At Least $n-1$ Intersection Points in a Connected Family of n Unit Circles in the Plane

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Abstract

Let $\mathcal{C}$ be a family of n different unit circles in the plane. We show that $\mathcal{C}$ determines at least $n - c$ intersection points, where c is the number of connected components in $\bigcup_{C \in \mathcal{C}} C$. This proves a conjecture of A. Bezdek.

1 Introduction

Let $\mathcal{C}$ be a family of n unit circles in the plane, no two of which touch each other. Assume further that there are no isolated circles (i.e., every circle intersects at least one other circle from $\mathcal{C}$). A beautiful argument of K. Bezdek and R. Connely shows that $\mathcal{C}$ determines at least n intersection points. We bring their proof here, both for its ingenuity and for its relation to what we further wish to prove in this paper.

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Theorem 1 (K. Bezdek & R. Connely). *Let $\mathcal{C}$ be a connected family of (at least two) unit circles in the plane, no two of which touch each other. Then $\mathcal{C}$ determines at least n intersection points.*

Proof: Denote by m the number of intersection points determined by $\mathcal{C}$. Give weight 1 to every intersection point. Then equally distribute the weight of each intersection point among the circles passing through it. Therefore, if say, k circles pass through a point P, then P gives a weight of $\frac{1}{k}$ to every such circle.

We now analyze the weight of each circle. Let $C \in \mathcal{C}$. Assume C contains k intersection points. Clearly, through every such intersection point P at most k circles pass, for every circle through P intersects C in another unique point. Hence C is given a total weight of at least 1. This shows that $m \geq n$. ∎

K. Bezdek and Connelly, (see [5]), considered coverings of the plane by unit disks. They noticed that the unit disks centered at the intersection points of the boundary circles of the original covering also form a covering of the plane. They conjectured that this second covering can not have smaller density than the original covering. They proved Theorem 1 above to settle this conjecture for the case in which no two of the disks in the covering touch each other. Later, A. Bezdek (see [2]) settled the conjecture of K. Bezdek and R. Connelly in its full generality without requiring no tangency. A. Bezdek's argument was also weight distribution argument but slightly more involved. While working out the details of his 8 pages long weight distribution argument, A. Bezdek noticed that probably the right question to ask is not the above one concerning coverings, but the following stated as a conjecture (see [2]. See also [3] for a related result):

Conjecture 2.: [A. Bezdek] There are at least $n-1$ intersection points in a connected family of n unit circles in the plane.

A second and very important part of his conjecture was also a list of cases in which there is equality. He conjectured that his list was complete.

The proof of Theorem 1 does not apply here because now circles are allowed to touch. This small difference creates a huge difficulty. In fact, a straight forward implication of the above argument to this case will give a lower bound of $\frac{n}{2}$ intersection points. This indeed can be attained if $\mathcal{C}$ is a union of disjoint pairs of touching circles.

In this paper we prove the first part of A. Bezdek's conjecture, stated in the next theorem. The second part of the conjecture is still open.

Theorem 3. *Let $\mathcal{C}$ be a connected family of n unit circles in the plane and let $\mathcal{P}$ be the set of their intersection points. Then $|\mathcal{P}| \geq n - 1$,*

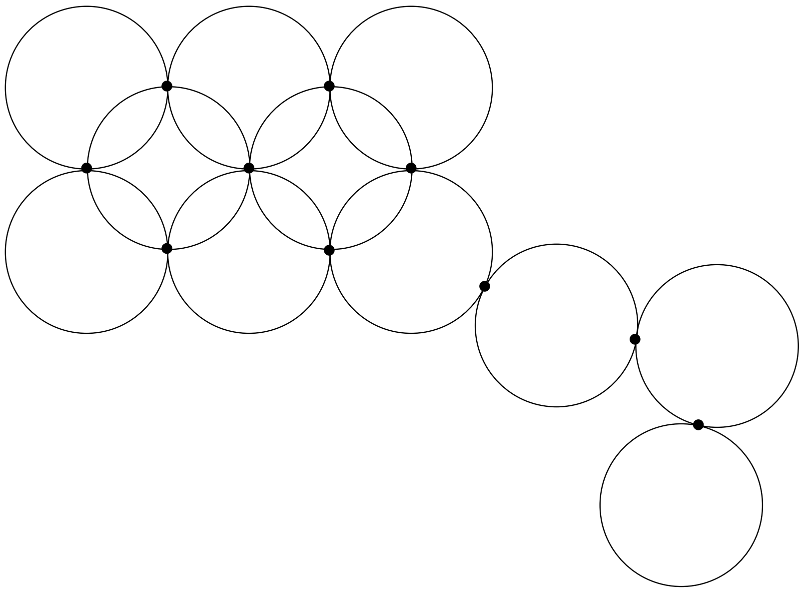

Figure 1: A connected family of $n = 11$ circles that determine just 10 intersection points.

Figure 1 shows one example illustrating that Theorem 3 is tight. For a further discussion of the extremal cases see [2]. It is instructive to note that in Theorem 1 almost no geometric property of unit circles was required. The theorem would hold (and the same proof will go through) for any collection of simple closed curves, as long as no three curves share two points of intersection in common and every two curves are either disjoint or intersect at two points. This is no longer true for Theorem 3. Here we will rely on intimate geometric properties of the unit circle. Figure 2 shows an example of a connected collection of 8 circles with just 6 intersection points. Observe that no three circles share two common intersection points and no three circles touch at the same point. By considering a chain of n such configurations each touches its predecessor at a point, we will obtain a connected collection of $8n$ circles with just $7n - 1$ intersection points.

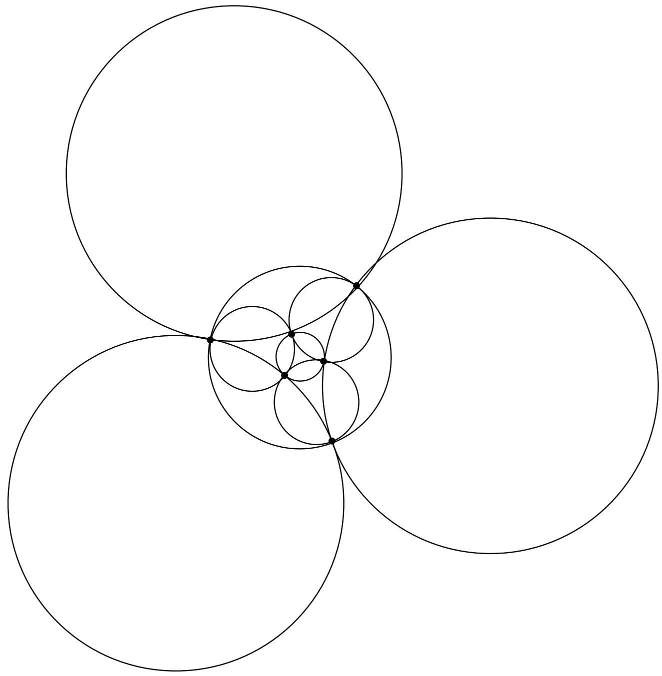

Figure 2: A connected family of $n = 8$ (non unit) circles that determine just 6 intersection points.

2 Definitions and notations

By a "circle", we always mean " a unit circle". We'll refer to points by capital letters and to circles by lower-case letters. For a point P and a circle c, whenever we write $P \in c$ we mean that $P \in \mathcal{P}$ lies on $c \in \mathcal{C}$, if P is not necessarily in $\mathcal{P}$ or c is not necessarily in $\mathcal{C}$, we will mention it explicitly.

The *degree* of a point $P \in \mathcal{P}$, denoted by $\deg P$, is the number of circles from $\mathcal{C}$ that pass through P. Similarly, the *degree* of a circle $c \in \mathcal{C}$, denoted by $\deg c$, is the number of intersection points from $\mathcal{P}$ that lie on C.

Notation 4. *Let $P, Q \in c$.*

- $\overline{P}^c$ *is the point on c that is antipodal to P.*
- $t(P, c)$ *is the circle that touches c at the point P.*

- $\overrightarrow{N}_c P$ *is the point of* $\mathcal{P}$ *that is the first to come after* P *on* c *in the clockwise direction.*
- $\overleftarrow{N}_c P$ *is the point of* $\mathcal{P}$ *that is the first to come after* P *on* c *in the counterclockwise direction. (See Figure 3.)*
- $\mathrm{arc}_c(PQ)$ *is the open arc bounded between* P *and* Q *on* c *in the clockwise direction. We say that* $\mathrm{arc}_c(PQ)$ *is* empty*, if there are no points of* $\mathcal{P}$ *on it.*
- $\mathrm{disk}(c)$ *is the closed disk bounded by* c*.*
- *if* $Q \neq \overline{P}^c$ *and* $Q \neq P$*, then* $\overline{c}(P, Q)$ *denotes the only circle other than* c *that passes through* P *and* Q*.*

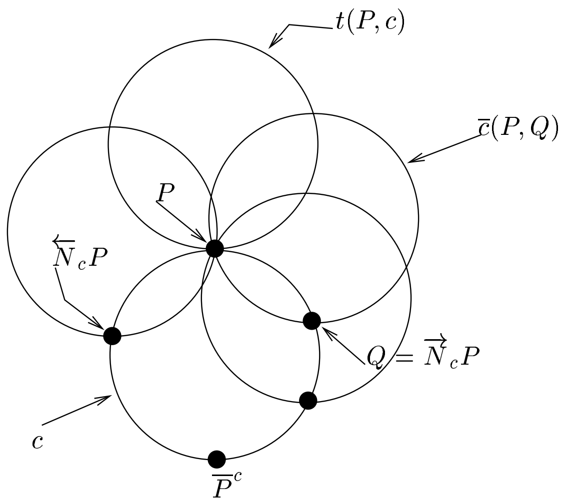

Figure 3: various notations

Definition 5. *Let* $P \in c$

- $Val(P, c) = \frac{1}{deg_P} - \frac{1}{deg_c}$.
- (P, c) *is* weak*, if* $Val(P, c) < 0$*.*
- (P, c) *is* strong*, if* $Val(P, c) > 0$*.*

- *P is* c-connected, *if for every $P \neq Q \in \mathcal{P}$ on c, $\overline{c}(P,Q) \in \mathcal{C}$.*

  (The intuition for the name is that when P is c-connected, there are circular arcs of circles in $\mathcal{C} \setminus \{c\}$ connecting it to each of the other intersection points on c.)

