On sets of points in general position that lie on a cubic curve in the plane and determine lines that can be pierced by few points

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Abstract

Let P be a set of n points in general position in the plane. Let R be a set of points disjoint from P such that for every $x, y \in P$ the line through x and y contains a point in R. We show that if $|R| < \frac{3}{2}n$ and $P \cup R$ is contained in a cubic curve c in the plane, then P a special algebraic structure on c, that is, P is contained in a coset of a subgroup H of c of cardinality at most $|R|$.

We use the same approach to show a similar result in the case where each of B and G is a set of n points in general position in the plane and every line through a point in B and a point in G passes through a point in R. This provides a partial answer to a problem of Karasev.

The bound $|R| < \frac{3}{2}n$ is best possible at least for part of our results. Our extremal constructions provide a counterexample to an old conjecture attributed to Jamison about point sets that determine few direction. Jamison conjectured that if P is a set of n points in general position in the plane that determines at most $2n - c$ distinct directions, then P is contained in an affine image of the set of vertices of a regular m-gon. This conjecture of Jamison is strongly related to our results in the case the cubic curve c is reducible and our results can be used to prove Jamison's conjecture at least when $m - n$ is in the order of magnitude of $O(\sqrt{n})$.
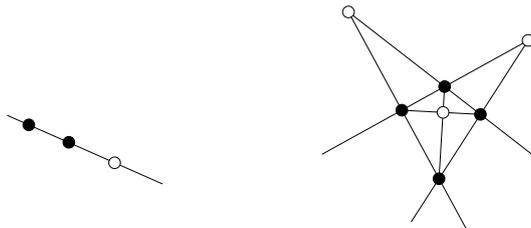
1   Introduction

Figure 1: Constructions with $|R| = n - 1$ for $n = 2, 4$. The points in P are colored black while the points in R are colored white.

In [5] Erdős and Purdy considered the following problem. Let P be a set of n points in the plane that lie in *general position* in the sense that no three points of P are collinear. Let R be another set of points disjoint from P such that every line through two points of P contains a point in R. Give a lower bound on $|R|$ in terms of n.

If n is odd it is easy to prove the tight bound $|R| \geq n$. This is because every point in R may be incident to at most $\frac{n-1}{2}$ of the $\binom{n}{2}$ lines determined by P. To observe that this bound is tight let P be the set of vertices of a regular n-gon and let R be the set of n points on the line at infinity that correspond to the directions of the edges (and diagonals) of P. This construction is valid also when n is even.

If n is even, a trivial counting argument shows that $|R|$ must be at least $n-1$. This is because every point in R may be incident to at most $n/2$ lines determined by P. This trivial lower bound for $|R|$ is in fact sharp in the cases $n = 2$ and $n = 4$, as can be seen in Figure 1. Is the bound $|R| \geq n - 1$ sharp also for larger values of n?

The following theorem proves a conjecture attributed to Erdős and Purdy [5].

Theorem 1.1 ([1, 13, 16])**.** *Let P be a set of n points in general position in the plane and, where $n > 4$ is even. Assume R is another set of points disjoint from P such that every line through two points of P contains a point from R. Then $|R| \geq n$.*

Theorem 1.1 was first proved in [1] (see Theorem 8 there), as a special case of the solution of the Magic Configurations conjecture of Murty [14]. The proof in [1] contains a topological argument based on Euler's formula for planar maps and the discharging method. An elementary (and long) proof of Theorem 1.1 was given by Milićević in [13]. Probably the "book proof" of the Theorem 1.1 can be found in [16].

Theorem 1.1 was proved also over $\mathbb{F}_p$ by Blokhuis, Marino, and Mazzocca [4].

As we have seen, there are constructions of sets P of n points in general position and sets R of n points not in P, such that every line determined by P passes through a point in R. One major question that arises here is to characterize those sets P in general position for which there exists a set R with $|R| = |P|$ such that every line that is determined by P passes through a point in R.

The following conjecture of Milićević in [13], came up in connection to the above mentioned Theorem 1.1.

Conjecture 1.2. *Let P be a set of n points in general position and let R be a set of n points disjoint from P. If every line determined by P passes through a point in R, then $P \cup R$ is contained in a cubic curve.*

A special case of Conjecture 1.2 is proved in [10], where Conjecture 1.2 is proved under additional assumption:

Theorem 1.3. *Suppose P is a set of n points in general position in the plane and R is another set of n points, disjoint from P. If for every $x, y \in P$ there is a point $r \in R$ on the line through x and y and outside the interval determined by x and y, then $P \cup R$ is contained in a cubic curve.*

We recall that given an irreducible cubic curve c in the plane, there is a natural abelian group structure on c. In this group structure the sum of three collinear points on c is equal to 0. Moreover, if there is a line ℓ that crosses c at a point A and is tangent to c at a point B, then $A + B + B = 0$.

