Large Sets Must Have Either a k-Edge or a $(k+2)$-Edge

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Abstract. Let $k \geq 0$ be a given integer. We show that if G is a set of points in the plane, not contained in a line, and $|G| \geq 2k+2$, then G has either a k-edge or a $(k+2)$-edge.

1. Introduction

Let G be a finite set of points in the plane. We say that a line l is *determined* by G, if l passes through at least two points of G. A line l, determined by G is called a k-edge if one of the two open half-planes bounded by l contains precisely k points of G. We say that G *skips* k, if G has no k-edge.

Given A, a set of nonnegative integers, we say that A is *skippable*, if there are arbitrary large finite noncollinear sets of points in the plane that skip k for every $k \in A$. Clearly, if G is contained in a line, then G skips every $k \neq 0$. Moreover, if G is a finite set of points, then G skips every k for $k > |G| - 2$. If G is a set of points in general position, namely no three points of G are collinear, then clearly G has a k-edge for every $0 \leq k \leq |G| - 2$. Therefore, in our study of skippable sets we will mainly be concerned with sets that are not in general position.

Figure 1 shows an example of a set that skips $k = 2$. In fact, by adding arbitrary large number of points very close to the center of the shape in Figure 1, we will obtain a set that skips $k = 2$, thus showing that $\{2\}$ is a skippable set. Similarly, the example in Figure 2 illustrates that $\{4, 5\}$ is a skippable set. The

2000 *Mathematics Subject Classification.* Primary 05D99.

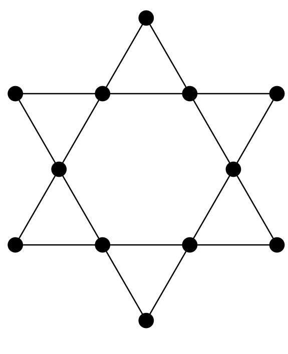

Figure 1. A set of points which skips $k = 2$

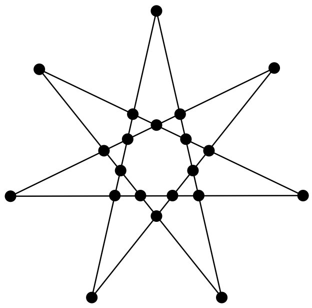

FIGURE 2. A set of points that skips $k = 4, 5$

study of skippable sets was originated by Kupitz and the first author. In [**KP**], they construct arbitrary large sets G, not contained in a line, that skip every k for $k = |G|, |G| - 1, \ldots, |G| - \log_2 |G|$.

In this paper we show that for every $k \geq 0$ $\{k, k+2\}$ is not skippable. In other words we prove the following theorem.

THEOREM 1.1. *Let G be a finite set of points in the plane that is not collinear. If $|G| \geq 2k+2$, then either G has a k-edge, or it has a $(k+2)$-edge.*

The proof of Theorem 1.1 is carried out through a careful analysis of G using the method of *allowable sequences*, invented by Goodman and Pollack. For convenience we use the term *flip array* to describe it. In Section 2 we give a quick review of this method together with some useful notation. For further discussion of the method consult [**GP93**] and [**GP84**].

2. Flip Arrays

Let G be a set of n points in the plane. Let L be a directed line through the origin which is not perpendicular to any of the lines determined by G. We arrange the points of G in a sequence $x_1, \ldots, x_n$ by the order of their projections on L from left to right. For every $0 \leq \theta \leq \pi$, let $L(\theta)$ be the line through the origin that arises from L by a rotation through angle θ in the positive direction (counterclockwise). Let $0 < \theta_1 < \ldots < \theta_m < \pi$ be all angles in $[0, \pi]$, so that each of $L(\theta_1), \ldots, L(\theta_m)$ is perpendicular to some line determined by G. Denote, for convenience, $\theta_0 = 0, \theta_{m+1} = \pi$.

For every $0 \leq \theta \leq \pi$ which is not one of $\{\theta_1, \ldots, \theta_m\}$, let Π_θ denote the permutation on $\{1, \ldots, n\}$, so that the projections of $x_{\Pi_\theta(1)}, \ldots, x_{\Pi_\theta(n)}$ on $L(\theta)$ are in that order from left to right. It is important to note that we think of a permutation Π as a sequence of n elements namely, $(\Pi(1), \ldots, \Pi(n))$. We then say that the *element* $\Pi(i)$ is at the *place* i in the permutation Π. If $\Pi^{-1}(i) < \Pi^{-1}(j)$ we say that i is *to the left* of j and that j is *to the right* of i.