- *P is* semi-c-connected *if for every $Q \in \mathcal{P}$ on c, which is different from P and $\overline{P}^c$, we have $\overline{c}(P,Q) \in \mathcal{C}$.*

Observe that $Val(P,c) < 0$ iff P is c-connected and $t(P,c) \in \mathcal{C}$. In this case, $\deg P = \deg c + 1$.

3 The proof of theorem 3

Theorem 3 is easily seen to be true when $n = 1, 2, 3$. Assume that Theorem 3 is not true and let $\mathcal{C}$ be a counterexample with minimum number n of circles.

Proposition 6. *Every intersection point $P \in \mathcal{P}$ has $\deg P \geq 3$.*

Proof: Assume that through a point P only $c_1, c_2 \in \mathcal{C}$ pass. Let $\mathcal{C}_2 \subset \mathcal{C}$ be the set of circles in the connected component of $\bigcup_{c \in \mathcal{C} \setminus \{c_1\}} c$ which contains c_2. Denote $\mathcal{C}_1 = \mathcal{C} \setminus \mathcal{C}_2$, and observe that $\mathcal{C}_1$ is a connected union of circles.

Let $n_1 = |\mathcal{C}_1|$, and $n_2 = n - n_1 = |\mathcal{C}_2|$. By the minimality of $\mathcal{C}$, the circles of $\mathcal{C}_1$ have at least $n_1 - 1$ intersection points and the circles of $\mathcal{C}_2$ have at least $n_2 - 1$ intersection points. Clearly, all those intersection points are different. Therefore $|\mathcal{P}| \geq n_1 - 1 + n_2 - 1 + 1 = n - 1$, for we count also the point P. ∎

Corollary 7. *The degree of each circle in $\mathcal{C}$ is at least two.*

Step 1 of the proof. Similar to the proof of Theorem 1, we give every intersection point in $\mathcal{P}$ a weight of 1 and we equally distribute the weight of each intersection point among the circles which pass through it. Therefore, for $P \in c$, the weight that P contributes to c is: $w_1(P,c) = \frac{1}{\deg P}$.

The existence of weak couples (P,c) is the reason that we cannot repeat the arguments of Theorem 1, since the total w_1-weight on a circle, may be smaller than 1. But, the point of light here is that while some couples (P,c) are weak, other couples (P',c') are strong. Therefore, we are able to transfer weight from strong couples to weak couples to get on each circle a total weight of at least 1. In particular, we have the following observation.

Observation 8. *If (P, c) is weak, then for every $Q \in c$ ($Q \neq P$), (Q, c') is strong, where $c' = \overline{c}(P, Q)$.*

(note that if (P, c) is weak, then $\overline{P}^c \neq \mathcal{P}$)
Proof: Every circle through Q, except for $t(Q, c')$, intersects c' in a unique point, other than P. It follows that $\deg Q \leq \deg c' + 1$. Note that $t(Q, c') \notin \mathcal{C}$ for otherwise it intersects c in the point $\overline{P}^c$ in contrast to that P is c-connected. Also, note that $t(P, c)$ intersects c' in the point $\overline{Q}^{c'}$. There is no circle, other than c', through Q and $\overline{Q}^{c'}$. Hence $\deg Q \leq \deg c' - 1$. ∎

Definition 9. *Let c be a circle with $\deg c = 2$ and $P, Q \in \mathcal{P}$ are the intersection points on it. If $\deg \overline{c}(P, Q)$ is 4 or 5, then c is called* problematic.

Step 2 of the proof.

In step 2 we define a modified weight $w_2(P, c)$ for every pair (P, c) where $P \in c$, $P \in \mathcal{P}$ and $c \in \mathcal{C}$ with the property that $\sum_{P \in c} w_2(P, c) = \sum_{P \in c} w_1(P, c)$ where the sums are taken over all pairs $P \in \mathcal{P}$ and $c \in \mathcal{C}$ such that $P \in c$. We aim to show that $\sum_{P \in c} w_2(P, c) \geq 1$, for every circle $c \in \mathcal{C}$ which is not problematic.

Definition 10. *Let (P, c) be a weak couple and (Q, e) be a strong couple. We say that (Q, e)* settles *(P, c), if $Val(P, c) + Val(Q, e) \geq 0$.*

Definition 11. *We say that (P, c)* clockwise-dominates *(Q, e) and write $(P, c) \longrightarrow_1 (Q, e)$, if (P, c) is weak, $Q = \overrightarrow{N}_c P$, and $e = \overline{c}(P, Q)$. We then say that (Q, e) is* clockwise-dominated *by (P, c).*

We say that (P, c) counterclockwise-dominates *(Q, e) and write $(P, c) \longrightarrow_2 (Q, e)$, if (P, c) is weak, $Q = \overleftarrow{N}_c P$, and $e = \overline{c}(P, Q)$. We then say that (Q, e) is* counterclockwise-dominated *by (P, c).*

If $(P, c) \longrightarrow_1 (Q, e)$ or $(P, c) \longrightarrow_2 (Q, e)$, we say that (P, c) dominates *(Q, e). We also say that (Q, e) is* dominated *by (P, c). If (Q, e) is not dominated by any weak couple, we say that (Q, e) is* independent. *(See Figure 4.)*

Observation 12. *Every strong couple (Q, e) is dominated by at most two weak couples.*

Definition 13. *Let (P, c) be a weak couple, $h = t(P, c) \in \mathcal{C}$. Suppose that*

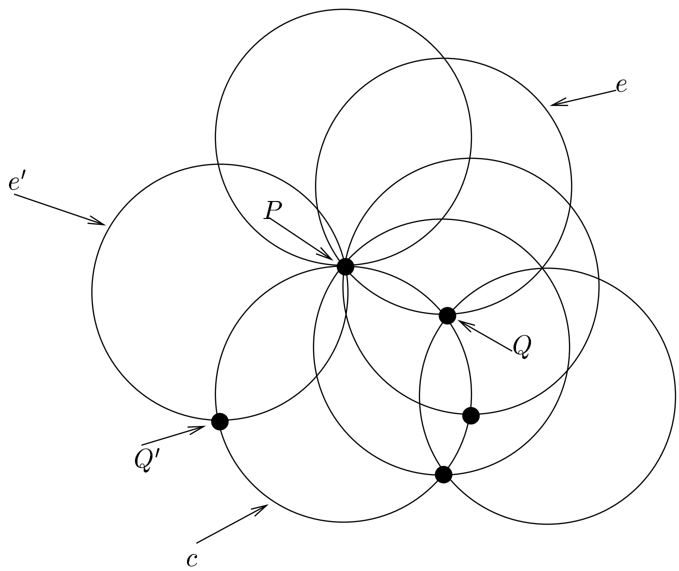

Figure 4: $(P, c) \longrightarrow_1 (Q, e)$ and $(P, c) \longrightarrow_2 (Q', e')$

1. *there exist two circles $d, e \in \mathcal{C}$, that pass through P and intersect in a different point that lies in the interior of* $\text{disk}(h)$.

2. *there exists $f \in \mathcal{C}$ that intersects h in two points, such that P lies in the interior of* $\text{disk}(f)$, *and no other circle in $\mathcal{C}$ but h passes through P and one of the intersection points of f and h.*

Then (P, h) is called the helper *of (P, c). (See Figure 5.)*

Observation 14. *If (P, h) is the helper of (P, c) then:*

1. *(P, h) is independent.*

2. *(P, h) is strong.*

Proof:

1. If (P', c') dominates (P, h) then c' intersects h and c in the points $P', \overline{P'}^{c'}$ respectively. Hence, P' is not c'-connected, a contradiction to the assumption that (P', c') is weak.

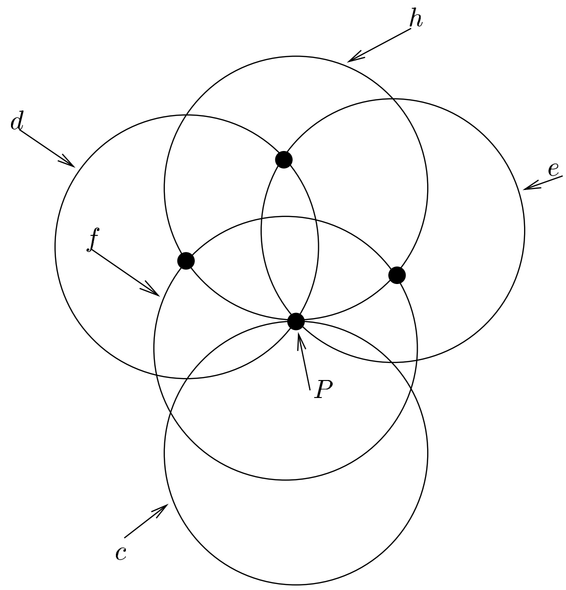

Figure 5: (P, h) is the helper of (P, c)

2. Denote $\deg P = x$. Since the two circles (other than h) that pass through P and the intersection points of f and h are not in $\mathcal{C}$, $\deg h \geq x + 1$. Hence, (P, h) is strong.

∎

A short overview of the proof: The idea of the proof is to find the right way (or just any way that works) to distribute the weight of the intersection points among the circles so that the total weight absorbed by each circle is at least 1. We do find such a way and define the new distribution in Definition 15, where the quantities $w_2(P, c)$ are defined. ¿From the definition it will be very easy to see that $\sum_{P,c} w_2(P, c) = \sum_{P,c} w_1(P, c) = |P|$. It is then a slightly tedious and very technical work to show that the new so called 'weights' satisfy the property that for every given circle c, $\sum_{P|P \in c} w_2(P, c) \geq 1$. Although very tedious, the important thing is that it works! The rest of the proof then follows as in the argument of Bezdek and Connely.

We will now redistribute the weights $w_1(P, c)$ of all couples (P, c), where $P \in c$, between themselves and call the resulting new weights $w_2(P, c)$. This is done according to certain guide-lines (at first we define $w_a(P,c)$, $w_{a'}(P,c)$, $w_b(P,c)$, $w_c(P,c)$, $w_d(P,c)$, $w_{d'}(P,c)$, $w_e(P,c)$ and then we define $w_2(P,c)$):

Definition 15. • **If nobody else wants it- take it all!**

a. *If* $(P,c) \longrightarrow_1 (Q,e)$ *and if* (Q,e) *is dominated exclusively by* (P,c), *then* $w_a(P,c) = Val(Q,e)$, $w_a(Q,e) = -Val(Q,e)$.

a'. *If* $(P,c) \longrightarrow_2 (Q,e)$ *and* $deg\, c \geq 3$, *and if* (Q,e) *is dominated exclusively by* (P,c), *then* $w_{a'}(P,c) = Val(Q,e)$, $w_{a'}(Q,e) = -Val(Q,e)$.