In this paper we show that if indeed $P \cup R$ in Conjecture 1.2 is contained in a cubic curve c in the plane and if $|P|$ is not too small, then there is a subgroup H of c such that both P and R are cosets of H. In fact, we can extend it as follows:

Theorem 1.4. *Let P be a set of $n > 6$ points in general position in the plane. Let R be another set of less than $\frac{3}{2}n$ points, disjoint from P such that any line through two points of P passes through a point in R. Assume that $P \cup R$ is contained in an irreducible cubic curve c in the plane. Then there is a subgroup H of c of size at most $|R|$ such that P is contained in a coset of H. If $|R| = n$, then both P and R are equal to cosets of H.*

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It is not hard to consider the situation in Theorem 1.4 also in the case where the cubic curve c is reducible. In such a case c is either a union of three lines, or a union of a quadric and a line. The former case is literally impossible as we assume P is in general position and large enough. The following easy theorem settles the case where c is a union of a quadric and a line:

Theorem 1.5. *Let P be a set of $n > 6$ points in general position in the plane. Let R be another set disjoint from P such that any line through two points of P passes through a point in R. Assume that $P \cup R$ is contained in a reducible cubic curve c that is a union of a quadric Q and a line ℓ. If $n \leq |R| < \frac{3}{2}n$, then Q must be an ellipse, $P \subset Q$, and up to an affine transformation P is a subset of the set of vertices of a regular m-gon for some $m \leq |R|$. The bound $\frac{3}{2}n$ in the statement of the theorem is best possible.*

We remark that for values of n smaller than or equal to 6 one can indeed find some sporadic examples that are contained in a union of three lines. For instance, the examples in Figure 1 are of sets of points contained in a union of at most three lines and satisfy in some respect the conditions in Theorem 1.4.

Another interesting remark is that the bound $|R| < \frac{3}{2}n$ in Theorem 1.5 cannot be improved, not even by one unit. We now present a simple construction showing this. Before presenting the construction, the following two very easy observations will be useful.

Observation 1.6. *Let S be the set of vertices of a regular m-gon. Then S determines lines that appear in precisely m distinct directions.*

Observation 1.7. *Let Q be a circle and let P_1 be the set of vertices of a regular m-gon inscribed in Q. Let P_2 the set of vertices of another regular m-gon inscribed in Q such that P_1 is disjoint from P_2. Then the lines connecting a point from P_1 to a point from P_2 may appear in one of at most m distinct directions (see Figure 2).*

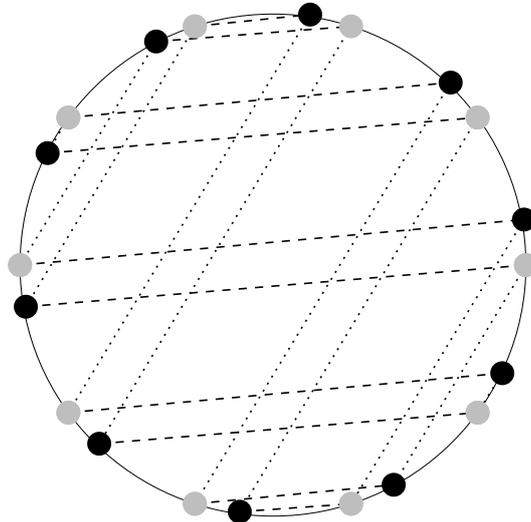We leave it to the reader to verify the validity of Observation 1.6 and Observation 1.7.

Figure 2: Illustration of Observation 1.7 for the case $m = 10$. The points in P_1 are colored black while the points in P_2 are colored gray. The diagonals connecting points of P_1 and points of P_2 can be partitioned into m families of pairwise parallel diagonals.

In order to see that the bound $|R| < \frac{3}{2}n$ in Theorem 1.5 is best possible consider a circle Q and let P_1 be the set of vertices of a regular $\frac{n}{2}$-gon inscribed in Q (we assume n is even). Let P_2 be a generic rotation (for example, by an irrational angle) of P_1 about the center of Q. Let P be the union $P_1 \cup P_2$. Therefore, P is a set of n points contained in Q. The lines passing through

two points of P_1 have precisely $\frac{n}{2}$ distinct directions. The same is true, by Observation 1.6, for lines passing through two points of P_1. From Observation 1.7 we know that the lines passing through a point of P_1 and a point of P_2 have precisely $\frac{n}{2}$ distinct directions. We conclude that the lines passing through two points of P have precisely $\frac{3}{2}n$ distinct directions. Define R to the the set of $\frac{3}{2}n$ points on the line at infinity that correspond to the $\frac{3}{2}n$ directions of lines determined by the points in P. Then every line through two points of P passes through a point of R. Moreover, P is not a subset of the set of vertices of any regular polygon.

