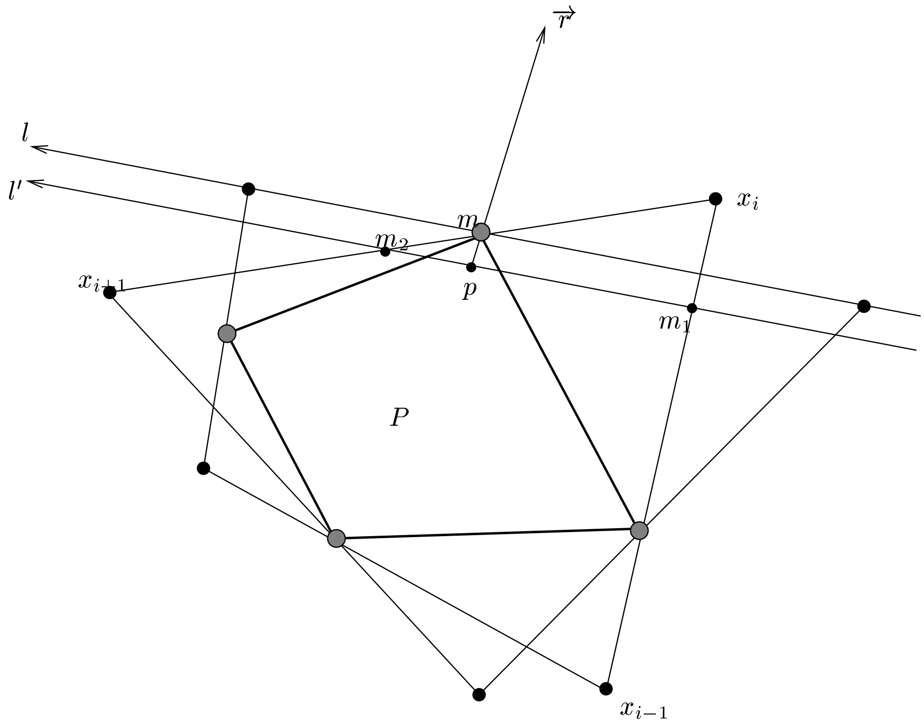

Figure 3: Lemma 2.3

Observe that if $x_i \in l$, then either the edge $[x_i, x_{x+1}]$, or the edge $[x_{i-1}, x_i]$ of $\mathcal{C}$ is included in l. Indeed, this follows because l is a tangent of P. Therefore, if k denotes the number of edges of $\mathcal{C}$ that are included in l, then $B(l) = 2k$. Clearly, every edge of $\mathcal{C}$ that is included in l is crossed by $\overrightarrow{r}$. Indeed, let $[x_i, x_{i+1}]$ be such an edge. By the definition of a tangency path, $[x_i, x_{i+1}] \supset l \cap P$. Therefore, $\overrightarrow{r}$ and $[x_i, x_{i+1}]$ meet at m. It follows that there are precisely $I(\mathcal{C}) - k$ edges of $\mathcal{C}$ that cross $\overrightarrow{r}$ and are not included in l.

If $[x_i, x_{i+1}]$ is an edge of $\mathcal{C}$ that crosses $\overrightarrow{r}$ and is not included in l, then precisely one of x_i and x_{i+1} is to the right of l. Indeed, if both x_i and x_{i+1} are to the right of l, then $[x_i, x_{i+1}] \cap P = \emptyset$. If none of x_i and x_{i+1} is to the right of l, then $[x_i, x_{i+1}] \cap \overrightarrow{r} = \emptyset$. In both cases we reach a contradiction.

On the other hand we claim that if x_i is in the half-plane to the right of l, then precisely one of the edges $[x_i, x_{i+1}]$ and $[x_{i-1}, x_i]$ crosses $\overrightarrow{r}$. Indeed, observe that both $[x_i, x_{i+1}]$ and $[x_{i-1}, x_i]$ must cross the line l, for otherwise their intersection with P is empty. By the choice of l', both $[x_i, x_{i+1}]$ and $[x_{i-1}, x_i]$ must cross also l'. Let m_1 denote the intersection point of $[x_{i-1}, x_i]$ with l', and let m_2 denote the intersection point of $[x_i, x_{i+1}]$ with l'. Now p must be strictly between m_1 and m_2 on l' because