- **b. Two clockwise-dominations equally share the loot.**

  If $(P_1,c_1) \longrightarrow_1 (Q,e)$ *and* $(P_2,c_2) \longrightarrow_1 (Q,e)$ *then* $w_b(P_1,c_1) = \frac{1}{2}Val(Q,e)$, $w_b(P_2,c_2) = \frac{1}{2}Val(Q,e)$ *and* $w_b(Q,e) = -Val(Q,e)$.

- **c. Two counterclockwise-domination equally share the loot.** *If* $(P_1,c_1) \longrightarrow_2 (Q,e)$ *and* $(P_2,c_2) \longrightarrow_2 (Q,e)$, *then* $w_c(P_1,c_1) = \frac{1}{2}Val(Q,e)$, $w_c(P_2,c_2) = \frac{1}{2}Val(Q,e)$ *and* $w_c(Q,e) = -Val(Q,e)$.

- **Clockwise-domination wins counterclockwise-domination.** *If* $(P_1,c_1) \longrightarrow_1 (Q,e)$ *and* $(P_2,c_2) \longrightarrow_2 (Q,e)$, *then let* D *be equal the quantity* $\min\{Val(Q,e), -Val(P_1,c_1)\}$.

  d. *define:* $w_d(P_1,c_1) = D$, *and* $w_d(Q,e) = -Val(Q,e)$.

  d'. *define:* $w_{d'}(P_2,c_2) = Val(Q,e) - D$.

- **e. A special bonus from the helper.**

  If (P,c) *is weak and* (P,h) *is the helper of* (P,c), *then* $w_e(P,c) = Val(P,h)$ *and* $w_e(P,h) = -Val(P,h)$.

- *For every pair* $P \in \mathcal{P}$ *and* $c \in \mathcal{C}$, *where* $P \in c$, *we define*

$$\begin{aligned} w_2(P,c) &:= w_1(P,c) + w_a(P,c) + w_{a'}(P,c) + w_b(P,c) + \\ &+ w_c(P,c) + w_d(P,c) + w_{d'}(P,c) + w_e(P,c). \end{aligned}$$

where any value among $w_a(P,c)$, $w_{a'}(P,c)$, $w_b(P,c)$, $w_c(P,c)$, $w_d(P,c)$, $w_{d'}(P,c)$, *and* $w_e(P,c)$ *is considered to be* 0 *unless it was specified by any of the previous guide-lines.*

Definition 16. *A couple (P,c) is called w_2-satisfied, if $w_2(P,c) \geq \frac{1}{deg\, c}$.*

In Proposition 20, the main proposition in the proof of Theorem 3, we will show that if c is not problematic, then $\sum_{P\in c} w_2(P,c) \geq 1$. To prove Proposition 20 we need the following observation and two lemmata.

Observation 17. *Let (P,c), (Q,e) be couples s.t $(P,c) \longrightarrow_1 (Q,e)$ and (P,c) is not w_2 satisfied. Then $w_a(P,c) + w_b(P,c) + w_d(P,c) \geq \frac{1}{2} \times Val(Q,e)$*

Proof: Suppose that the inequality does not hold. Then there exist (P',c') satisfying $(P',c') \longrightarrow_2 (Q,e)$ and $w_d(P,c) = min\{Val(Q,e), -Val(P,c)\} = -Val(P,c)$. Hence, (P,c) is w_2 satisfied, a contradiction. ∎

Lemma 18. *Let (P,c) be a weak couple, $Q \in c$, $d = \overline{c}(P,Q)$ such that (Q,d) does not settle (P,c). Then: (Q,c) is weak, and either $deg\, d = deg\, c + 2$ and Q is semi-d-connected, or c is problematic, or both.*

Recall that (Q,c) is weak iff Q is c-connected and $t(Q,c) \in \mathcal{C}$

Proof: Let $x = \deg c$. Since (P,c) is weak, $\deg P = x+1$ and $c' = t(P,c) \in \mathcal{C}$. Since P is c connected, then $t(Q,d) \notin \mathcal{C}$ (for otherwise $\overline{P}^c$ is an intersection point on c). Let $y = \deg d$ and $r = \deg Q$. Observe that $r \leq x+1$, and moreover, $y \geq r+1$.

Since (Q,d) does not settle (P,c), we have:

$$\frac{1}{x+1} - \frac{1}{x} + \frac{1}{r} - \frac{1}{y} < 0. \tag{1}$$

Since $y \geq r+1$, it follows that

$$\frac{1}{x+1} - \frac{1}{x} + \frac{1}{y-1} - \frac{1}{y} < 0.$$

Therefore, we must have $y \geq x+2$. Assume that $y \geq x+3$, then this, together with $r \leq x+1$, implies that

$$\frac{1}{x+1} - \frac{1}{x} + \frac{1}{x+1} - \frac{1}{x+3} < 0.$$

¿From this we conclude that $x = 2$.

Therefore, either $y = x+2$ (and then it is easy to see that $r = x+1$), or $x = 2$.

In the former case, since $r = x + 1$ and $\deg c = x$, it follows that $c_1 = t(Q, c) \in \mathcal{C}$, and that Q is c connected. This shows that (Q, c) is weak. The fact that Q is semi-d-connected follows from $\deg Q = r = x + 1 = y - 1 = \deg d - 1$.

In the latter case ($x = 2$), if $y \geq 6$ (keeping in mind that $r \leq x + 1 = 3$) $\frac{1}{x+1} - \frac{1}{x} + \frac{1}{r} - \frac{1}{y} \geq 0$ contradicting (1). Hence, we conclude that if $x = 2$, then either $y = 4$ or $y = 5$ (and thus c is problematic). Here, again $r = x + 1$ and therefore (Q, c) is weak. ∎

Lemma 19. *Let (P, c) be a week couple, $deg\, c \geq 3$. Suppose that $(P, c) \longrightarrow_1 (Q_1, e_1)$ and $(P, c) \longrightarrow_2 (Q_2, e_2)$. Then $w_1(P, c) + Val(Q_1, e_1) + Val(Q_2, e_2) \geq \frac{1}{deg\, c}$*

Proof: If (Q_1, e_1) or (Q_2, e_2) settles (P, c) then we are done. Assume then that none of (Q_1, e_1) and (Q_2, e_2) settles (P, c). Let $x = \deg c$. By Proposition 18, $\deg e_1 = \deg e_2 = x + 2$, and $\deg Q_1 = \deg Q_2 = \deg P = x + 1$.

Therefore,

$$\begin{aligned} w_1(P, c) &= \frac{1}{x+1} \\ Val(Q_1, e_1) &= \frac{1}{x+1} - \frac{1}{x+2} \\ Val(Q_2, e_2) &= \frac{1}{x+1} - \frac{1}{x+2} \end{aligned}$$

Therefore,

$$w_1(P, c) + Val(Q_1, e_1) + Val(Q_2, e_2) = \frac{x+4}{x^2 + 3x + 2}.$$

This expression is greater than or equal to $\frac{1}{x} = \frac{1}{\deg c}$ for every $x \geq 2$. ∎

We now move on to the main proposition:

Proposition 20. *For every couple (P, c) such that c is not problematic, if (P, c) is not w_2-satisfied, then $\sum_{P \in c} w_2(P, c) \geq 1$.*

Note that proposition 20 implies that $\sum_{P \in c} w_2(P, c) \geq 1$ for every circle c which is not problematic.

Proof: Let (P, c) be a couple such that c is not problematic.

If (P, c) is not weak, then by definition 15, (P, c) is w_2-satisfied. So, from now on, we assume that (P, c) is weak.

If all the couples (Q, e) that are dominated by (P, c), are dominated by (P, c) alone, then there are two possibilities:

- (I) (P, c) is dominating a single couple, (Q, e) which is not dominated by any other weak couple. If (Q, e) settles (P, c), then (P, c) is w_2-satisfied and we are done. Otherwise, (Q, e) doesn't settle (P, c). Since (P, c) is dominating a single couple, $\deg c = 2$. By Lemma 18, c is problematic, a contradiction.

- (II) (P, c) is dominating two couples (Q_1, e_1) and (Q_2, e_2), and none of them is not dominated by any other weak couple. Then, by Lemma 19, (P, c) is w_2-satisfied.

Therefore, we may assume that there is additional weak couple, (P_1, c_1) such that both (P, c) and (P_1, c_1) dominate the same couple, say (Q, e).

Then, $e = \overline{c}(P, Q) = \overline{c_1}(P_1, Q)$.

We claim that exactly one of $\text{arc}_c(PQ)$ and $\text{arc}_{c_1}(P_1Q)$ has length greater than π while the other has length smaller than π.

Suppose to the contrary that both $\text{arc}_c(PQ)$ and $\text{arc}_{c_1}(P_1Q)$ have lengths greater than π. Let $Q' = \overline{Q}^e$. Then both points P and P_1 must either lie on $\text{arc}_e(Q'Q)$, or on $\text{arc}_e(QQ')$. Assume without loss of generality that they both lie on $\text{arc}_e(Q'Q)$ and $P_1 \in \text{arc}_e(PQ)$. Then $t(P_1, c_1)$ intersects both $\text{arc}_c(PQ)$ and $\text{arc}_c(QP)$ in contrast to fact that one of them must be empty in order for (P, c) to dominate (Q, e).

We argue similarly if both $\text{arc}_c(PQ)$ and $\text{arc}_{c_1}(P_1Q)$ have lengths smaller than π.

We will assume without loss of generality that the length of $\text{arc}_c(PQ)$ is smaller than π and the length of $\text{arc}_{c_1}(P_1Q)$ is greater than π. At the same time, because of the asymmetry, we will have to prove that if (P, c) is not w_2-satisfied, then $\sum_{P' \in c} w_2(P', c) \geq 1$ and if (P_1, c_1) is not w_2-satisfied, then $\sum_{P' \in c_1} w_2(P', c_1) \geq 1$

We consider two main cases: (1) c and c_1 intersect in two points (one of which is Q); (2) c and c_1 touch at the point Q.

case (1): c and c_1 intersect in two points.