It is extremely interesting to remark about the relation between Theorem 1.4 and Theorem 1.5 and an old conjecture attributed to Jamison in [8]. By a well known theorem of Ungar ([18]), proving a conjecture of Scott ([17]), every set of n points in the plane that is not collinear must determine lines appearing in at least $2\lfloor \frac{n}{2} \rfloor$ distinct directions. In [8] Jamison studied the case where the set P of n points is in general position in the sense that no three of its points are collinear. In this case it is very easy to prove that P determines at least n distinct directions. Jamison showed that in the extremal case if P is a set of n points in general position that determines lines in precisely n distinct directions, then P is, up to an affine transformation, the set of vertices of a regular n-gon. Jamison mentioned in [8] that is it believed that if a set P of n points in general position determines $m \le 2n - c$ distinct directions (for some large enough absolute constant c), then up to an affine transformation P is contained in the set of vertices of a regular m-gon. This conjecture mentioned in [8] was recently addressed in [15], where the conjecture is proved for the case $m = n + 1$.

Having shown that the bound $|R| < \frac{3}{2}n$ in Theorem 1.5 is best possible we constructed a set P of n points (where we assumed n is even) that determines precisely $\frac{3}{2}n$ distinct directions. This set P that we constructed was contained in a circle but was not a subset of any set of vertices of a regular m-gon. This shows that Jamison's conjecture is false for $m \ge \frac{3}{2}n$ and one may hope to prove it only for smaller values of m.

One possible way to approach Jamison's conjecture is to show that if P is a set of n points that determines less than $\frac{3}{2}n$ distinct directions, then P is contained in a conic. Then we can take R to be the set of points on the line at infinity that correspond to the directions determined by P and Theorem 1.5 will automatically imply that P must be contained (up to an affine transformation) in the set of vertices of a regular m-gon for some $m < \frac{3}{2}n$.

Such an approach is carried out in an ongoing work by the second author and Alexandr Polyanskii in the case where P determines at most $n + O(\sqrt{n})$ distinct directions. One can show that such a set P must be contained in a conic. Consequently, together with Theorem 1.5, this extends significantly the result in [15] and in fact proves Jamison's conjecture in these cases.

Next we turn to present the second part of our paper that is of similar nature. A bipartite version of the situation that appears in Conjecture 1.2 was raised by Karasev: Assume P is a set of $2n$ points that is the union of a set B of n blue and a set G of n green points. The set R is a set of n red points and we assume that the sets B, G, and R are pairwise disjoint. We also assume that the set $P = B \cup G$ is in general position in the sense that no three of its points are collinear. Assume that every line through a point in B and a point in G contains also a point from R. The problem of characterizing the set $B \cup G \cup R$ was raised by Karasev (in a dual version) in [9] (see Problem 6.2 there).

In this context the following analogous conjecture to Conjecture 1.2 appears in [10]:

Conjecture 1.8 ([10])**.** *Let B, G, and R be three sets of points in the plane, each of which is in general position and has size n. Assume that every line passing through two points from two different sets passes also through a point from the third set. Then $B \cup G \cup R$ lies on a cubic curve.*

Conjecture 1.8 is proved in [10] under a similar additional assumption as in Theorem 1.3.

Theorem 1.9. *[10] Let $B, G,$ and R be three pairwise disjoint sets of points in the plane. Assume that $B \cup G$ is in general position and $|B| = |G| = |R| = n$. If every line through a point $b \in B$ and a point $g \in G$ contains a point $r \in R$ that does not lie between b and g, then $B \cup G \cup R$ lies on some cubic curve.*

Similar to Theorem 1.4, we can prove something about the algebraic structure of $B \cup G \cup R$ in Conjecture 1.8 and in particular in Theorem 1.9.

Theorem 1.10. *Let $B, G,$ and R be three pairwise disjoint sets of points in the plane such that $B \cup G$ is in general position and every line through a point in B and a point in G passes through a point in R. Assume $|B| = |G| = n$, $|R| < \frac{3}{2}n$ and $B \cup G \cup R$ is contained in an irreducible cubic curve c, then the sets B, G are contained in cosets of the same subgroup H of c of cardinality at most $|R|$.*

We can consider Theorem 1.10 also in the case c is a reducible cubic curve. Here too if c is a union of three lines and $|B|$ and $|G|$ are greater than 6 we get a contradiction to the assumption that both B and G are in general position.