P and therefore also p is to the left of both directed lines $\overrightarrow{x_i x_{i+1}}$ and $\overrightarrow{x_{i-1} x_i}$. It now follows that $\overrightarrow{r}$ crosses precisely one of $[x_i, x_{i+1}]$ and $[x_{i-1}, x_i]$ as required. This is because $\overrightarrow{r}$ meets precisely two edges of the triangle whose vertices are m_1, x_i, and m_2. One edge met by $\overrightarrow{r}$ is $[m_1 m_2]$, the other is either $[m_1, x_i]$, or $[m_2, x_i]$.

We can therefore conclude that $I(\mathcal{C}) - k = A(l)$. Combining this with $B(l) = 2k$ we obtain the desired result, namely, $A(l) + B(l)/2 = I(\mathcal{C})$. ■

The following corollary is thus an immediate consequence of Lemma 2.3.

Corollary 2.4. *Let P be a convex polygon and let $\mathcal{C}_1, C_2, \ldots, \mathcal{C}_t$ be vertex disjoint tangency paths for P. Let G denote the set of vertices of all paths together. For every directed line l that touches P, so that P is contained in the open half-plane to the left of l, we have*

$$A_G(l) + B_G(l)/2 = \sum_{i=1}^{t} I(\mathcal{C}_i). \quad ■$$

The next lemma relates between tangency paths and the property of skipping k for a certain value of k.

Lemma 2.5. *Let P be a convex polygon and let $J \subset P$ be a finite set of points. Let $\mathcal{C}_1, \ldots, \mathcal{C}_t$ be a collection of vertex disjoint tangency paths for P, with the property that every edge of P is contained in at least one edge of some tangency path. Let G denote the set of vertices of all paths together. Then the set $S = G \cup vert(P) \cup J$ skips $k = I(\mathcal{C}_1) + \ldots + I(\mathcal{C}_t)$.*

Proof. Let l be a directed line determined by S. It is enough to show that the number of points of S in the open half-plane to the right of l is different from k.

Let l' be the directed line with the same direction as that of l, such that l' touches P and P is contained in the closed half-plane to the left of l' (see Figure 2).

Case 1. $l = l'$. Then by Corollary 2.4, $A_G(l) + B_G(l)/2 = k$. Observe that $B_G(l) > 0$. Indeed, this is true if l contains an edge of P because then there is an edge of some tangency path $\mathcal{C}_i$ that lies on l. If l touches P at a vertex then it must pass through at least one more point of S which therefore belongs to G. It follows that $A_G(l) < k$. However, $A_G(l)$ is exactly the number of points of S in the open half-plane to the right of l.

Case 2. l is to the left of l'. Then the number of points of G in the open half-plane to the right of l is at least $A_G(l') + B_G(l') \geq A_G(l') + B_G(l')/2 = k$. Moreover, since l' passes through at least one vertex of P, we obtain that the number of points of G in the open half-plane to the right of l is at least $k + 1$.

Case 3. l is to the right of l'. Then the number of points of S that are on or to the right of l is at most $A_G(l') \leq k$. However, l passes through at least two points of S, and therefore the number of points of G in the open half-plane to the right of l is at most $k - 2$. ■

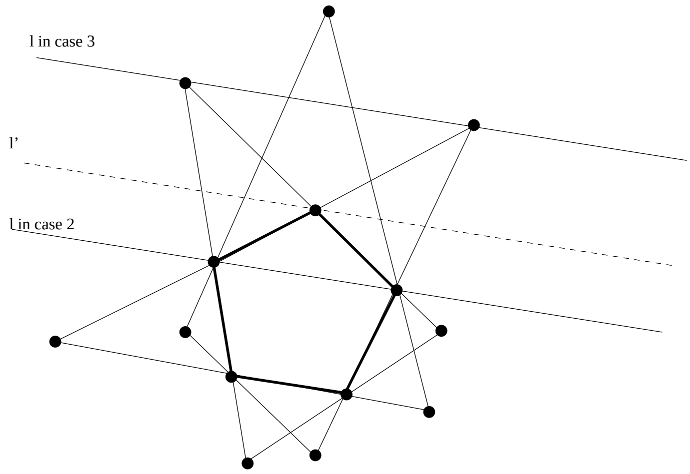

Figure 4: Lemma 2.5

We can impose another simple condition on the tangency paths $\mathcal{C}_1, \ldots \mathcal{C}_t$ in Lemma 2.5, so that the resulting set S skips two consecutive values:

Lemma 2.6. *Let P be a convex polygon and let $J \subset P$ be a finite set of points. Let $\mathcal{C}_1, \ldots, \mathcal{C}_t$ be a collection of vertex disjoint tangency paths for P, with the following two properties:*

- *Every edge of P is contained in at least two edges of some tangency paths.*
- *Every edge of a tangency path is collinear with at least one more edge of a (possibly another) tangency path.*

Let G denote the set of vertices of all paths together. Then $S = G \cup vert(P) \cup J$ skips k and $k-1$, where $k = I(\mathcal{C}_1) + \ldots + I(\mathcal{C}_t)$.

Proof. The proof goes exactly along the same lines as the proof of Lemma 2.5. Except that now whenever $B(l') > 0$ we may conclude that $B(l') \geq 4$.

The example in Figure 5 shows such a case where the polygon P is the inner regular 7-gon. ■

3 Construction of Large Skippable Sets

In this section we will use Lemma 2.5 to construct arbitrary large skippable sets. We will also show that $\{k\}$ is skippable for every $k \geq 2$, by constructing suitable arbitrary large sets of points that do not have a k-edge for a fixed value of $k \geq 2$. This is the goal of our next theorem.

We will need the following terminology in the sequel. If P is a convex polygon, then a *diagonal* of P is a segment connecting two vertices of P. We say that the *order* of that diagonal is k, if one of the open half-planes bounded by the line that contains this diagonal, includes precisely $k-1$ vertices of P. Thus for example, an edge of P is a diagonal of order 1.

For a convex polygon P and a point x outside P, the angle at which x *sees* P is the angle between the two tangents to P that pass through x.

It easy to see (and also follows immediately from Claim 4.1 in [PP1]) that $\{0\}$ and $\{1\}$ are not skippable. In other words, every non-collinear set of points S, that is large enough, must have a 0-edge and a 1-edge (in fact one can drop the 'large enough' condition here). In view of this we can now characterize precisely the skippable sets that consist of one element only.

Theorem 3.1. *The set $\{k\}$ is skippable iff $k \geq 2$.*

Proof. We already observed the 'only if' part. For the 'if' part fix some $k \geq 2$. Let Q be a regular $(2k+1)$-gon in the plane. Denote the vertices of Q in a cyclic counterclockwise order by $x_0, x_1, \ldots, x_{2k}$. Start from x_0 and draw a segment to x_k and from there to x_{2k} and from there to x_{3k} and so forth, where the indices are taken modulo $2k+1$. Since k and $2k+1$ are relatively prime, we will obtain a closed path $\mathcal{C}$ of length $2k+1$. This path is in fact combined from the diagonals of Q of order k. The intersections of all half-planes bounded by those diagonals and contain the center of Q, form a smaller copy of a $(2k+1)$-gon that we denote by P. Let J be an arbitrary large set of points inside P.

P, J, and $\mathcal{C}$ satisfy the conditions of Lemma 2.5. Therefore $S = vert(P) \cup J \cup vert(Q)$ skips $I(\mathcal{C})$. This index is easy to calculate. Every vertex of Q sees P at an angle of $\pi/(2k+1)$. Therefore, the index of the path $\mathcal{C}$ with respect to any point inside P is $I(\mathcal{C}) = \frac{(2k+1)(\pi - \pi/(2k+1))}{2\pi} = k$. ■

Remark. The construction in the proof of Theorem 3.1 was in fact suggested much earlier by Perles. The present notion of a tangency path gives us a convenient environment for presenting an elegant proof for the validity of the construction.

Using Lemma 2.5 as our main tool, we can also show very easily that there are arbitrary large skippable sets. This is the content of the next theorem. We will omit the very specific details of the construction, but include enough for the reader to be able to complete the proof.