Let Q and R be those points. We first show that $Q = \overrightarrow{N}_c P$. Indeed, assume not, then $Q = \overleftarrow{N}_c P$ and $Q = \overrightarrow{N}_{c_1} P_1$. We conclude that $\mathrm{arc}_c(QP)$ and $\mathrm{arc}_{c_1}(P_1Q)$ are empty. Therefore, $\deg Q = 3$, as any additional circle through Q, other than c, c_1, e, intersects either $\mathrm{arc}_c(QP)$ or $\mathrm{arc}_{c_1}(P_1Q)$ or both. Similarly, $\deg R = 3$, since every circle through R, other than $c, c_1, \overline{c}(P, R)$, intersects either $\mathrm{arc}_c(QP)$ or $\mathrm{arc}_{c_1}(P_1Q)$. Hence, by considering $\mathcal{C} \setminus \{c, c_1\}$, we get a connected collection of $n-2$ circles without Q and R as intersection points (see Figure 6). This is a contradiction to the minimality of $\mathcal{C}$.

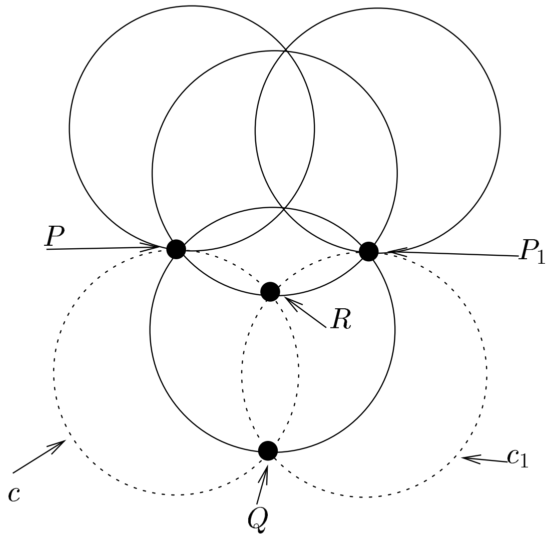

Figure 6: the case where $\mathrm{arc}_c(QP)$ and $\mathrm{arc}_{c_1}(P_1Q)$ are empty

$Q = \overrightarrow{N}_c P$ implies that R lies outside disk(e) and $Q = \overleftarrow{N}_{c_1} P_1$. Hence, $\mathrm{arc}_c(PQ)$ and $\mathrm{arc}_{c_1}(QP_1)$ must be empty.

We claim that both arcs $\mathrm{arc}_{c_1}(RQ)$ and $\mathrm{arc}_c(QR)$ are empty. We show this for $\mathrm{arc}_{c_1}(RQ)$ (the argument for $\mathrm{arc}_c(QR)$ is symmetric). Assume that $\mathrm{arc}_{c_1}(RQ)$ is not empty and let $S \in \mathrm{arc}_{c_1}(RQ)$ be such that $\mathrm{arc}_{c_1}(SQ)$ is empty. As P_1 is c_1-connected, the circle $\overline{c_1}(P_1, S) \in \mathcal{C}$ intersects c at a point S'. Now, since P is c-connected, $\overline{c}(P, S') \in \mathcal{C}$ will intersect c_1 at a point S'' on $\mathrm{arc}_{c_1}(SQ)$. A contradiction. (See Figure 7.)

We will therefore assume that $\mathrm{arc}_c(QR)$ and $\mathrm{arc}_{c_1}(RQ)$ are empty. Let $d = \overline{c}(P, R) = \overline{c_1}(P_1, R)$ and recall that $e = \overline{c}(P, Q) = \overline{c_1}(P_1, Q)$.

There are two sub-cases to be considered, $\deg c_1 > 3$ and $\deg c_1 = 3$:

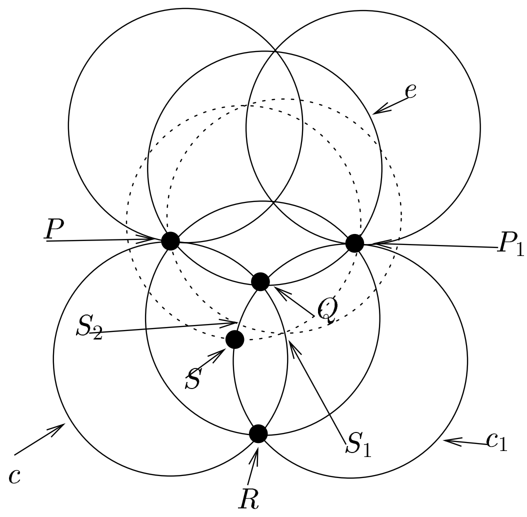

Figure 7: $\text{arc}_{c_1}(RQ)$ and $\text{arc}_c(QR)$ must be empty

Case 1.1. $\deg c_1 > 3$

In this case, let $Q_1 = \overrightarrow{N}_{c_1} P_1$, $e_1 = \overline{c_1}(P_1, Q_1)$.

Let $Q' = \overline{Q}^{c_1}$, $R' = \overline{R}^{c_1}$, and $P_1' = \overline{P_1}^{c_1}$. Since (P_1, c_1) and (P, c) are weak, P_1 is c_1-connected and P is c-connected.

1.1(i) We claim that (P_1, c_1) is w_2-satisfied.

First, we show that (Q_1, e_1) settles (P_1, c_1). Indeed, otherwise, by Lemma 18, Q_1 is c_1 connected. We will now obtain a contradiction by considering all possible locations of Q_1 on c_1.

- Q_1 cannot be equal to R' or Q' or P_1', because otherwise it is not c_1-connected.
- Q_1 cannot be equal to R, as $\deg c_1 > 3$.
- Q_1 cannot belong to $\text{arc}_{c_1}(P_1 Q') \setminus \{R'\}$, for otherwise, since it is c_1-connected, $\overline{c_1}(Q, Q_1) \in \mathcal{C}$. But $\overline{c_1}(Q, Q_1)$ intersects $\text{arc}_c(PR)$, which must be empty.
- Q_1 can not belong to $\text{arc}_{c_1}(Q' P_1')$. Since otherwise, $\overline{c_1}(Q, Q_1)$ intersects $\text{arc}_c(R\overline{P}^c)$ in a point, L, and then $\overline{c}(P, L)$ intersects $\text{arc}_{c_1}(QP_1)$, which is empty.

- $Q_1 \notin \text{arc}_{c_1}(P'_1R)$. Since otherwise $\overline{c_1}(P_1, Q_1)$ would intersect $\text{arc}_c(PQ)$, which is empty.

We conclude that (Q_1, e_1) settles (P_1, c_1).

Since $(P_1, c_1) \longrightarrow_1 (Q_1, e_1)$, if (Q_1, e_1) is exclusively dominated by (P_1, c_1), or if it is counterclockwise dominated by additional weak couple, then (P_1, c_1) is w_2-satisfied. We show here that even when there exists additional weak couple (P', c') s.t $(P', c') \longrightarrow_1 (Q_1, e_1)$, we still have that (P_1, c_1) is w_2-satisfied.

Let (P', c') be a weak couple different from (P, c), s.t $(P', c') \longrightarrow_1 (Q_1, e_1)$ (see Figure 8). $(P', c') \longrightarrow_1 (Q_1, e_1)$ implies that $c' = t(Q_1, c_1)$ and also that P' is the intersection point of e_1 and c'.

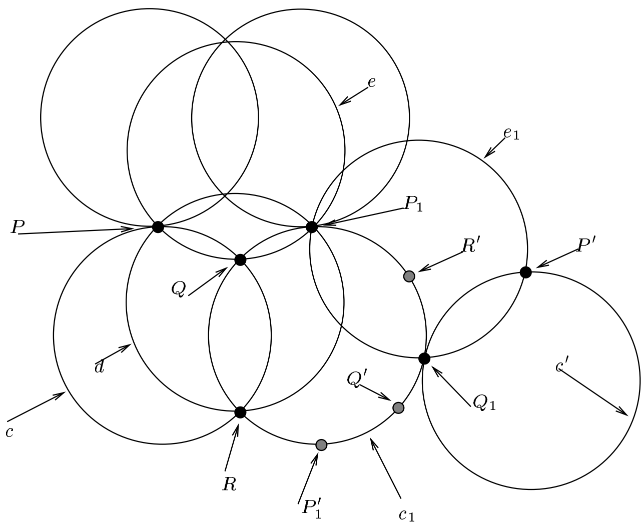

Figure 8: the case where $\deg c_1 > 3$ and $(P', c') \longrightarrow_1 (Q_1, e_1)$

Our purpose is to show that $\deg Q_1 = 3$. If we can show this our argument will continue as follows: $\deg e_1 \geq 5$ because c', c_1, and $t(P_1, c_1)$ intersect e_1 in four points, while e or d intersect e_1 in at least one more different point.

Hence, $\deg Q_1 = 3$ implies that $Val(Q_1, e_1) \geq \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$ and since $\deg c_1 \geq 4$ (which implies $|Val(P_1, c_1)| \leq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$), we have:

$$w_b(P_1, c_1) = \frac{1}{2} \times Val(Q_1, e_1) \geq \frac{1}{15} \geq |Val(P_1, c_1)|$$

This implies that (P_1, c_1) is w_2-satisfied.

We will now show that $\deg Q_1 = 3$. Note that $Q_1 \in \text{arc}_{c_1}(P_1 P_1')$ (for otherwise $\overline{c_1}(P_1, Q_1)$ intersects $\text{arc}_c(PQ)$). Moreover, $(P', c') \longrightarrow_1 (Q_1, e_1)$ implies that $\deg c_1 = 4$, because for every $X \in \text{arc}_{c_1}(Q_1 P_1')$, $\overline{c_1}(P_1, X)$ intersects $\text{arc}_{c'}(P'Q_1)$, which is empty.

If $\deg Q_1 > 3$, then $\overline{c_1}(Q_1, Q) \in \mathcal{C}$ or $\overline{c_1}(Q_1, R) \in \mathcal{C}$.

Suppose first that $\overline{c_1}(Q_1, Q) \in \mathcal{C}$. Note that the intersection point of $\overline{c_1}(Q_1, Q)$ and d which lies inside disk(c_1), is of degree 2. This is because any third circle passing through it intersects c_1 at a point different from Q_1, P_1, Q, and R. This is a contradiction to $\deg c_1 = 4$.