It is interesting to remark that when $|B| = |G| \leq 6$ we may get examples satisfying the conditions in Theorem 1.10. One easy example is the case $n = 1$ of three collinear points (one of B, one of G, and one of R). A more interesting example is Pappus' Theorem that is illustrated in Figure 3. In this example each of the sets $G, B,$ and R consists of 3 points and they satisfy the conditions in Theorem 1.10. The union $B \cup G \cup R$ is contained in a union of three lines.
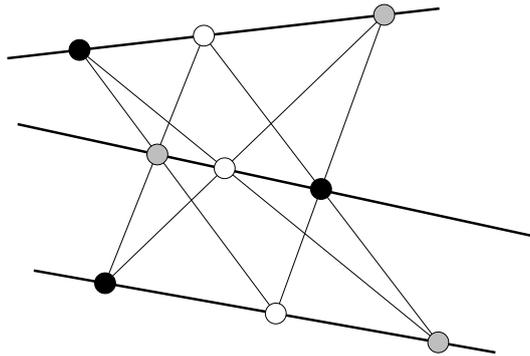

Figure 3: Pappus' Theorem gives rise to a small example for Theorem 1.10 that is contained in three lines. The points of B are colored black, the points of G are colored gray, and the points of R are colored white.

In the case $n > 6$, $B \cup G$ cannot be contained in a union of three lines and the only possibility for a reducible cubic curve containing $B \cup G$ is where c is the union of a quadric and a line. This case is studied in the following theorem.

Theorem 1.11. *Let $B, G,$ and R be three pairwise disjoint sets of points in the plane such that $B \cup G$ is in general position and every line through a point in B and a point in G passes through a point in R. Assume $|B| = |G| = n > 6$, $|R| < \frac{3}{2}n$, and $B \cup G \cup R$ is contained in a reducible cubic curve c that is a union of a quadric Q and a line ℓ, then the sets $B, G \subset Q$, and Q must be an ellipse. Moreover, up to an affine transformation that takes Q to a circle, each of B and G is a subset of a set of vertices of some regular m-gon contained in Q, where $m \leq |R|$. The bound $\frac{3}{2}n$ on $|R|$ in the statement of the theorem is best possible.*

We remark that by Observation 1.7, if Q is a circle and each of A and B is a subset of the set of vertices of a regular m-gon contained in Q, then the number of distinct directions of lines passing through a point in A and a point in B is at most m. This is equivalent to saying that one can find a set R of m points on the line at infinity such that every line through a point in A and a point in B passes through a point in R.

Another important remark is about the tightness of the bound $|R| < \frac{3}{2}n$ in the statement Theorem 1.11. We now present the construction showing this. Let Q be a circle and let k be an odd integer. Let Z be the set of vertices of a regular k-gon contained in Q. Let Z' be a generic rotation of Z about the center of Q. Then set $B = Z \cup (-Z')$ and $G = -Z \cup Z'$. We have $|B| = |G| = 2k$ and set $n = 2k$. We claim that the set of lines passing through a point in B and a point in G may appear in one of at most $3k$ distinct directions. Once this is verified, let R be the set of $3k$ points on the line at infinity that correspond to the $3k$ distinct directions of lines passing through a point in B and a point in G. Then every line passing through a point of B and a point of G will pass also through a point of R. We have $|R| = \frac{3}{2}n$ and neither B or G is contained in the set of vertices of a regular m-gon. To see that indeed the number of distinct directions of lines passing through a point in B and a point in G is equal to $3k = \frac{3}{2}n$ we use Observation 1.7. Notice that each of $Z, Z', -Z$, and $-Z'$ is the set of vertices of a regular k-gon inscribed in Q. Recall that $B = Z \cup (-Z')$ and $G = (-Z) \cup Z'$ By Observation 1.7, the lines passing through a point of Z and a point of $-Z$ have precisely k distinct directions. The same is true for the lines passing through a point of $-Z'$ and a point of Z'. The crucial observation is that the set of directions of lines passing through a point of $-Z'$ and a point of $-Z$ (and there are precisely k such distinct directions) is precisely the same set of directions of the lines passing through a point of Z' and a point of Z. This implies that the number of distinct directions of lines passing through a point of $B = Z \cup (-Z')$ and a point of $G = (-Z) \cup Z'$ is equal to $3k = \frac{3}{2}n$, as desired (see Figure 4).
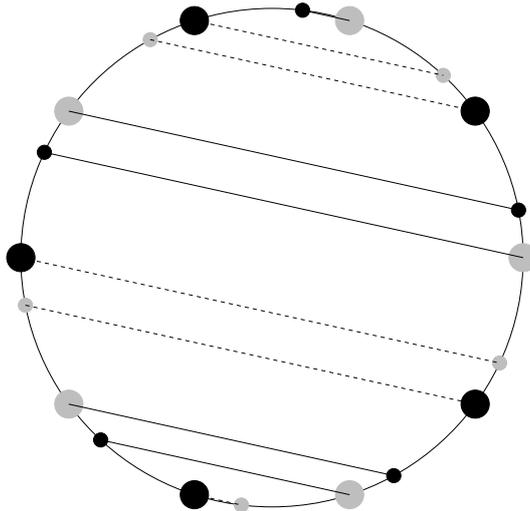