Theorem 3.2. *There are arbitrary large skippable sets.*

Proof. We start with a set G_1 that skips just one value k_1 and constructed just like in the proof of Theorem 3.1. The construction consists of a polygon P_1 together with some points outside. We may add any number of points inside P_1 to get a larger set that also skips k_1.

Our construction for the proof of Theorem 3.2 is recursive. Assume that we have already constructed a set G_n that skips the values $k_1, \ldots, k_n$. Assume that G_n includes the set of vertices of a regular polygon P_n, such that no matter how many points we add to G_n inside P_n, the resulting set will still skip $k_1, \ldots, k_n$.

To construct G_{n+1} we will add points to G_n but only inside the polygon P_n. We add a very small copy of a set that skips say $k = 5$ and is constructed just like in the proof of Theorem 3.1. Namely, we add the set of vertices of an 11-gon plus the intersection of every two of its consecutive diagonals of order 5. These intersection points are the vertices of another (smaller) 11-gon that we denote by P_{n+1}. Then we add a set S of additional points outside (but very close to) P_{n+1}, so that they are still inside P_n, and such that $G_n \cup S$ may be regarded as a vertex disjoint union of tangency paths for P_{n+1}. One can easily be convinced that this can be done by adding at most say 10 extra points for each point of G_n. We thus get a resulting set G_{n+1} that can be regarded as G_n together with some extra points inside P_n, and therefore it still skips $k_1, \ldots, k_n$. However it can also be regarded as a union of tangency paths for P_{n+1} and thus skips another value that we denote by k_{n+1}. This value must be greater than k_n since the sum of the indices of all tangency paths to P_{n+1} is clearly greater than that of the tangency paths for P_n. Moreover, we can add any number of points inside P_{n+1} and the resulting set of points will still skip $k_1, \ldots, k_{n+1}$. This shows that $\{k_1, \ldots, k_{n+1}\}$ is skippable, and also concludes the induction step. ■

Observe that the construction brought in the proof of Theorem 3.2 is exponential in the number of values that are skipped. This is clear from the proof. We can thus in general construct arbitrary large sets of points G that skip $\Omega(\log |G|)$ values of k. It is interesting to note that the totally different construction by Kupitz and Perles of arbitrary large sets G that skip the values $|G|, |G|-1, \ldots, |G| - \log_2 |G|$ is yet another example of a different nature for a set of n elements that skips roughly $\log n$ values between 1 and n. It is not known what is the maximum number of values (between 1 and n) that a non-collinear set of points of cardinality n can skip. A nontrivial (with constant multiplier less than 1) linear upper bound follows from the result in [PP1], which says that a non-collinear set of points cannot skip both k and $k + 2$, provided that its cardinality is at least $2k + 4$.

We thus leave this question open with no conjecture.

Problem: What is the maximum number of values that a non-collinear set of n points in the plane can skip?

Of course, in terms of the order of magnitude, the answer can be anything between $\log n$ and n.

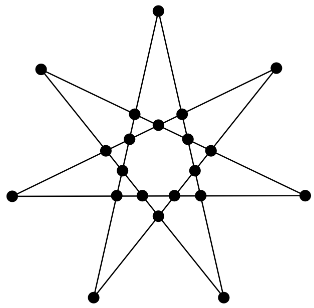

Figure 5: A set of points that skips $k = 4, 5$

4 Skippable Sets of Two Elements

In this section we will characterize all skippable sets of two elements. It shown in [PP1] that $\{k, k+2\}$ is not skippable for any k. If follows from Theorem 3.1 that any set $\{k, l\}$ that contains either 0 or 1 is not skippable. We will show that apart from another two sets of two elements that are not skippable, all others are skippable.

Theorem 4.1. *For every $k \geq 4$, the set $\{k, k+1\}$ is skippable.*

Proof. We will use a construction that satisfies the conditions of Lemma 2.6. Fix $k \geq 4$. Let P be a regular $(2k-1)$-gon and let $l_0, \ldots, l_{2k-2}$ denote the lines containing the edges of P in a cyclic order. Let S_1 denote the intersection points $l_j \cap l_{j+2}$ for $j = 0, \ldots, 2k-1$ (where the indices are taken modulo $2k-1$). S_1 is in fact the set of vertices of a regular $(2k-1)$-gon that contains P. Let S_2 denote the intersection points $l_j \cap l_{j+k-1}$ for $j = 0, \ldots, 2k-1$. S_2 is the set of vertices of a regular $(2k-1)$-gon that contains S_1 inside. Observe that we need $k \geq 4$ in order for S_1, S_2, and the vertices of P to be pairwise different.