Let x_i and x_j be two points of G and let $\alpha < \beta$ be two angles which are not from $\{\theta_1, \ldots, \theta_m\}$. The relative order of i and j in the permutations Π_α and Π_β is the same if and only if the vector $\overrightarrow{x_i x_j}$ is not perpendicular to any of the lines $\{L(\theta) | \alpha < \theta < \beta\}$. It follows that if $0 \leq k \leq m$ and $\theta_k < \alpha < \beta < \theta_{k+1}$, then $\Pi_\alpha = \Pi_\beta$. This justifies the following notation.

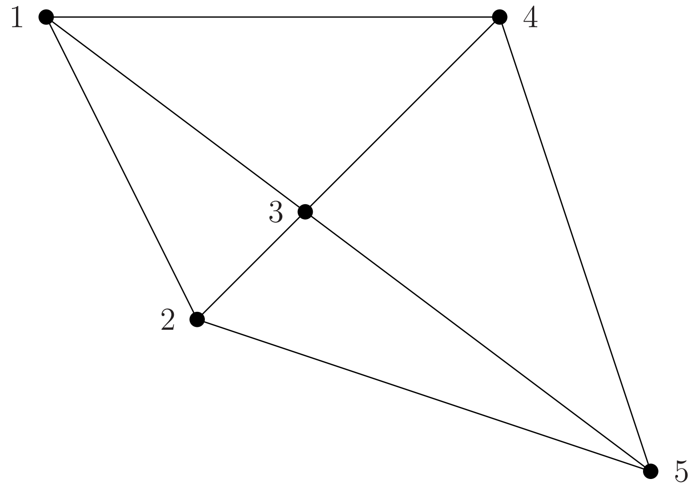

FIGURE 3

DEFINITION 2.1. For every $0 \le j \le m$, let Q_j be the permutation Π_α where α is any angle such that $\theta_j < \alpha < \theta_{j+1}$.

Clearly, Q_0 is the identity permutation, and for every two indices $1 \le i < j \le n$, the relative order between i and j changes exactly for one value of k $(0 \le k < m)$, when going from Q_k to Q_{k+1}. Eventually, Q_m is the permutation $(n, n-1, \ldots, 1)$.

For a fixed k $(0 \le k < m)$, we would like to find out how exactly Q_{k+1} arises from Q_k. For every $1 \le i < j \le n$, the relative order between i and j in the permutation Q_{k+1} is different from that in Q_k if and only if the vector $\overrightarrow{x_i x_j}$ is perpendicular to $L(\theta_{k+1})$. Let M be a line spanned by points of G which is perpendicular to $L(\theta_{k+1})$. Let $x_{i_1}, \ldots, x_{i_s}$ be all the points of $G \cap M$, where we assume that $i_1 < i_2 < \ldots < i_s$. Since for every $1 \le u < v \le s$ the relative order between i_u and i_v *changes* when going from Q_k to Q_{k+1}, we conclude that in Q_k the relative order of $i_1, \ldots, i_s$ is the same as in Q_0, namely, the natural order. Let j be an index which is not from $\{i_1, \ldots, i_s\}$. When going from Q_k to Q_{k+1} we do not change the relative order between j and i_1 and, between j and i_s. Therefore, in permutation Q_k, j cannot be between i_1 and i_s, or, in other words, either $Q_k^{-1}(j) < Q_k^{-1}(i_1)$ or $Q_k^{-1}(j) > Q_k^{-1}(i_s)$. This shows that in Q_k the elements $i_1, \ldots, i_s$ come one after the other and form a monotone increasing sequence. It follows then that in Q_{k+1} those same elements form a monotone decreasing sequence. In other words, some subsequence of consecutive elements in Q_k appears flipped in Q_{k+1}.

A *block* in a permutation Π is a sequence of consecutive elements in Π. We say that a block B is *monotone increasing* if the elements in that block form a monotone increasing sequence from left to right. We define a *monotone decreasing* block similarly.

DEFINITION 2.2. Let $1 \le a < b \le n$. We denote by $[a, b]$ the block which consists of the positions $a, a+1, \ldots, b$ in a permutation on n elements.

Therefore, we can say that Q_{k+1} arises from Q_k by flipping blocks in Q_k. Every such block represents a line, determined by the points of G, which is perpendicular to $L(\theta_{k+1})$. Every pair of blocks that flip when going from Q_k to Q_{k+1} represent two parallel lines, and therefore are disjoint, so we can treat them as if they were flipped one after the other.