Figure 4: The construction with $|B| = |G| = 10$. The points of $B = Z \cup (-Z')$ are colored black. The points of Z are drawn by smaller black discs and the points of $-Z'$ by bigger black discs. The points of $G = (-Z) \cup Z'$ are colored gray. The points of $-Z$ are drawn by smaller gray discs and the points of Z' by bigger gray discs.

The case of Theorem 1.10 is simpler than the one of Theorem 1.4. For this reason we start with the proof of Theorem 1.10 and Theorem 1.11 in Section 2 and continue to the proof of Theorem 1.4 and Theorem 1.5 in Section 3.

2 Proof of Theorem 1.10 and Theorem 1.11.

We will rely on the following result about subsets of abelian groups. This is one of several variation of, by now a classical, result of Frieman [6] who proved a similar result for general groups. The Lemma as we bring it below is from [2] (see Proposition 2.1 there).

Lemma 2.1 ([2]). *Suppose A_1 and A_2 are two finite subsets of an abelian group F. Let H denote the subgroup of F that is the stabilizer of $A_1 + A_2$. That is, $H = \{x \in F \mid x + A + B = A + B\}$. If $|A_1| \geq |A_2|$ and $|A_2| \geq \frac{3}{4}|A_1|$, and if $|A_1 + A_2| < \frac{3}{2}|A_1|$ and then $A_1 + A_2$ is equal to a coset of H. Consequently, each of A_1 and A_2 is contained in a coset of H.*

Lemma 2.1 is stated in [2] for *finite* abelian groups although only the finiteness of A_1 and A_2 is required for the proof. Moreover, the statement of Lemma 2.1 in [2] only says that such a subgroup H exists but in the proof H is taken to be the stabilizer of $A_1 + A_2$. We will not make use of the fact that H is the stabilizer of $A_1 + A_2$. The easy proof of Lemma 2.1 in [2] relies on Kneser's theorem (see [12]).

We start with the proof of Theorem 1.10, where we assume that the cubic curve c is irreducible. In this case we have the group action that is naturally defined on c, where three points on c are collinear if and only if their sum is equal to 0. Because every line through a point in B and a point in G passes through a point in R we conclude that $R \supset -(B + G)$.

Let H be the subgroup of c that is the stabilizer of $B + G$. By Lemma 2.1, where B and G are in the role of A_1 and A_2, both B and G are subsets of a coset of H. Moreover, $B + G$ is equal to a coset of H. In particular, $|H| \leq |R|$.

If $|R| = n$ it follows that $B + G = -R$. Consequently, R is a coset of H and because $|B| = |G| = |H| = n$ both B and G must be equal to cosets of H. This completes the proof of Theorem 1.10

We remark that the case $|R| = n$ in Theorem 1.10 is extremely easy even without using Lemma 2.1. We give here the easy argument because it is very short. Choose $b \in B$ and $g \in G$ and let $B' = -B + b$ and $G' = -G + g$. Then $0 \in B', G'$ and we have $R + (b + g) = B' + G'$. Because $|B'| = |G'| = |B' + G'| = n$ and because $B', G' \subset B' + G'$ (this is because $0 \in B', G'$), then it must be that $B' = G' = B' + G'$ and we denote this set by H. H must be a subgroup because $H + H = H$ and $0 \in H$. Now the result follows because $B = -B' + b = -H + b = H + b$, $G = -G' + g = -H + g = H + g$, and $R = B' - (b + g) = H + b - (b + g) = H - g$. This completes the proof of Theorem 1.10 in the case $|R| = n$. ∎

We now move on to the proof of Theorem 1.11. We will need the following lemma that can be found in [7].