Suppose that $\overline{c_1}(Q_1, R) \in \mathcal{C}$. Then, $\overline{c_1}(Q_1, R)$ intersects $\text{arc}_{c'}(P'Q_1)$ which is empty.

We conclude that $\deg Q_1 = 3$, hence (P_1, c_1) is w_2-satisfied. We now move on to take care of (P, c).

1.1(ii) We claim that if (P, c) is not w_2-satisfied, then $\sum_{P \in c} w_2(P, c) \geq 1$. Suppose that (P, c) is not w_2-satisfied. Since $(P, c) \longrightarrow_1 (Q, e)$ and $(P_1, c_1) \longrightarrow_2 (Q, e)$, then it follows from Definition 15 part (d) that if (Q, e) settles (P, c), then (P, c) is w_2-satisfied.

Hence, we assume that (Q, e) doesn't settle (P, c). By Lemma 18, Q is c-connected.

We claim that $\deg c$ is 3 or 4 and if $\deg c = 4$ then S,the forth point on c (other than P,Q, and R) is $R'' = \overline{R}^c$.

Indeed, keeping in mind that Q is c-connected, this follows by considering the possible locations of an intersection point S on c.

- S cannot belong to $\text{arc}_c(PR)$ since it is empty.
- S cannot belong to the arc $\text{arc}_c(RP'')$ since otherwise, $\overline{c}(P, S)$ would intersect $\text{arc}_{c_1}(QP_1)$, which is empty.
- S cannot belong to $\text{arc}_c(P''Q'')$ since otherwise, $\overline{c}(Q, S) \in \mathcal{C}$ as Q is c-connected. $\overline{c}(Q, S)$ intersects c_1 at a point L on $\text{arc}_c(P_1'R)$, but then $\overline{c_1}(P_1, L)$ intersects $\text{arc}_c(PQ)$ which is empty.
- S cannot belong to $\text{arc}_c(Q''P) \setminus \{R''\}$ since otherwise $\overline{c}(Q, S)$ intersects $\text{arc}_{c_1}(RP_1)$, which is empty.

Let $Q_2 = \overleftarrow{N}_c P$ and $e_2 = \overline{c}(P, Q_2)$. Then $(P, c) \longrightarrow_2 (Q_2, e_2)$.

If (Q_2, e_2) is dominated exclusively by (P, c), then by Lemma 19 and Definition 15, parts (a) and (d), (P, c) is w_2-satisfied.

So assume that there exists additional couple, (P_3, c_3), that also dominates (Q_2, e_2).

Let $P'' = \overline{P}^c$, $Q'' = \overline{Q}^c$, $R'' = \overline{R}^c$. We know that $Q_2 = R''$ or $Q_2 = R$.

1.1(ii) Sub-case a. $Q_2 = R''$. Then $\deg c = 4$, $w_1(P, c) = \frac{1}{5}$. We show that (P, c) is w_2-satisfied by showing that $w_c(P, c) + w_d(P, c) \geq \frac{1}{20}$ (hence, $w_2(P, c) \geq \frac{1}{5} + \frac{1}{20} = \frac{1}{4}$).

Estimation of $w_d(P, c)$. Recall that $(P, c) \longrightarrow_1 (Q, e)$, and observe that since $\deg c = 4$, we have $\deg Q = 5$. (See Figure 9.) Hence, $Val(Q, e) \geq \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$. By Definition 15 part (d), $w_d(P, c) \geq \frac{1}{30}$.

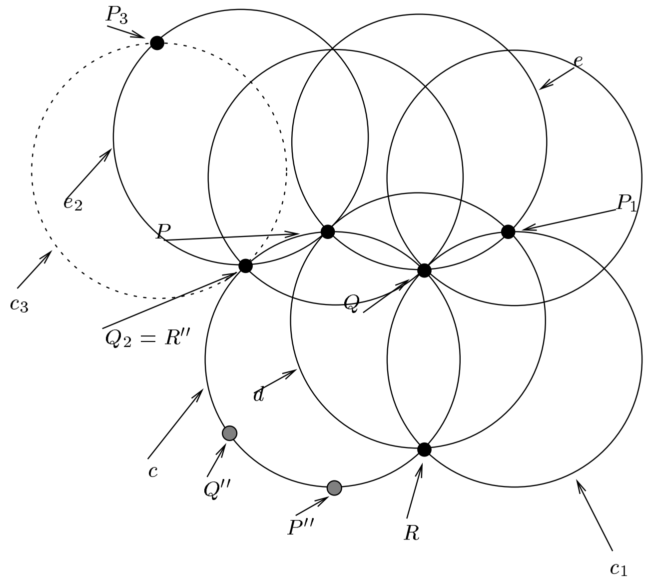

Figure 9: the case where $\deg c_1 > 3$ and $Q_2 = R''$

Estimation of $w_c(P, c)$. Recall that $(P, c) \longrightarrow_2 (R'', e_2)$. Since R is on c , we have $\deg R'' \leq 4$. Hence, $Val(R'', e_2) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. Recall that (P_3, c_3) also dominates (R'', e_2). We show here that $(P_3, c_3) \longrightarrow_2 (R'', e_2)$ and hence, by Definition 15 part (c), $w_c(P, c) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$. Indeed $(P_3, c_3) \longrightarrow_2 (R'', e_2)$, since otherwise $(P_3, c_3) \longrightarrow_1 (R'', e_2)$ and so $\text{arc}_{c_3}(P_3 R'')$ is empty. But, c_3 touches c (otherwise, there is a fifth point on c) and so, $\overline{c}(Q, R'')$ intersects $\text{arc}_{c_3}(P_3 R'')$, a contradiction.

We get: $w_c(P, c) + w_d(P, c) \geq \frac{1}{30} + \frac{1}{40} = \frac{7}{120} > \frac{1}{20}$.

1.1(ii) Sub-case b. $Q_2 = R$.

In this case $e_2 = d$, $\deg c = 3$, and $c_3 = t(R, c)$. Observe that $(P_3, c_3) \longrightarrow_2 (R, d)$. (See Figure 10.)

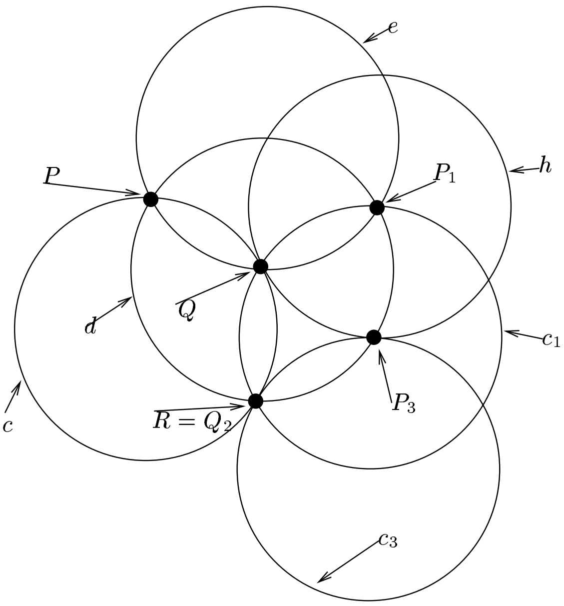

Figure 10: the case where $\deg c_1 > 3$ and $Q_2 = R$

We will show that $\sum_{P' \in c} w_2(P', c) \geq 1$. We first consider $w_2(P, c)$.

Estimation of $w_c(P, c)$. We know that $(P, c) \longrightarrow_2 (R, d))$. Moreover, $\deg R = 4$, which implies $Val(R, d) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. Since $(P_3, c_3) \longrightarrow_2 (R, d)$, by Definition 15 part (c), $w_c(P, c) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$.

Estimation of $w_d(P, c)$. We know that $(P, c) \longrightarrow_1 (Q, e)$. $\deg Q \leq 4$, which implies $Val(Q, e) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. Since $(P_1, c_1) \longrightarrow_2 (Q, e)$, by Definition 15 part (d), $w_d(P, c) \geq \frac{1}{20}$.

We can now find an estimate for $w_2(P, c)$ (recall that $w_1(P, c) = \frac{1}{4}$), $w_2(P, c) \geq \frac{1}{4} + \frac{1}{40} + \frac{1}{20} = \frac{39}{120}$.

Next we estimate $w_2(Q, c)$.

Estimation of $w_e(Q, c)$ Denote $h = t(P_3, c_3)$. Since h cannot intersect $\text{arc}_c(PR)$ or $\text{arc}_{c_1}(RP_1)$, we have $h = t(Q, c)$. Since $\deg c = 3$ it is easy to check that (Q, h) is strong and moreover, it is the helper of (Q, c). Note that $\deg Q = 4$. Also, note that $\deg h \geq 6$ because the two touching circles c_1 and $t(P_1, c_1)$ intersect h in 4 points while the circles d and e intersect h in additional 2 points. Hence, $Val(Q,h) \geq \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$ and by Definition 15 part (e), $w_e(Q,c) = Val(Q,h) \geq \frac{1}{12}$.

Estimation of $w_a(Q,c) + w_b(Q,c) + w_d(Q,c)$. We know that $(Q,c) \longrightarrow_1 (R,d)$. Moreover, $\deg R = 4$ and hence $Val(R,d) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. By Definition 15 parts (a), (b), and (d), $w_a(Q,c) + w_b(Q,c) + w_d(Q,c) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$.

Estimation of $w_{a'}(Q,c) + w_c(Q,c)$. We know that $(Q,c) \longrightarrow_2 (P,e)$. We have $\deg P = 4$ and hence $Val(P,e) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. If additional weak couple (P',c') dominates (P,e), then since $\deg c = 3$, we have $c' = t(P,c)$. Circle d intersects $\text{arc}_{c'}(P'P)$ and hence, $(P',c') \longrightarrow_2 (P,e)$. By Definition 15 parts (a) and (c), $w_{a'}(Q,c) + w_c(Q,c) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$.

We can now estimate $w_2(Q,c)$. Since $w_1(Q,c) = \frac{1}{4}$, we have $w_2(Q,c) \geq \frac{1}{4} + \frac{1}{40} + \frac{1}{40} + \frac{1}{12} = \frac{46}{120}$.

Finally, we estimate $w_2(R,c)$.

Estimation of $w_a(R,c) + w_b(R,c) + w_d(R,c)$. We know that $(R,c) \longrightarrow_1 (P,d)$. Now, $\deg P = 4$ and hence $Val(P,d) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. By Definition 15 parts (a), (b), and (d), $w_a(R,c) + w_b(R,c) + w_d(R,c) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$.

