A Theorem on Unit Segments on the Real Line

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April 24, 2017

Abstract

Let $n \geq 1$ be an odd integer. For every $1 \leq i \leq n$ let $s_i = (a_i, b_i)$ be an open unit segment on the real line. Let $0 \leq \epsilon < \frac{1}{2}$ be fixed. Color by green all the points (numbers) on the real line of the form $a_i + \epsilon$ and $b_i - \epsilon$. Then there exists at least one green point that belongs to an odd number of the segments $s_1, \ldots, s_n$.

1 Introduction.

Given a family S of an odd number n of unit segments (that is, segments of length 1) on the real line it is not hard to see that there must be many points that belong to an odd number of segments in S. In fact one can show that the set of all these points has Lebesgue measure at least 1 (see [1]).

If the segments in S are closed, it is easy to see that there is at least one left endpoint of a segment that belongs to an odd number of the segments in S. Indeed, we may first assume that no two segments in S are identical or else we delete two identical segments from S and conclude by induction. Now it is enough to observe that the left endpoint of the leftmost segment in S is contained in precisely one segment in S and 1 is an odd number. The same argument works for a right endpoint of some segment in S.

A slightly more interesting example is to consider the centers of the segments in S. It is true that there is at least one center of a segment in S that belongs to an odd number of the segments in S.

Here we give an extremely short and elegant proof of this fact that is due to Barry Balof (personal communication in August 2013). We define a graph whose vertices are the segments in S. We connect two distinct segments by an edge if and only if each contains the center of the other. (Notice that for two segments s and s' of the same length it is true that s contains the center of s' if and only if s' contains the center of s.) We obtain a graph on an odd number n of vertices and therefore there must be a vertex (that is, a segment in S) whose degree in the graph is even. The center of this segment s is contained in an even number of the other segments in S and therefore in a total of an odd number of segments in S (including the segment s itself).

One might be tempted to think that for every $0 < \epsilon < 1$ there exists a point of the form $a_i + \epsilon$ for some $1 \leq i \leq n$ such that $a_i + \epsilon$ is contained in an odd number of the segments in S. Somewhat surprisingly, this is not true. In Figure 1 one can see one such simple example consisting of just 3 unit segments. For convenience we draw the segments on different layers and one should imagine that they are all projected on the x-axis. The points $a_i + \epsilon$ are drawn as small disks.

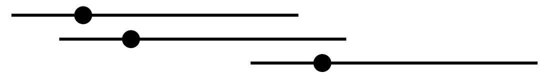

Figure 1: An example where every point $a_i + \epsilon$ lies in an even number of segments.

The main result in this paper is the following theorem:

Theorem 1. *Let $n \geq 1$ be an odd integer and consider a family S of n open unit segments $S = \{s_i = (a_i, b_i) \mid i = 1, \dots, n\}$. For every point $x \in \mathbb{R}$ denote by $d(x)$ the number of segments in S that contain x. Let $0 \leq \epsilon < \frac{1}{2}$ and color by green all the points (numbers) of the form $a_i + \epsilon$ and $b_i - \epsilon$. Then there exists at least one green point g with $d(g)$ odd.*

As we shall see later there are families S and specific ϵ's for which one cannot find more than one green point g with $d(g)$ odd. However, we are able to strengthen Theorem 1 as follows:

Theorem 2. *Let $n \geq 1$ be an odd integer and consider a family S of n open unit segments $S = \{s_i = (a_i, b_i) \mid i = 1, \dots, n\}$. For every point $x \in \mathbb{R}$ denote by $d(x)$ the number of segments in S that contain x. Then for every $\epsilon \in (0, \frac{1}{2})$, with at most $\binom{n}{2}$ exceptions, it is true that if we color by green all the points (numbers) of the form $a_i + \epsilon$ and $b_i - \epsilon$, then there exists at least* **two** *green points g with $d(g)$ odd.*

In general, even for some ϵ, one cannot guarantee more than two such green points. This is definitely the case where S consists of only one unit segment (or an arbitrarily large odd number of the same unit segment), and one can construct easily many other examples where S is arbitrarily large.

We remark that one can conclude from Theorem 1 the same result for closed segments. Indeed, If S is a family of closed unit segments, then extend each segment $[a_i, b_i]$ to an open segment $(a_i - \delta, b_i + \delta)$ where $\delta > 0$ is so small so that no green point lies in any of $(a_i - \delta, a_i)$ and $(b_i, b_i + \delta)$.

Another immediate consequence of Theorem 1 is the discrete version of it: Let S be a collection of n sets each consisting of k consecutive integers. Fix a positive integer $\ell < \lfloor \frac{k+1}{2} \rfloor$ and color an integer by green if it is the ℓth element in some set in S or it is the $(k - \ell + 1)$th element in some set in S. Then there is at least one green integer that belongs to an odd number of the sets in S.

2 Proof of Theorem 1.

For $n = 1$ the theorem is clearly true. We may assume without loss of generality that $a_1 < a_2 < \dots < a_n$. This is because if two of the segments s_i and s_j coincide, then one can ignore them and conclude the theorem by induction on n.