A careful look in the construction of S_1 and S_2 shows that each of the sets S_1 and S_2 can be regarded as a set of vertices of a union of tangency paths for P. Moreover, every edge of P is contained in (exactly) two edges of these tangency paths, and every edge of a tangency path contains an edge of P. The sum of the indices of these tangency paths with respect to any point in P is also easy to calculate (keeping in mind that all the paths are counterclockwise oriented). Every point of S_1 sees P at an angle of $4\pi/(2k-1)$ and every point of S_2 sees P at an angle of $2(k-1)\pi/(2k-1)$. Therefore the sum of the indices of all paths with respect to P is

$$\frac{\sum_{i=1}^{2k-1}(\pi - 4\pi/(2k-1)) + \sum_{i=1}^{2k-1}(\pi - 2(k-1)\pi/(2k-1))}{2\pi} = k+1.$$

By Lemma 2.6, the set that consists of the vertices of P together with S_1 and S_2 skips both k and $k+1$. This is still true if we add an arbitrary number of points inside P. (Figure 5 shows the resulting construction in Theorem 4.1 for $k = 4$.) ■

Next we show, in Theorems 4.2 and 4.3, that the sets $\{2,3\}$ and $\{3,4\}$ are not skippable. In the proofs we use the method of flip arrays also known as allowable sequences, that was invented by Goodman and Pollack (see e.g. [GP93]). We refer the reader to the corresponding section of [PP1] for a detailed description and useful notation. Briefly, one can encode a set of points G in the plane as a sequence of permutation on n elements. This is done by keeping track of the order of the projections of the points from left to right on a directed line that changes its direction graduately counterclockwise until it reverses its direction. If we number the points according to the order of their projection on the line in its initial position, the recorded different ordering of the projections of the points form a sequence of permutation on $\{1,\ldots,n\}$. The first permutation is the identity while the last is $(n, n-1, \ldots, 1)$. Each permutation is obtained from its predecessor by flipping a contiguous block of monotone increasing elements. Each such flip corresponds to a line determined by the set of points. The important observation is that a flip of the block $[a,b]$ (namely, the elements in the places $a, a+1, \ldots, b$ in a permutation) represents a line, determined by G, that passes through $b-a+1$ points of G. This line has $a-1$ points of G in one open half-plane bounded by it and $n-b$ points of G in the other. Another important property of the sequence of permutations is that any pair of elements from $\{1,2,\ldots,n\}$ change order exactly once.

Theorem 4.2. *If G is a non-collinear set of at least 8 points in the plane, then G cannot skip both $k=2$ and $k=3$. In particular, $\{2,3\}$ is not skippable.*

Proof. Assume to the contrary that $|G| \geq 8$ and G does not have a 2-edge nor a 3-edge. We carefully analyze the flip array of G. Let $n = |G|$. The flip array of G consists of a sequence of permutation in S_n. Each permutation is obtained from its predecessor by flipping a contiguous monotone increasing block of elements. Observe that we are not allowed to flip blocks of the form $[3,b]$ (where $b>3$) nor $[4,b]$ (where $b>4$). Indeed, such blocks represent a 2-edge or a 3-edge respectively of G.

Define an *interesting flip* as a flip of a block that contains the block $[2,5]$. Observe that, in view of the forbidden flips, the only way to take an element that is in one of the first 4 places of a permutation to a place within $[5,n]$ is by an interesting flip. The element 1 must eventually move to position n. Hence, there must be at least one interesting flip. The block of the first interesting flip must be of the form $[2,b]$ for some $b \geq 5$, because if it contained the block $[1,5]$ that would means that the convex hull of G contains an edge with at least 5 points and this implies easily that G has an r-edge for every $r<5$.