Lemma 2.2 (Proposition 7.3 in [7]). *Let c be a cubic curve that is a union of a quadric Q and a line ℓ. Then there is an abelian group F and two mapping $\phi_Q : F \to Q$ and $\phi_\ell : F \to \ell$ such that for $x, y, z \in F$ $x + y + z = 0$ if and only if $\phi_Q(x), \phi_Q(y)$, and $\phi_\ell(z)$ are collinear. Moreover, if Q is a hyperbola, then F is isomorphic $\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}$, if Q is a parabola, then F is isomorphic to $(\mathbb{R}, +)$, and if Q is an ellipse, then F is isomorphic to $(\mathbb{R}/\mathbb{Z}, +)$.*

In the case of Theorem 1.11 the cubic curve c is a union of a quadric Q and a line ℓ. We claim that no point of $B \cup G$ may lie on ℓ. To see this, assume to the contrary and without loss of generality that $b \in B$ lies on ℓ. Then by considering the n lines through b and the n points in G we conclude that there must be at least $n - 1$ points of R not on ℓ and that means there are at least $n - 1$ points of R on Q. On the other hand because both B and G are in general position, there cannot be more than two points of B and two points of G on ℓ. Therefore, there are at least $n - 2$ points of B on Q and at least $n - 2$ points of G on Q. Every line through a point of B on Q and a point of G on Q must contain a point of R on ℓ. This implies at least $n - 2$ points of R on ℓ. This is a contradiction as we assume $|R| < \frac{3}{2}n < 2n - 4$.

Having shown that $B \cup G \subset Q$ it follows that essentially $R \subset \ell$. This is because no point on Q may be collinear with a point of B and a point of G (both also on Q).

We now use Lemma 2.2. We conclude that there is an abelian group F and two mapping $\phi_Q : F \to Q$ and $\phi_\ell : F \to \ell$ such that for $x, y, z \in F$ $x + y + z = 0$ if and only if $\phi_Q(x), \phi_Q(y)$, and $\phi_\ell(z)$ are collinear. We know moreover, if Q is a hyperbola, then F is isomorphic $\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}$, if Q is a parabola, then F is isomorphic to $(\mathbb{R}, +)$, and if Q is an ellipse, then F is isomorphic

to $(\mathbb{R}/\mathbb{Z}, +)$. We know that every point of B (on Q) and a point of G (on Q) are collinear with a point of R. Therefore, taking $\tilde{B} = \phi_Q^{-1}(B)$, $\tilde{G} = \phi_Q^{-1}(G)$, and $\tilde{R} = \phi_\ell^{-1}(R)$, we have $\tilde{B} + \tilde{G} \subset -\tilde{R}$ and consequently $|\tilde{B} + \tilde{G}| \leq \frac{3}{2}n$.

We apply Lemma 2.1 with the abelian group F taking A_1 and A_2 in Lemma 2.1 to be $A_1 = \tilde{B}$ and $A_2 = \tilde{G}$, respectively. Lemma 2.1 implies (because we have $|\tilde{B} + \tilde{G}| < \frac{3}{2}n$) that $\tilde{B} + \tilde{G}$ is equal to a coset of a subgroup H of F and each of $\tilde{B}$ and $\tilde{G}$ is contained in a coset of H. Notice that the cardinality of H is smaller than or equal to $|R|$ and consequently, if $|R| = n$, then each of $\tilde{B}$ and $\tilde{G}$ are in fact *equal* to a coset of H.

We can now further continue and give a more concrete description of B and G. Project ℓ to the line at infinity. The points of R correspond to a collection of $|R|$ distinct directions and every line through a point of B and a point of G has one of these directions.

We claim that Q cannot be a parabola or a hyperbola. To see this, recall that by Lemma 2.2, if Q is a hyperbola or a parabola, then the abelian group F is either $\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}$, or $(\mathbb{R}^*, .)$, respectively. In either case it cannot have a finite subgroup H of size greater than 2.

Having shown that Q cannot be a parabola or a hyperbola we conclude that Q must be an ellipse. In this case the abelian group F is isomorphic to $(\mathbb{R}/\mathbb{Z}, +)$ and the only finite subgroups it has are isomorphic to $\mathbb{Z}_k$. By applying an affine transformation we may assume that Q is a circle. We may assume without loss of generality that $0 \in H$ and therefore H is a finite subgroup of F. Hence H is isomorphic to $(\mathbb{Z}_k, +)$ for some $k < \frac{3}{2}n$. We know that $|\tilde{B}| = |\tilde{G}| = n$ and $\tilde{B} + \tilde{G} = H$.