To summarize, given a set G of n points in the plane, we derive from it a sequence of permutations on the numbers $1, \ldots, n$, with the following properties. The first permutation is the identity permutation, the last one is permutation $(n, n-1, \ldots, 1)$ and each permutation arises from its predecessor by flipping a block which is monotone increasing (right before the flip). Every such block represents a line determined by the points of G. We derive from it a *flip array*, that is, a sequence of permutations on the elements $\{1, 2, \ldots, n\}$. Each permutation arises from its predecessor by a flip T of a block B. For every element $x \in B$, we say that *x takes part in the flip T*. For example, the flip array of the set presented in Figure 3 is $(1,2,3,4,5)$, $(1,2,3,5,4)$, $(2,1,3,5,4)$, $(2,5,3,1,4)$, $(5,2,3,1,4)$, $(5,2,3,4,1)$, $(5,4,3,2,1)$.

We say that two elements $x, y \in \{1, 2, \ldots, n\}$ *change order* in a flip F, if x and y belong to the block of the flip F and thus their relative order reverses right after F.

We note the following two simple observations.

Observation 2.3. Let S_G be a flip array of a set G of n points in the plane. Every two elements change order in some flip in the flip array S_G. From that point on (i.e., in all permutations that come afterwards in S_G) they are always in reversed order.

Observation 2.4. Let S_G be a flip array of a set G of n points in the plane. If a line L, determined by G, is represented by a flip of the block $[a, b]$, then there are exactly $a-1$ points of G in one open half-plane bounded by L, and $n-b$ points in the other half-plane bounded by L. There are exactly $b-a+1$ points of G on L.

3. Proof of Theorem 1.1

We will consider the flip array S_G of the set G of n points in the plane. Assume that $|G| \geq 2k+2$ and that G does not have a k-edge nor a $(k+2)$-edge. Observe that this condition implies that in the flip array S_G no permutation is obtained from its predecessor by a flip of a block $[a, b]$ where $a \in \{k+1, k+3\}$ and $b > a$.

We denote by conv G the convex hull of G. We need the following very easy observation that will simplify the proof.

Claim 3.1. If one of the edges of conv G contains (at least) r points, then G has a k-edge for every $0 \leq k \leq r-1$.

Proof. It is enough to show that G has a $(r-1)$-edge. Let l be a directed line supporting conv G in an edge with at least r points. We assume that l is directed so that G is to its right. Let x denote the rth point of G on l from left to right. Rotate l counterclockwise around x until it hits a point of G. Since G is not contained in a line, then, as soon as this happens, l will have exactly $r-1$ points of G in the open half-plane to its left. □

It follows immediately from Claim 3.1 that G has a 0-edge and a 1-edge. Therefore, we may assume that $k \geq 2$.

The structure of the proof is as follows. We will define three classes of permutations on $n = |G|$ elements. We will show that if $\sigma \in S_G$ belongs to one of these classes then the next permutation in S_G must also belong to one of these three classes. We will obtain a contradiction by showing that the last permutation in S_G, namely $(n, n-1, \ldots, 1)$, does not belong to any of the three classes, even though there is a permutation of S_G which belongs to one of these classes.

Now we define three classes of permutations mentioned above. S_n denotes the set of all permutations on n elements.

Definition 3.2. We say that a permutation $\sigma \in S_n$ belongs to Class I with parameter x if

(1) x at the position $k+1$ in σ,
(2) $x \leq k+1$
(3) the block $[k+1, k+3]$ in σ is not monotone increasing.

Definition 3.3. We say that a permutation $\sigma \in S_n$ belongs to Class II with parameter (x, z) if

(1) $z < x \leq k+1$,
(2) z is at the position $k+1$ in σ,
(3) x belongs to the block $[1, k]$ in σ, and
(4) every y which is between x and z in σ satisfies $y < x$.

Definition 3.4. We say that a permutation $\sigma \in S_n$ belongs to Class III with parameter (x, z) if

(1) $x \leq k+1$,
(2) z is at the position $k+1$ in σ,
(3) x belongs to the block $[1, k]$ in σ,
(4) every y which is between x and z in σ, satisfies $y < x$ or $y > z$, and
(5) the block $[k+1, k+3]$ in σ is monotone decreasing.

The following claim shows, as promised, that the last permutation of S_G does not belong to any of the classes defined above.

Claim 3.5. The permutation $\sigma = (n, n-1, n-2, \ldots, 1)$ does not belong to any of the classes I, II, or III.

Proof. Since $n \geq 2k+2$, there is no element which is less than or equal to $k+1$ in the block $[1, k+1]$ in σ. However, it is a requirement in the definition of all classes I, II, and III, that there is some element which is less than or equal to $k+1$ in the block $[1, k+1]$ of a permutation. $\square$

We will now show, in a series of three lemmata (Lemma 3.9, Lemma 3.10, and Lemma 3.11), that if $\sigma \in S_G$ belongs to one of the classes I, II, or III, then the next permutation in S_G must also belong to one of these classes.