Estimation of $w_{a'}(R,c) + w_{d'}(R,c)$. We know that $(R,c) \longrightarrow_2 (Q,c_1)$. Moreover, $\deg Q = 4$ and hence $Val(Q,c_1) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. If additional weak couple (P',c') dominates (Q,c_1), then since $\text{arc}_{c_1}(QP_1)$ is empty, we have $P' = P_1$ and $c' = e$. We show here that $-\frac{1}{30} \leq Val(P_1,e) < 0$, and then by Definition 15 part (d'), $w_{a'}(Q,c) + w_{d'}(Q,c) \geq \frac{1}{20} - \frac{1}{30} = \frac{1}{60}$. Indeed, $\deg e \geq 5$ because the circles c and c_1 intersect e in three points, while h and $t(P_1,c_1)$ intersect e in additional two points. Hence, $-\frac{1}{30} = \frac{1}{5} - \frac{1}{6} \leq Val(P_1,e) < 0$.

We can now estimate $w_2(R,c)$: Since $w_1(R,c) = \frac{1}{4}$, we have $w_2(R,c) \geq \frac{1}{4} + \frac{1}{40} + \frac{1}{60} = \frac{35}{120}$.

We conclude that $\sum_{P' \in c} w_2(P',c) = \frac{35}{120} + \frac{46}{120} + \frac{39}{120} = 1$.

<u>**Case 1.2**</u>. $\deg c_1 = 3$.

In this case $\overrightarrow{N}_{c_1} P_1 = R$. Recall that $d = \overline{c_1}(P_1,R) = \overline{c}(P,R)$

<u>**1.2(i)**</u> We claim that (P,c) is w_2-satisfied. Indeed, observe that (Q,e) settles (P,c), since otherwise, by Proposition 18, $t(Q,c) \in \mathcal{C}$ (see Figure 11). But, $t(Q,c)$ intersects c_1 at a point different from P_1, Q, and R, contradicting the assumption that $\deg c_1 = 3$.

Since $(P,c) \longrightarrow_1 (Q,e)$, $(P_1,c_1) \longrightarrow_2 (Q,e)$, and (Q,e) settles (P,c), we conclude that $w_d(P,c) = |Val(P,c)|$. Hence, (P,c) is w_2-satisfied.

<u>**1.2(ii)**</u> We claim that if (P_1,c_1) is not w_2-satisfied, then $\sum_{P' \in c_1} w_2(P',c_1) \geq 1$.

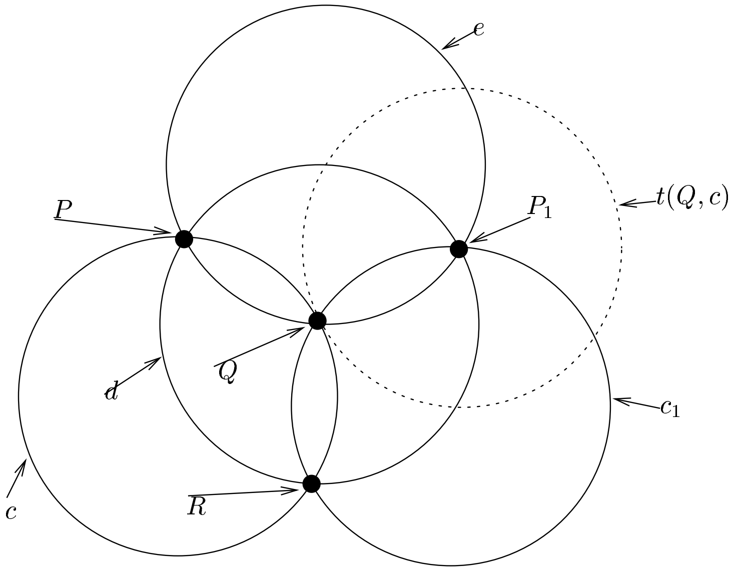

Figure 11: (P, c) is w_2-satisfied when $\deg c_1 = 3$

We consider two sub-cases: $\deg Q = 3$, $\deg Q = 4$.

1.2(ii) Sub-case a. $\deg Q = 3$.

In this case (see Figure 12) we show here that (P_1, c_1) is w_2-satisfied.

Estimation of $w_a(P_1, c_1)$. Recall that $(P_1, c_1) \longrightarrow_1 (R, d)$. $\deg c_1 = 3$ implies that $\deg R \leq 4$ and hence, $Val(R, d) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. Note that there is no weak couple (P_2, c_2), other than (P_1, c_1), that clockwise-dominates (R, d). Indeed, suppose to the contrary that (P_2, c_2) dominates (R, d). (P_2, c_2) is different from (P, c) because if (P, c) dominates (R, d), then $(P, c) \longrightarrow_2 (R, d)$. Since $\deg c_1 = 3$, we have $c_2 = t(R, c_1)$, and P_2 is the intersection point of c_2 and d different from R. Since (P_2, c_2) is weak, then $g = t(P_2, c_2) \in \mathcal{C}$. Observe that $P \in \text{disk}(g)$ (because the length of $\text{arc}_d(RP)$ is less than π). This implies that $g = t(Q, c_1)$, since otherwise g would intersect one of the empty arcs $\text{arc}_c(PQ)$ and $\text{arc}_c(QR)$). $t(Q, c_1) \in \mathcal{C}$ is a contradiction to $\deg Q = 3$.

We conclude that $w_a(P_1, c_1) + w_d(P_1, c_1) \geq \frac{1}{20}$.

Estimation of $w_{d'}(P_1, c_1)$. Recall that $(P_1, c_1) \longrightarrow_2 (Q, e)$. $\deg Q = 3$ implies that $Val(Q, e) \geq \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

If (P_1, c_1) is w_2-satisfied, then we are done. Otherwise, (P_1, c_1) is not w_2-satisfied, which implies that $c_2 = t(R, c_1) \in \mathcal{C}$. The reason for this, is that if $c_2 \notin \mathcal{C}$, then (P_1, c_1) dominates (R, d) exclusively or $(P, c) \longrightarrow_2 (R, d)$.

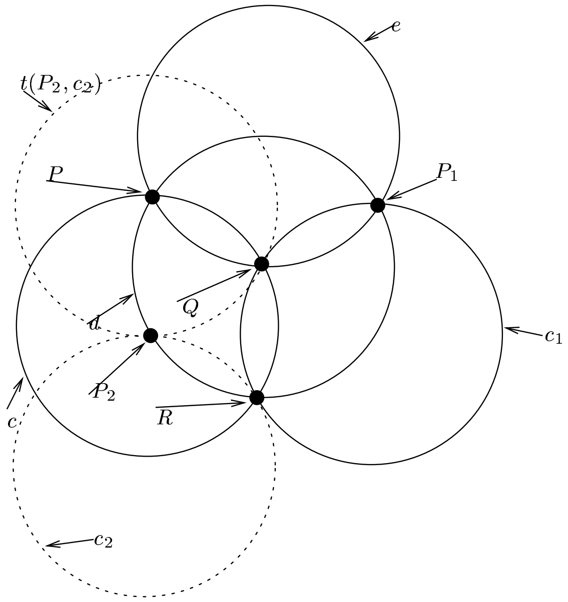

Figure 12: $\deg c_1 = 3$ and $\deg Q = 3$

Also, $c_2 \notin \mathcal{C}$ implies that $\deg R = 3$ and hence $Val(R, d) \geq \frac{1}{12}$. Hence, $w_a(P_1, c_1) + w_d(P_1, c_1) \geq \frac{1}{12}$ and so, (P_1, c_1) is w_2-satisfied.

$c_2 = t(R, c_1) \in \mathcal{C}$ implies that $\deg c \geq 4$. Hence, $|Val(P, c)| \leq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. Thus, we have:

$w_{d'}(P_1, c_1) \geq \frac{1}{12} - \frac{1}{20} = \frac{1}{30}$.

We can now estimate $w_2(P_1, c_1)$: $w_2(P_1, c_1) \geq \frac{1}{4} + \frac{1}{20} + \frac{1}{30} = \frac{1}{3}$, namely (P_1, c_1) is w_2-satisfied.

1.2(ii) Sub-case b. $\deg Q = 4$. Here we show that if (P_1, c_1) is not w_2-satisfied, then $\sum_{P' \in c_1} w_2(P', c_1) \geq 1$.

As in sub-case a, $c_2 = t(R, c_1) \in \mathcal{C}$. Let P_2 be the intersection point of c_2 and d other than R. Let P_3 be the intersection point of c_2 and c other than R. Let $h = t(Q, c_1)$ and $e = \overline{c_1}(P_1, Q) = \overline{c}(P, Q)$. (See Figure 13.)

Since $\deg Q = 4$, $h \in \mathcal{C}$. Observe also that $P_3 = \overline{Q}^c$, which implies that h intersects c at a point different from P_3. Hence, $\deg c \geq 5$.

We will show that $\sum_{P' \in c_1} w_2(P', c_1) \geq 1$. To this end, we first estimate $w_2(P_1, c_1)$.

Estimation of $w_a(P_1, c_1) + w_b(P_1, c_1)$. Recall that $(P_1, c_1) \longrightarrow_1 (R, d)$.

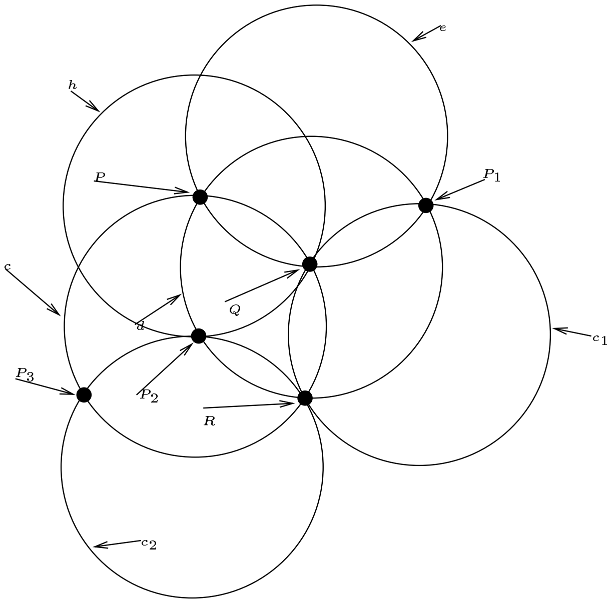

Figure 13: $\deg c_1 = 3$ and $\deg Q = 4$

Now $\deg R = 4$, which implies $Val(R, d) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. By Definition 15, $w_b(P_1, c_1) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$.