We claim that $\phi_Q(H)$ must be the set of vertices of a regular k-gon. To see this observe that H is a finite subgroup of F that is isomorphic to $(\mathbb{Z}_k, +)$. From Lemma 2.2 we know that if $x + y + z = 0$ in F, then the line through $\phi_Q(x)$ and $\phi_Q(y)$ is parallel to the direction $\phi_\ell(z)$ on the line ℓ at infinity. Therefore, the lines through two points of $\phi_Q(H)$ can be only in $|H| = k$ distinct directions. It now follows from a well known result of Jamison (Theorem 2 in [8]), and also very easy to show directly because we know that $\phi_Q(H)$ is contained in a circle, that $\phi_Q(H)$ is equal to the set of vertices of a regular k-gon. Because each of $\tilde{B}$ and $\tilde{G}$ is a subset of a coset of H we conclude that each of B and G is a rotation about the center of Q of some (could be different for B and for G) subset of size n of the set of vertices of a regular k-gon, where $k < \frac{3}{2}n$. This concludes the proof of Theorem 1.11. In the remarks following the statement of Theorem 1.11 it is shown why the bound $|R| < \frac{3}{2}n$ in the statement of the theorem cannot be improved even by one unit. ∎

3   Proof of Theorem 1.4 and Theorem 1.5

The proof of Theorem 1.4 is a bit more involved than the proof of Theorem 1.10. The reason is that from the fact that a line through two points in P passes through a point in R we cannot conclude that $P + P \subset -R$. This is because we have no information about the sum of a point in P with itself.

For a subset A of a group we denote by $A \dot{+} A$ the set $\{a + a' \mid a, a' \in A, \ a \neq a'\}$. Therefore, the conditions in Theorem 1.4 imply only $P \dot{+} P \supset -R$ rather than $P + P \subset -R$. This difference makes the proof of Theorem 1.4 a bit more challenging.

We will need the following result from [11] about subsets of abelian groups.

Theorem 3.1 (Theorem 3 in [11])**.** *Suppose F is a group, $A \subset F$, and $|A \dot{+} A| \leq \frac{1+\sqrt{5}}{2}|A| - (L + 2)$, where L, called the doubling constant of F, is the maximum number of solutions to $x + x = a$ over all $a \in F$. Then $A \dot{+} A = A + A$.*

We recall that P is a set of n points in general position in the plane. R be another set of n points, disjoint from P such that any line through two points of P passes through a point in R. We assume that $P \cup R$ is contained in a cubic curve c in the plane. We need to show

there is a subgroup H of c of size at most $|R|$ such that $P\dotplus P$ is equal to a coset of H. We have $P\dotplus P \subset -R$ and consequently $P\dotplus P < \frac{3}{2}n$. We apply Theorem 3.1, where the abelian group F is the cubic curve c with its prescribed group structure. We notice that the doubling constant of the group F is not greater than 6. This is because a solution to $x + x = a$ for $a \in F$ means that there is a line through the point a that touches c at x with multiplicity 2. For an irreducible cubic curve there cannot be more than 6 tangent lines through any given point a. To this elementary fact one can project a to infinity and assume without loss of generality that all the lines through a are horizontal. Then at any point (x, y) on c in which the tangent line is horizontal we must have $\frac{\partial c}{\partial x}(x, y) = 0$. However, $\frac{\partial c}{\partial x}(x, y) = 0$ is a quadric and, by Bezout theorem, it intersects the cubic c in at most 6 points (because c is irreducible).

Having verified that the doubling constant of F is at most 6, Theorem 3.1 implies (because we have $|P\dotplus P| < \frac{3}{2}n < \frac{1+\sqrt{5}}{2}n - 8$, where we assume n is large enough) that $P\dotplus P = P + P$. Now we apply Lemma 2.1, where we take $A_1 = A_2 = P$ in Lemma 2.1. We have $|P+P| = |P\dotplus P| < \frac{3}{2}|P|$. Therefore, by Lemma 2.1 $P\dotplus P = P + P$ is equal to a coset of a subgroup H and P is contained in a coset of H. Notice that the cardinality of H is smaller than or equal to $|R|$ and consequently, if $|R| = n$, then P is equal to a coset of H. ∎

We now move on to the proof of Theorem 1.5. We start by following the proof of Theorem 1.11. The cubic curve c is a union of a quadric Q and a line ℓ. We claim that no point of P may lie on ℓ. To see this, assume to the contrary that $x \in P$ lies on ℓ. Then by considering the $n - 1$ lines through x and the $n - 1$ other points in P we conclude that there must be at least $n - 2$ points of R not on ℓ and that means there are at least $n - 2$ points of R on Q. On the other hand because P is in general position, there cannot be more than two points of P on ℓ. Therefore, there are at least $n - 2$ points of P on Q. Every line through two points of P on Q must contain a point of R on ℓ. This implies at least $n - 3$ points of R on ℓ. This is a contradiction as we assume $|R| < \frac{3}{2}n < (n - 2) + (n - 3)$.

Having shown that $P \subset Q$ it follows that essentially $R \subset \ell$. This is because a line through two points (of P) on Q cannot be collinear with another point (of R) on Q.