For every $1 \leq i \leq n$ we let $r_i = b_i - \epsilon$ and let $\ell_i = a_i + \epsilon$ and we say that these two green points are associated with s_i. We denote $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$.

We notice the following important observation whose very easy proof we omit (see Figure 2):

Observation 1. *For every i and j we have $r_i \in s_j$ if and only if $\ell_j \in s_i$.*

We remark that in view of Observation 1, one can reformulate the conclusion of Theorem 1 under the same conditions in the following, perhaps more elegant, way (although we will not make use of it later):

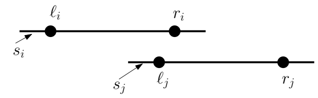

Figure 2: $r_i \in s_j$ if and only if $\ell_j \in s_i$.

Either there exists a point of the form $a_i + \epsilon$ that belongs to an odd number of the segments in S, or there is a segment in S containing an odd number of points of the form $a_i + \epsilon$.

An immediate consequence of Observation 1 is the following statement whose proof we leave to the reader.

Claim 1. *The sum $\sum_{i=1}^{n} d(\ell_i) + d(r_i)$ is even.*

In order to prove Theorem 1, assume to the contrary that every green point is contained in an even number of segments in S.

Claim 2. *For every two green points x and y such that $x < y$ it must be that $|A \cap [x,y)| + |B \cap (x,y]|$ is even.*

Proof. Indeed, if $|A \cap [x,y)| + |B \cap (x,y]|$ is odd, then the parity of the number of segments in S containing x is different from the parity of the number of segments in S containing y. This would imply that one of x and y is contained in an odd number of segments in S, contradicting our assumption. ∎

The idea of the proof of Theorem 1 is to find, under the contrary assumption, a pair of segments s_i, s_{i+1} in S such that the symmetric difference of s_i and s_{i+1} contains no green point associated with any of the other segments s_k for k different from i and $i+1$. Then we can conclude the theorem by the induction hypothesis on $S \setminus \{s_i, s_{i+1}\}$.

Claim 3. *There exists an index i such that there is no green point in the interval $(a_i, a_{i+1}]$.*

Proof. We will show that $i = 1$ is such an index. We consider the two leftmost segments $s_1 = (a_1, b_1)$ and $s_2 = (a_2, b_2)$. We claim that $a_2 < \ell_1$. Indeed, if this is not the case, then the green point ℓ_1 belongs to an odd number of segments from S, namely only to s_1, and we are done.

Therefore, we have $a_2 < \ell_1$. Because ℓ_1 is the leftmost green point, there are no green points in the segment $(a_1, a_2]$, as desired. ∎

Going back to the proof of Theorem 1. Let $1 \leq i < n$ be an index such that $(a_i, a_{i+1}]$ contains no green point and such that $a_{i+1} - a_i$ is minimum. Notice in particular that $a_{i+1} - a_i < 1$, for otherwise the green point ℓ_i lies in $(a_i, a_{i+1}]$.

We claim that the symmetric difference of s_i and s_{i+1} contains no green point r_k or ℓ_k for k different from i and $i+1$. Once we show this we are done because we can conclude Theorem 1 by induction on n after removing s_i and s_{i+1} from S.

We will need the following observation:

Observation 2. *None of the segments $(\ell_i, \ell_{i+1}]$ and $(r_i, r_{i+1}]$ contains a point of B.*

Proof. Suppose there is a point $b_j \in B$ in one of the segments $(\ell_i, \ell_{i+1}]$ and $(r_i, r_{i+1}]$.

Case 1. $b_j \in (r_i, r_{i+1}]$. In this case because $r_{i+1} - r_i = a_{i+1} - a_1 < 1$ it must be that $r_i \in s_j$ and $r_{i+1} \notin s_j$. By Observation 1, $\ell_j \in s_i$ and $\ell_j \notin s_{i+1}$. This implies that the green point ℓ_j lies in $s_i \setminus s_{i+1} = (a_i, a_{i+1}]$, contradicting our assumption.

Case 2. $b_j \in (\ell_i, \ell_{i+1}]$. In this case (keeping in mind that $\ell_{i+1} - \ell_i = a_{i+1} - a_i < 1$) we must have $\ell_i \in s_j$ and $\ell_{i+1} \notin s_j$. By Observation 1, $r_j \in s_i$ and $r_j \notin s_{i+1}$. This implies that the green point r_j lies in $s_i \setminus s_{i+1} = (a_i, a_{i+1}]$, contradicting our assumption. ∎

As a consequence of Observation 2, we have the following.

Claim 4. *None of the segments* $[\ell_i, \ell_{i+1})$ *and* $[r_i, r_{i+1})$ *contains a point of* A.

Proof. We prove the claim for the segment $[\ell_i, \ell_{i+1})$ and the proof for the segment $[r_i, r_{i+1})$ is the same.