The following definition will be useful.

Definition 3.6. Suppose that the permutation σ_2 is obtained from a permutation σ_1 by a flip F. F is called *interesting*, if it involves an element in the block $[k+1, k+3]$ of σ_1. Otherwise it is called a *non-interesting* flip.

Let F be an interesting flip whose block is $B = [a, b]$. F is said to be of *type 1*, if B contains the position $k+1$ but not the position $k+3$. F is said to be of *type 2*, if B contains all places $k+1, k+2, k+3$. F is said to be of *type 3*, if B contains the position $k+3$ but not the position $k+1$.

Observe that any interesting flip must be either of type 1 or of type 2 or of type 3. Moreover, if F is of type 3, then the block of F must be of the form $[k+2, b]$ where $b > k+2$. Indeed, since G does not have a $(k+2)$-edge, there is no flip whose block is of the form $[k+3, b]$ where $b > k+3$.

In order to simplify the rest of the proof we will need the following two auxiliary claims.

Claim 3.7. Let $\sigma_1, \sigma_2 \in S_G$ be two consecutive permutations, so that σ_2 is obtained from σ_1 by an interesting flip F of type 3. If the element x in the position $k+1$ in σ_1 is less than or equal to $k+1$, then σ_2 belongs to class I.

Proof. The block of F includes places $k+2$ and $k+3$, while x does not belong to the block of F, as F is of type 3. Therefore, in σ_2 the block $[k+1, k+3]$ is not monotone increasing and hence σ_2 belongs to class I with parameter x. □

Claim 3.8. Let $\sigma_1, \sigma_2 \in S_G$ be two consecutive permutations, so that σ_2 is obtained from σ_1 by an interesting flip F of type 1. If the element x in the position $k+1$ in σ_1 is less than or equal to $k+1$, then σ_2 belongs either to class I or to class II.

Proof. If the block of F is $[k, k+2]$, then x is at the position $k+1$ in σ_2, and the block $[k+1, k+3]$ in σ_2 is not monotone increasing. Therefore, σ_2 belongs to class I with parameter x.

Otherwise, the block of F is of the form $[a, k+2]$ where $a < k$. Let z denote the element at the position $a+1$ in σ_1. Then z belongs to the block of F and $z < x$ since the block of F is monotone increasing in σ_1. In σ_2, z is at the position $k+1$, and x is at the position $a+1 \in [1, k]$. Every element y which is between x and z in σ_2 satisfies $z < y < x$. Therefore, σ_2 is in class II with parameter (x, z). □

Lemma 3.9. *Assume that $\sigma_1, \sigma_2 \in S_G$ are two consecutive permutations in S_G, and that σ_2 is obtained from σ_1 by a flip F. If σ_1 belongs to class I with parameter x, then σ_2 belongs either to class I or to class II.*

Proof. Assume that σ_1 belongs to class I with parameter x. Therefore, x is at the position $k+1$ in σ_1. Let $B = [a, b]$ denote the block of the flip F. If F is not an interesting flip, then the block $[k+1, k+3]$ of σ_2 is identical with that of σ_1 and therefore σ_2 also belongs to class I.

Observe that F cannot be of type 2 since in σ_1 the block $[k+1, k+3]$ is not monotone increasing, by Definition 3.2. If F is of type 3, then, by Claim 3.7, σ_2 belongs to class I with parameter x. If F is of type 1, then, by Claim 3.8, σ_2 belongs either to class I or to class II. □

Lemma 3.10. *Assume that $\sigma_1, \sigma_2 \in S_G$ are two consecutive permutations in S_G, so that σ_2 is obtained from σ_1 by a flip F. If σ_1 belongs to class II with parameter (x, z), then σ_2 belongs either to class I, or to class II, or to class III.*

Proof. Assume that σ_1 belongs to class II with parameter (x, z) (recall Definition 3.2). First, consider the case when F is a not an interesting flip. In this case z does not belong to the block of F, so it is at the position $k+1$ also in σ_2. Moreover, in σ_2, x is in the block $[1, k]$, since x and z do not change order by the flip F. Let y be an element between x and z in σ_2. If y is between x and z also in σ_1, then $y < x$. Otherwise, in σ_1, y is to the left of x. As y and x change order in F, we conclude that $y < x$. Therefore, σ_2 also belongs to class II with parameter (x, z).