We now use Lemma 2.2. We conclude that there is an abelian group F and two mapping $\phi_Q : F \to Q$ and $\phi_\ell : F \to \ell$ such that for $x, y, z \in F$ $x + y + z = 0$ if and only if $\phi_Q(x), \phi_Q(y)$, and $\phi_\ell(z)$ are collinear. We know moreover, if Q is a hyperbola, then F is isomorphic $\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}$, if Q is a parabola, then F is isomorphic to $(\mathbb{R}, +)$, and if Q is an ellipse, then F is isomorphic to $(\mathbb{R}/\mathbb{Z}, +)$. We know that every two points of P (on Q) are collinear with a point of R (on ℓ). Therefore, taking $\tilde{P} = \phi_Q^{-1}(P)$ and $\tilde{R} = \phi_\ell^{-1}(R)$, we have $\tilde{P}\dotplus\tilde{P} \subset -\tilde{R}$ and consequently $|\tilde{P}\dotplus\tilde{P}| \leq \frac{1+\sqrt{5}}{2}n - 4$. We can now continue similar to the proof of Theorem 1.11. We apply Theorem 3.1 with the abelian group F. We notice that the doubling constant of the group F is not greater than 2. Indeed, consider a solution to $x + x = a$ for $a \in F$. This means that $x + x + (-a) = 0$ but this means that $\phi_\ell(-a)$ is a point on ℓ and the line through it and $\phi_Q(x)$ is tangent to Q. As Q is a conic there cannot be more than two such points x. Having verified that the doubling constant of F is at most 2, Theorem 3.1 implies (because we have $|\tilde{P}\dotplus\tilde{P}| < \frac{3}{2}n \leq \frac{1+\sqrt{5}}{2}n - 4$, where n is large enough) that $\tilde{P}\dotplus\tilde{P} = \tilde{P} + \tilde{P}$. Now we apply Lemma 2.1, where we take $A_1 = A_2 = \tilde{P}$ in Lemma 2.1. We have $|\tilde{P} + \tilde{P}| = |\tilde{P}\dotplus\tilde{P}| < \frac{3}{2}|\tilde{P}|$. Therefore, by Lemma 2.1, $\tilde{P}\dotplus\tilde{P} = \tilde{P} + \tilde{P}$ is equal to a coset of a subgroup H of F and $\tilde{P}$ is contained in a coset of H. Notice that the cardinality of H is smaller than or equal to $|R|$ and consequently, if $|R| = n$, then $\tilde{P}$ is equal to a coset of H.

We can now further continue and give a more concrete description of P. Project ℓ to the line at infinity. The points of R correspond to a collection of $|R|$ distinct directions and every line through two points of P has one of these directions.

We claim that Q cannot be a parabola or a hyperbola. To see this, recall that by Lemma 2.2, if Q is a hyperbola or a parabola, then the abelian group F is either $\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}$, or $(\mathbb{R}^*, .)$, respectively. In either case it cannot have a finite subgroup H of size greater than 2.

Having shown that Q cannot be a parabola or a hyperbola we conclude that Q must be an ellipse. In this case the abelian group F is isomorphic to $(\mathbb{R}/\mathbb{Z}, +)$ and the only finite subgroups it has are isomorphic to $\mathbb{Z}_k$. By applying an affine transformation we may assume that Q is a circle. We may assume without loss of generality that $0 \in \tilde{P}$ and therefore, H is a finite subgroup of F. Hence H is isomorphic to $(\mathbb{Z}_k, +)$ for some $k < \frac{3}{2}n$. We know that $|\tilde{P}| = n$ and $\tilde{P} + \tilde{P} = H$. The geometric consequence is now clear. P must be a subset of the set of vertices of a regular k-gon on Q. To see this observe that H is a finite subgroup of F that is isomorphic to $(\mathbb{Z}_k, +)$. From Lemma 2.2 we know that if $x + y + z = 0$ in F, then the line through $\phi_Q(x)$ and $\phi_Q(y)$ is parallel to the direction $\phi_\ell(z)$ on the line ℓ at infinity. Therefore, the lines through two points of $\phi_Q(H)$ can be only in $|H|$ distinct directions. As in the proof of Theorem 1.11, it now follows from a well known result of Jamison ([8]), and also very easy to show directly, that $\phi_Q(H)$ is affinely equivalent to the set of vertices of a regular $|H|$-gon. This concludes the proof of Theorem 1.5. In the remarks following the statement of Theorem 1.5 it is shown why the bound $|R| < \frac{3}{2}n$ in the statement of the theorem cannot be improved even by one unit. ∎

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