Assume that there is a point of A in $[\ell_i, \ell_{i+1})$. Because both ℓ_i and ℓ_{i+1} are by definition green points, one can find two consecutive green points $g_1 < g_2$ in the segment $[\ell_i, \ell_{i+1}]$ such that there is a point of A in $[g_1, g_2)$. By Claim 2, $|A \cap [g_1, g_2)| + |B \cap (g_1, g_2]|$ is even. By Observation 2, $|B \cap (g_1, g_2]| \leq |B \cap (\ell_i, \ell_{i+1}]| = 0$. Because we assume $|A \cap [g_1, g_2)| \geq 1$, it follows that $|A \cap [g_1, g_2)| \geq 2$. Therefore we can find two consecutive points a_j and a_{j+1} of A in $[g_1, g_2)$. Because g_1 and g_2 are two consecutive green points, the segment $(a_j, a_{j+1}]$ does not contain any green point. In addition, $a_{j+1} - a_j < g_2 - g_1 \leq \ell_{i+1} - \ell_i = a_{i+1} - a_i$. We thus get a contradiction to the choice of the index i. ∎

As a consequence of Claim 4, we will now show that there is no green point in the segment $[b_i, b_{i+1})$. Indeed, suppose that $q \in [b_i, b_{i+1})$ is a green point. We consider two possible cases.

Case 1. $q = \ell_j$ for some j. Because $\ell_j \in [b_i, b_{i+1})$, we have $\ell_j \in s_{i+1}$ and $\ell_j \notin s_i$. By Observation 1, $r_{i+1} \in s_j$ and $r_i \notin s_j$. Because $r_i < r_{i+1}$, this implies $a_j \in [r_i, r_{i+1})$. This is a contradiction to Claim 4.

Case 2. $q = r_j$ for some j. Because $r_j \in [b_i, b_{i+1})$, we have $r_j \in s_{i+1}$ and $r_j \notin s_i$. By Observation 1, $\ell_{i+1} \in s_j$ and $\ell_i \notin s_j$. Because $\ell_i < \ell_{i+1}$, this implies $a_j \in [\ell_i, \ell_{i+1})$. This is a contradiction to Claim 4.

We conclude that there is no green point in $[b_i, b_{i+1})$. By the choice of i, there is no green point in $(a_i, a_{i+1}]$. In other words, the symmetric difference of s_i and s_{i+1} contains no green point which is what we had to show. Now remove s_i and s_{i+1} from S and conclude Theorem 1 by induction on n. ∎

Remark. Theorem 1 says that for every $0 < \epsilon < \frac{1}{2}$ there exists a point x such that $d(x)$ is odd and x is either of the form $x = a_i + \epsilon$ or $x = b_i - \epsilon$ for some $1 \leq i \leq n$.

By Claim 1, it is always true (regardless of whether n is odd) that the sum $\sum_{i=1}^{n} d(a_i+\epsilon)+d(b_i-\epsilon)$ is even.

It follows now from Theorem 1 that under the same conditions of Theorem 1 one can find **two** points x such that $d(x)$ is odd and x is either of the form $a_i + \epsilon$ or $b_i - \epsilon$, unless we are unlucky and the same point x can be represented both as $a_i + \epsilon$ and as $b_j - \epsilon$ for two distinct indices i and j.

If we are indeed unlucky, then it must be that $b_j - a_i = 2\epsilon$ for some i and j. In particular, there is only a finite number of ϵ's (that is, at most $\binom{n}{2}$ of them) for which we may be unlucky. This proves Theorem 2.

It is very interesting to note that one cannot generalize Theorem 2 for **every** $\epsilon \in (0, \frac{1}{2})$ with no exceptions. It is indeed possible to find a family S of odd number of unit segments and a specific $\epsilon \in (0, \frac{1}{2})$ such that if we color green the points $a_i + \epsilon$ and $b_i - \epsilon$, as in the statement of Theorem 2, then there is only one green point g with $d(g)$ odd.

Figure 3 shows one such example in the discrete case where each segment in S consists of 7 units and we color green the third unit from the left and the third from the right. (To get a non-discrete construction, just replace the colored square with its center point.) For convenience we draw the segments on different layers but one should imagine that they are all projected on the x-axis. One can see in the figure that there is a unique column of squares that contains a black square (that corresponds to the green color) and altogether an odd number of squares (white or black) on that column (this corresponds to $d(g)$ being odd).

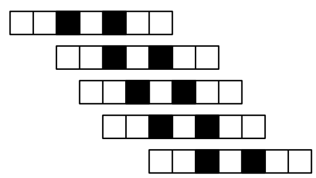

Figure 3: An example with only one green point.

In Figure 4, there is a construction where each discrete segment consists of 10 units. This is the smallest example with even number of units per segments that we could find.

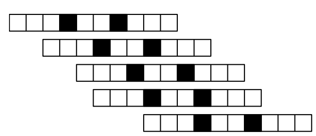

Figure 4: An example with only one green point.

All of the above constructions, although seemingly simple, were found with the aid of a computer. Such examples are not frequent at all and are quite difficult to find.

Acknowledgments

We thank Barry Balof for the short and elegant argument presented in the introduction. This elegant fact inspired the topic of this paper.

The author was supported by ISF grant (grant No. 409/16).

References

[1] R. Pinchasi, Points covered an odd number of times by translates. *Amer. Math. Monthly* **121** (2014) 632–636.

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