Therefore, after the first interesting flip there is at least one element from $\{1,2,3,4\}$ that remains in the region $[1,4]$. Since $|G| \geq 8$, this element must eventually move to the region $[5,n]$. Therefore, there must be a second interesting flip. The block of the second interesting flip will again contain 3 elements from the region $[1,4]$. That implies that at least 2 elements that took part in the first interesting flip will take part also in the second interesting flip. We reached a contradiction, as any two elements must change order exactly once. ■

Theorem 4.3. *If G is a non-collinear set of at least 10 points, then G has either a 3-edge or a 4-edge. In particular, the set $\{3, 4\}$ is not skippable.*

Proof. Assume to the contrary that $|G| \geq 10$ and that G does not have a 3-edge nor a 4-edge. Once again we will make use of the flip array of the set G. Let $n = |G|$. This time the flips of blocks of the form $[4, b]$ and $[5, b]$ are not allowed. We will also make use of the observation that flips of blocks of the form $[a, n-4]$ and $[a, n-5]$ are not allowed.

We will call a flip *interesting of type I*, if the block of this flip contains $[3, 6]$. A flip will be called *interesting of type II*, if the block of the flip contains $[n-5, n-2]$. Observe that in view of the forbidden flips, an element can move from the region $[1, 5]$ to the region $[6, n]$ (and vice versa) only by an interesting flip of type I. Similarly an element can move from the region $[n-4, n]$ to the region $[1, n-5]$ (and vice versa) only by an interesting flip of type II.

We claim that it is not possible to have a flip that is an interesting flip of both types I and II. Indeed, this would necessarily mean that the block of such a flip must contain the region $[3, n-2]$ in a permutation. This implies that G determines a line with at least $n-4$ points, having at most two points in each open half-plane bounded by it. It is easily seen by inspection, taking in account that $|G| \geq 10$, that G must then have either a 3-edge or a 4-edge.

Just like in the proof of Theorem 4.2, there must be at least two interesting flips of type I, and similarly two interesting flips of type II. Let us consider the first interesting flip of type I. This flip includes three elements from $\{1, 2, 3, 4, 5\}$ in the positions $[3, 5]$. Right after this flip the elements in the region $[3, 5]$ are in decreasing order. Therefore the second interesting flip of type I may include just one of them in its block. The other two elements must therefore be at the positions $[1, 2]$ and remain untouched while the second interesting flip happens. Right after the second interesting flip we have two elements in positions $[1, 2]$ that already changed order with each other, and three elements in positions $[3, 5]$ every two of which already changed order. Therefore a third interesting flip of type I is not possible (for it must include three elements from the region $[1, 5]$, no two of which already changed order). Similarly, there are just two interesting flips of type II.

Since $|G| \geq 10$, the elements $\{1, 2, 3, 4, 5\}$ must all end at the region $[n-4, n]$. We know that three elements from $\{1, 2, 3, 4, 5\}$ belong to the block of the first interesting flip. Those three elements must move from the region $[1, n-5]$ to $[n-4, n]$ and this can be done only by interesting flips of type II. There are at most two interesting flips of type II, and we obtain a contradiction since two of the three elements from $\{1, 2, 3, 4, 5\}$ that were flipped during the first interesting flip of type I, must be flipped during the same interesting flip of type II. ■

The next theorem will complete the picture as for the skippable sets of two elements.

Theorem 4.4. *For every $k \geq 2$ and every $l \geq k + 3$ the set $\{k, l\}$ is skippable.*

Proof. For any k and l that satisfy the conditions in the theorem, we must show that there are arbitrary large sets of points that do not have a k-edge nor an l-edge.