Estimation of $w_{d'}(P_1, c_1)$. Recall that $(P_1, c_1) \longrightarrow_2 (Q, e))$. Now $\deg Q \leq 4$, which implies $Val(Q, e) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. Note that $(P, c) \longrightarrow_1 (Q, e)$ and since $\deg c \geq 5$, $|Val(P, c)| \leq \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$. By Definition 15, $w_{d'}(P, c) \geq \frac{1}{20} - \frac{1}{30} = \frac{1}{60}$.

We can now estimate $w_2(P_1, c_1)$. Since $w_1(P_1, c_1) = \frac{1}{4}$, $w_2(P_1, c_1) \geq \frac{1}{4} + \frac{1}{40} + \frac{1}{60} = \frac{35}{120}$.

Next, we estimate $w_2(Q, c_1)$.

Estimation of $w_e(Q, c_1)$. Recall that $h = t(Q, c_1)$. Since $\deg c_1 = 3$, it is easy to check that (Q, h) is strong and moreover, it is the helper of (Q, c). Note that $\deg h \geq 6$ because the two touching circles c and $t(P, c)$ intersect h in four points while the circles d and e intersect h at two additional different points. Hence, $Val(Q, h) \geq \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$ and by Definition 15, $w_e(Q, c) = Val(Q, h) \geq \frac{1}{12}$.

Estimation of $w_{a'}(Q, c_1) + w_c(Q, c_1)$. Recall that $(Q, c_1) \longrightarrow_2 (R, c)$. $\deg R = 4$ and hence $Val(R, d) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. Since $\deg c_1 = 3$, if additional weak couple (P', c') dominates (R, c), then $c' = c_2$, $P' = P_3$ and $(P_3, c_2) \longrightarrow_2 (R, c)$. By Definition 15, $w_{a'}(Q, c_1) + w_c(Q, c_1) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$.

Estimation of $w_a(Q, c_1) + w_b(Q, c_1)$. $(Q, c_1) \longrightarrow_1 (P_1, e)$ and $\deg P_1 = 4$, hence $Val(P_1, e) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. By Definition 15, $w_a(Q, c_1) + w_b(Q, c_1) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$.

We can now estimate $w_2(Q, c_1)$. Since $w_1(Q, c) = \frac{1}{4}$, we have: $w_2(Q, c_1) \geq \frac{1}{4} + \frac{1}{40} + \frac{1}{40} + \frac{1}{12} = \frac{46}{120}$.

Finally we estimate $w_2(R, c_1)$.

Estimation of $w_{a'}(R, c_1) + w_c(R, c_1)$. $(R, c_1) \longrightarrow_2 (P_1, d)$ and $\deg P_1 = 4$, hence $Val(P, d) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$.

Suppose that additional weak couple (P', c') dominates (P_1, d). Then, as $\deg c_1 = 3$, we have $c' = t(P_1, c_1)$, and P' is the intersection point of d and c' which is not P_1. Moreover $(P', c') \longrightarrow_2 (P_1, d)$. By Definition 15, $w_{a'}(R, c_1) + w_c(R, c_1) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$.

Estimation of $w_a(R, c_1) + w_d(R, c_1)$. $(R, c_1) \longrightarrow_1 (Q, c)$ and $\deg Q = 4$, hence $Val(Q, c) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$.

Suppose that additional weak couple (P', c') dominates (Q, c). Then, as $\deg c_1 = 3$, we have $c' = e$, $P' = P$, and $(P, e) \longrightarrow_2 (Q, c)$.

By Definition 15, $w_a(R, c_1) + w_d(R, c_1) \geq \frac{1}{20}$.

We can now estimate $w_2(R, c_1)$. Since $w_1(R, c_1) = \frac{1}{4}$, we have: $w_2(R, c_1) \geq \frac{1}{4} + \frac{1}{40} + \frac{1}{20} = \frac{39}{120}$.

We conclude that $\sum_{P' \in c_1} w_2(P', c_1) = \frac{35}{120} + \frac{46}{120} + \frac{39}{120} = 1$.

Case (2). We now move to analyze the case where c and c_1 touch each other at the point Q. Observe that we may assume that $\deg c, \deg c_1 \geq 3$. Indeed, assume for instance that $\deg c = 2$. In this case $\deg P = \deg Q = 3$. Observe that $\mathcal{C} \setminus \{c, e\}$ is a connected collection of circles that contradicts the minimality of $\mathcal{C}$. The case where $\deg c_1 = 2$ is symmetric.

Next, we will distinguish four cases, namely, where either $(P, c) \longrightarrow_1 (Q, e)$ or $(P, c) \longrightarrow_2 (Q, e)$ and either $(P_1, c_1) \longrightarrow_1 (Q, e)$ or $(P_1, c_1) \longrightarrow_2 (Q, e)$.

Case 2.1: $(P, c) \longrightarrow_1 (Q, e)$ and $(P_1, c_1) \longrightarrow_1 (Q, e)$.

Let $Q_1 = \overleftarrow{N}_{c_1} P_1$ and $e_1 = \overline{c_1}(P_1, Q_1)$. Then $(P_1, c_1) \longrightarrow_2 (Q_1, e_1)$.

Let $Q_2 = \overleftarrow{N}_c P$ and $e_2 = \overline{c}(P, Q_2)$. Then $(P, c) \longrightarrow_2 (Q_2, e_2)$.

First, we prove that $\deg Q \leq 4$. Otherwise, let f be a circle that passes through Q, which is different from c, c_1, and e. Let F be the intersection point of f and c which is not Q (see Figure 14). Observe that $F \in \mathrm{arc}_c(Q''P)$, where $Q'' = \overline{Q}^c$ (otherwise, f intersects $\mathrm{arc}_{c_1}(P_1 Q)$ which is empty).

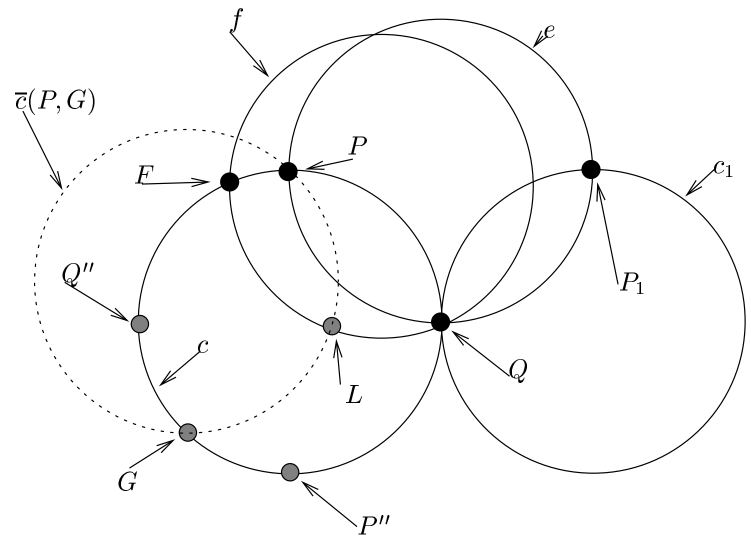

Figure 14: case 2.1: $\deg Q \leq 4$

Observe that $\text{arc}_c(QF)$ is empty. Indeed, suppose $G \in \text{arc}_c(QF)$ s.t. $\text{arc}_c(QG)$ is empty. Observe that G must lie on $\text{arc}_c(P''P)$, where $P'' = \overline{c}^P$, since otherwise $\overline{c}(P,G)$ intersects either $\text{arc}_c(PQ)$ or $\text{arc}_{c_1}(P_1Q)$ which are empty. Now, $\overline{c}(P,G)$ intersects f in a point L inside $\text{disk}(c)$. $\deg L = 2$, for any third circle through L intersects $\text{arc}_c(PQ)$ or $\text{arc}_c(QG)$. Since any point in $\mathcal{P}$ has a degree of at least three, this is a contradiction.

Since f is an arbitrary circle that passes through Q, which is not c, c_1, or e, our claim that $\text{arc}_c(QF)$ is empty implies that $\deg Q \leq 4$.

In what follows we show that (i): (P,c) is w_2-satisfied, and (ii): if (P_1, c_1) is not w_2-satisfied, then $\sum_{P \in c_1} w_2(P, c_1) \geq 1$.

2.1(i). We claim that (P,c) is w_2-satisfied.

In order to prove this, we need to consider two sub-cases: $\deg Q = 4$ and $\deg Q = 3$.

2.1(i) Sub-case a. $\deg Q = 4$.

Let f and F be as defined above: f is the circle that passes through Q, and is different from c, c_1, e and F is the intersection point of f and c which is not Q (see Figure 15).

Our claim that $\text{arc}_c(QF)$ is empty implies that $h = t(P,c)$ is the helper of (P,c). The reason for this is that $P \in \text{disk}(f)$ and f intersects h in two points, L and M s.t. there is no circle of $\mathcal{C}$ passing through P and one of

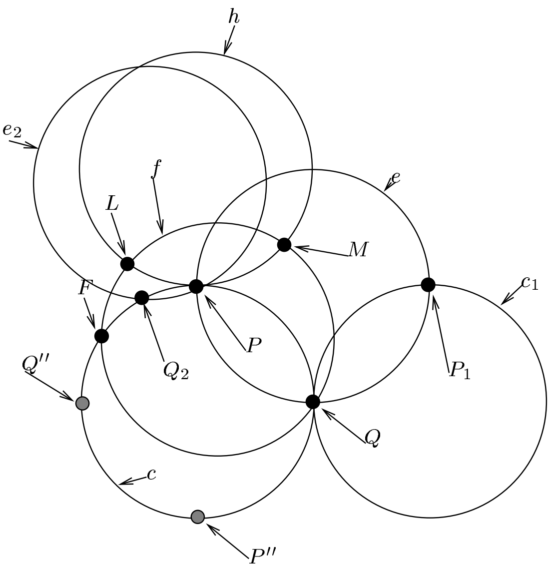

Figure 15: case 2.1 (i) a: $\deg Q = 4$

those points (since otherwise this circle would intersects $\mathrm{arc}_c(QF)$).

Suppose that $\deg P = x$. $x \geq 4$ because $\deg c \geq 3$. Also, $|Val(P,c)| = \frac{1}{x-1} - \frac{1}{x}$.