If F is an interesting flip of type 3, then by Claim 3.7, σ_2 belongs to class I with parameter z. If F is of type 1, then by Claim 3.8, σ_2 belongs either to class I or to class II.

Finally, Assume that F is of type 2. Let w denote the element at the position $k+1$ in σ_2. Observe that since $z < x$ and x is to the left of z in σ_1, then x does not belong to the block of F (for this block must be monotone increasing in σ_1). Therefore, in σ_2, x belongs to the block $[1, k]$. Let y be an element which is between x and w in σ_2. If y belongs to the block of F, then $y > w$. Otherwise, y must be between x and z in the permutation σ_1, and therefore $y < x$. Moreover, since F is of type 2, the block $[k+1, k+3]$ is monotone decreasing in σ_2. Therefore, σ_2 belongs to class III with parameter (x, w). □

LEMMA 3.11. *Assume that $\sigma_1, \sigma_2 \in S_G$ are two consecutive permutations in S_G, so that σ_2 is obtained from σ_1 by a flip F. If σ_1 belongs to class III with parameter (x, z), then σ_2 belongs either to class I, or to class III.*

Proof. Assume that σ_1 belongs to class III with parameter (x, z). First, consider the case when F is a not an interesting flip. Then, it is clear that in σ_2 the block $[k+1, k+3]$ remains as it is in σ_1, namely monotone decreasing with the element z at the position $k+1$. Moreover, the element x remains in the block $[1, k]$ as it cannot change order with z in σ_2. We claim that σ_2 in this case still belongs to class III with parameter (x, z). For this it remains to show that any element y which is between x and z in σ_2 satisfies either $y < x$ or $y > z$. This is clearly true if y is between x and z in σ_1 (since σ_1 itself belongs to class III with parameter (x, z)). Otherwise, y must be to the left of x, in σ_1. Then, it follows that y and x change order in σ_2 and, therefore, $y < x$.

Next, we consider the case when F is an interesting flip. Since the block $[k+1, k+3]$ is monotone decreasing in σ_1, it is easy to see that the block of F must be of the from $[a, k+1]$ where $a \leq k$ (observe that it cannot be of the form $[k+3, b]$, as we assume that G does not have a $(k+2)$-edge).

Let w denote the element at the position a in σ_1. We claim that $w \leq x$. Indeed, if x belongs to the block $[a, k+1]$, then as this block is monotone increasing in σ_1, we must have $w \leq x$. Otherwise, x is to the left of w in σ_1, or, in other words, in σ_1, w is between x and z. However, then either $w > z$ or $w < x$. The former case is not possible because w and z belong to the same monotone increasing block in σ_1 (namely, $[a, k+1]$), while w is to the left of z. We thus showed that $w \leq x$ and therefore $w \leq k+1$. Since w is at the position $k+1$ in σ_2 and $[k+1, k+3]$ is not monotone increasing in σ_2, we conclude that σ_2 belongs to class I with parameter w. □

We are now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By Claim 3.5, the last permutation in S_G, namely $(n, n-1, n-2, \ldots, 1)$, does not belong to any of the classes I, II, or III. In view of Lemmata 3.9, 3.10, and 3.11, we will obtain a contradiction if we show that some permutation in S_G must belong to one of the classes I, II, or III.

Indeed, let σ_1, σ_2 be the first two consecutive permutations in S_G such that σ_2 is obtained from σ_1 by an interesting flip. Observe that such an interesting flip must exist since, for example, the elements $k+1$ and $k+2$ must change order at some point.

If F is of type 1, then, by Claim 3.7, σ_2 belongs to class I. If F is of type 3, then, by Claim 3.8, σ_2 belongs either to class I or to class II.

It remains to consider the case when F is of type 2. In this case the block of F contains the block $[k+1, k+3]$ and therefore, this block is monotone decreasing in σ_2. Let $[a, b]$ denote the block of F. Observe that $a > 1$ since otherwise $[a, b] \supset [1, k+3]$ and, thus, the flip F corresponds to a line supporting conv G which contains at least $k+3$ points of G. By Claim 3.1, this implies that G has a k-edge as well as a $(k+2)$-edge.

Let x denote the element at the position $a-1$ in σ_1, and let z denote the element at the position $k+1$ in σ_2. In σ_2, every element y between x and z satisfies $y > z$, since y belongs to the block of F and it is to the left of z in σ_2. Moreover, observe that $x \leq k+1$. This is true because F is the first interesting flip and therefore the elements in the block $[1, k]$ in σ_1 are exactly $\{1, 2, \ldots, k\}$.

Therefore, σ_2 belongs to class III with parameter (x, z). This concludes the proof of the theorem. □

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