We will make use of Lemma 2.5 to show the validity of our construction. Fix $k \geq 2$, and assume first that $l = k+3$. Let Q_1 be a regular $(2k+2)$-gon. Let Q_2 be the regular $(2k+2)$-gon whose vertices are the intersection points of consecutive diagonals of order k of Q_1. Clearly, the vertices of Q_1 can be regarded as the union of vertices of tangency paths for Q_2. Each vertex of Q_1 sees Q_2 at the angle of $\pi/(k+1)$ thus the sum of the indices of all tangency paths is $\frac{(2k+2)(\pi-\pi/(k+1))}{2\pi} = k$. Observe that by the construction every edge of Q_2 is contained in an edge of some tangency path (just like in the proof of Theorem 3.1). Hence by Lemma 2.5, the set that consists of the union of the vertices of Q_1 and Q_2 skips k. This remains true if we add points inside Q_2. Let Q_3 be the $(2k+2)$-gon whose vertices are the intersection points of consecutive diagonals of order 2 of Q_2. Clearly, $Q_3 \subset Q_2$. The vertices of Q_2 can be regarded as the set of vertices of a union of tangency paths to Q_3 and each edge of Q_3 is contained in one edge of a tangency some tangency path. Since every point of Q_2 sees Q_3 at angle $\pi(k-1)/(k+1)$, the sum of the indices of these paths with respect to Q_3 is $\frac{(2k+2)(\pi-\pi(k-1)/(k+1))}{2\pi} = 2$. It is thus enough to show that we can add points inside Q_2 but outside Q_3 so that together with the points of Q_1 they will constitute the vertex set of a union of tangency paths of a total index $k+1$. This can indeed be done. For every two opposite points of Q_1 we add two points in $Q_2 \setminus Q_3$ so that they form a tangency path to Q_3 with index 1 for Q_3. This is illustrated in Figure 4.

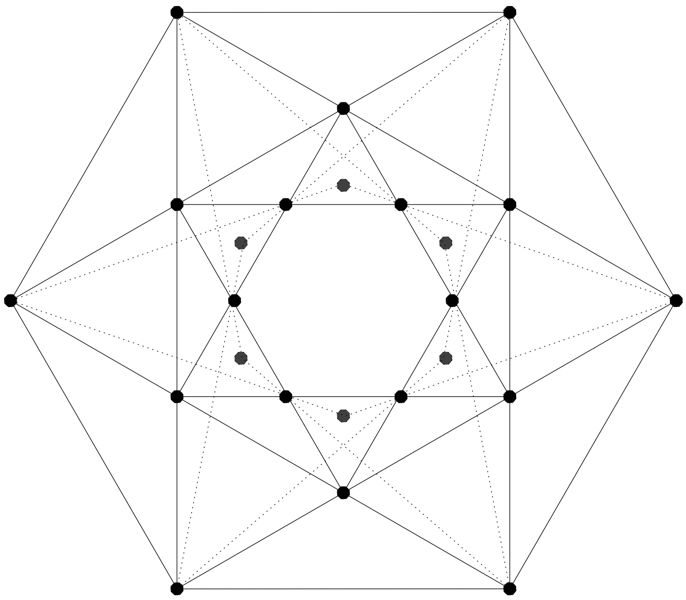

Figure 6: Theorem 4.4 (the case $k = 2$ and $l = k+3 = 5$)

We thus obtained a set G that skips k and $k+3$. This remains true if we add arbitrary number of points inside Q_3. This shows that $\{k, k+3\}$ is skippable. If $l > k+3$ we add to G l tangency paths of index 1 that are contained in $Q_2 \setminus Q_3$. This completes the proof of the theorem. ■

We can now summarize our results in the following theorem that characterizes all skippable sets of two elements.

Theorem 4.5. *Let $k < l$ be nonnegative integers. The set $\{k, l\}$ is skippable iff one of the following two conditions is satisfied:*

- $k \geq 2$ *and* $l \geq k + 3$*, or*
- $k \geq 4$ *and* $l = k + 1$

References

[GP84] J. F. Goodman and R. Pollack, On the number of k-sets of a set of n points in the plane, *J. Combin. Theory Ser. A,* **36** (1984), 101-104.

[GP93] J. E. Goodman and R. Pollack, Allowable sequences and order types in discrete and computational geometry, Chapter V in: *New Trends in Discrete and Computational Geometry (J. Pach, ed.)*, Springer-Verlag, Berlin, 1993, 103–134.

[K93] Y. S. Kupitz, Separation of a finite set in R^d by spanned hyper-planes, *Combinatorica,* **13** (1993), 249-258.

[K94] Y. S. Kupitz, Spanned k-supporting hyper-planes of finite sets in R^d, *J. Combin. Theory Ser. A,* **65** (1994), 117-136.

[KP] Y. S. Kupitz and M. A. Perles, personal communication.

[PP1] M. A. Perles and R. Pinchasi, Large sets must have either a k-edge or a $(k+2)$-edge. Towards a theory of geometric graphs, 225–232, Contemp. Math., 342, Amer. Math. Soc., Providence, RI, 2004.