Here we show that (P,c) is w_2-satisfied:

Estimation of $w_b(P,c)$: Recall that $(P,c) \longrightarrow_1 (Q,e)$. Observe that $\deg e \geq x+1$. There are two reasons for this. One is that $P_1 = \overline{P}^e$ and the other is that $t(P,e) \notin \mathcal{C}$ (because $t(P,e)$ intersect c in the point Q'' and since $F \in \mathrm{arc}_c(Q''P)$, $Q'' \in \mathrm{arc}_c(QF)$, which is is empty).

Since $\deg Q = 4$, $Val(Q,e) \geq \frac{1}{4} - \frac{1}{x+1}$. Recall that $(P_1,c_1) \longrightarrow_1 (Q,e)$, hence, $w_b(P,c) = \frac{1}{2} \times Val(Q,e) \geq \frac{1}{2} \times (\frac{1}{4} - \frac{1}{x+1})$

Estimation of $w_e(P,c)$: Since no circle of $\mathcal{C}$ passes through P and one of the two intersection points of f and h, we have $\deg h \geq x+1$. Hence, we have: $w_e(P,c) = Val(P,h) \geq \frac{1}{x} - \frac{1}{x+1}$.

Estimation of $w_{a'}(P,c) + w_c(P,c)$: Recall that $(P,c) \longrightarrow_2 (Q_2,e_2)$. Since $\deg P = x$, $\deg Q_2 \leq x$ and hence, $Val(Q_2,e) \geq \frac{1}{x} - \frac{1}{x+1}$.

We claim that $w_{a'}(P,c) + w_c(P,c) \geq \frac{1}{2} \times Val(Q_2,e_2) \geq \frac{1}{2} \times (\frac{1}{x} - \frac{1}{x+1})$.

It is enough to show that if (P_2,c_2) is a weak couple, different from (P,c), that dominates (Q_2,e_2), then $(P_2,c_2) \longrightarrow_2 (Q_2,e_2)$.

Indeed, suppose that (P_2,c_2) is a weak couple, different from (P,c), s.t

$(P_2, c_2) \longrightarrow_1 (Q_2, e_2)$. Then c_2 intersects $\mathrm{arc}_c(QF)$ which is empty. To observe that indeed c_2 intersects $\mathrm{arc}_c(QF)$, note that the circles e_2 and c intersect in two points that lie in the interior of disk(f). This implies that F lies in the interior of disk$(\overline{e_2}(Q_2, T))$, where T is the intersection point of e_2 and f, that lies outside disk(h).

Hence, $(P_2, c_2) \longrightarrow_2 (Q_2, e_2)$.

Estimation of $w_2(P, c)$:

$$\begin{aligned} w_2(P,c) &\geq w_b(P,c) + w_e(P,c) + w_{a'}(P,c) + w_c(P,c) \geq \\ &\geq \frac{1}{2} \times (\frac{1}{4} - \frac{1}{x+1}) + \frac{1}{x} - \frac{1}{x+1} + \frac{1}{2} \times (\frac{1}{x} - \frac{1}{x+1}) \end{aligned}$$

The right hand side is greater than $\frac{1}{x-1} - \frac{1}{x}$, for every $x \geq 4$.

Hence, $w_2(P, c) > \frac{1}{x-1} = \frac{1}{\deg c}$, which implies that (P, c) is w_2-satisfied.

2.1(i) Sub-case b. $\deg Q = 3$.

Assume first that $\deg P = 4$, then $\deg c = 3$. Since $\deg Q = 3$, and c, c_1, and e pass through Q, there is no circle in $\mathcal{C}$ passing through Q_2 and Q. Hence, $\deg Q_2 \leq 3$. Since the degree of any intersection point is at least three, we have $\deg Q_2 = 3$ and $c_2 = t(Q_2, c) \in \mathcal{C}$. Let P_2 be the intersection point of c_2 and e_2 (see Figure 16).

$\deg Q_2 = 3$ implies that $Val(Q_2, e_2) \geq \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

It also implies that if (P_3, c_3) is a weak couple, different from (P, c), that dominates (Q_2, e_2), then $P_3 = P_2$ and $c_3 = c_2$. We consider two cases: $t(P_2, c_2) \notin \mathcal{C}$ and $t(P_2, c_2) \in \mathcal{C}$.

If $t(P_2, c_2) \notin \mathcal{C}$, then (P, c) dominates exclusively (Q_2, e_2). Hence,

$$|Val(P,c)| \leq \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \leq Val(Q_2, e_2) = w_{a'}(P,c)$$

This implies that (P, c) is w_2-satisfied.

If $t(P_2, c_2) \in \mathcal{C}$, then we take the circles c, e, e_2 out of $\mathcal{C}$ and get a new connected family of $n - 3$ circles without the intersection points Q, P, Q_2, a contradiction to the minimality of $\mathcal{C}$.

It is left to consider the case $\deg P \geq 5$. Let g be a circle through P other than $c, t(P, c), e$, and e_2.

Estimation of $w_b(P, c)$: Recall that $(P, c) \longrightarrow_1 (Q, e))$. The four circles $c, c_1, t(P, c)$, and $t(P_1, c_1)$, intersect e in four points, while e_2 and g intersect

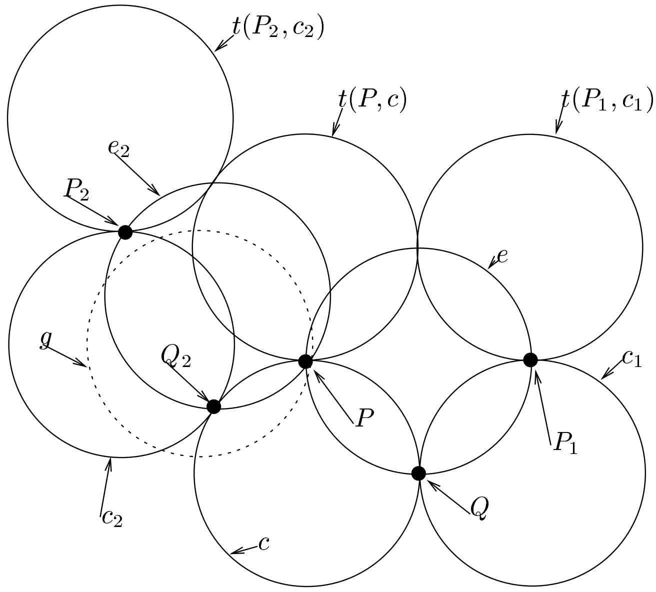

Figure 16: case 2.1 (i) b: $\deg Q = 3$

e in at least one more different point. Hence, $\deg e \geq 5$ and $Val(Q, e) \geq \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$. Since both $(P, c) \longrightarrow_1 (Q, e)$ and $(P_1, c_1) \longrightarrow_1 (Q, e)$, we have $w_b(P, c) \geq \frac{1}{2} \times \frac{2}{15} = \frac{1}{15}$. Using the above and $\deg P \geq 5$, we have: $|Val(P, c)| \leq \frac{1}{4} - \frac{1}{5} = \frac{1}{20} < \frac{1}{15} \leq w_b(P, c)$.

We conclude that (P, c) is w_2-satisfied.

2.1(ii). We claim that (P_1, c_1) is w_2-satisfied, or that $\sum_{P \in c_1} w_2(P, c_1) \geq 1$.

Recall that $Q_1 = \overleftarrow{N}_{c_1} P_1$, $e_1 = \overline{c_1}(P_1, Q_1)$, and $(P_1, c_1) \longrightarrow_2 (Q_1, e_1)$.

If (Q_1, e_1) settles (P_1, c_1) and (P_1, c_1) dominates (Q_1, e_1) exclusively, then (P_1, c_1) is w_2-satisfied. So, assume that (Q_1, e_1) does not settle (P_1, c_1) or that (Q_1, e_1) is dominated by additional weak couple (P', c').

Let L be the intersection point of e_1 and h which lies inside disk(e). Let E be the intersection point of e and e_1, that is different from P_1.

If (Q_1, e_1) does not settle (P_1, c_1), then, by Lemma 18, Q_1 is semi e_1-connected, hence if $L \neq Q_1$, then $\overline{e_1}(L, Q_1) \in \mathcal{C}$.

If (Q_1, e_1) is dominated by additional weak couple (P_2, c_2), then $P_2 \in \text{arc}_{e_1}(Q_1 E) \cup \{E\}$, since otherwise e intersects both $\text{arc}_{c_2}(Q_1 P_2)$ and $\text{arc}_{c_2}(P_2 Q_1)$, which contradicts the assumption that (P_2, c_2) dominates (Q_1, e_1).

Note that in each of these two cases, there exists a circle $g \in \mathcal{C}$ that passes through Q_1 and a point on $\text{arc}_{e_1}(Q_1E) \cup \{E\}$ (see Figure 17).

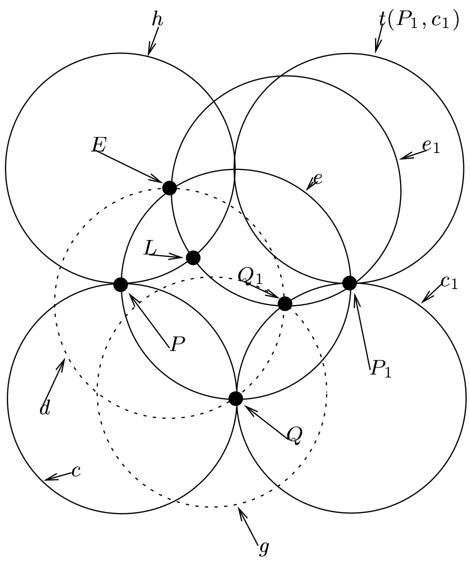

Figure 17: case 2.1(ii)

The above implies that Q_1 lies on $h = t(P, c)$. Since otherwise, $L \neq Q_1$ and hence, g does not pass through E. We claim that g intersects $\text{arc}_{c_1}(P_1Q)$ which is empty, a contradiction. To verify that indeed g intersects $\text{arc}_{c_1}(P_1Q)$, observe that the circle $\overline{e_1}(Q_1, E)$ passes through Q, which implies that any circle g that passes through Q_1 and a point on $\text{arc}_{e_1}(Q_1E)$, satisfies $Q \in \text{disk}(g)$. Hence, g intersects $\text{arc}_{c_1}(P_1Q)$.

We claim that the circles c, c_1, h mutually touch each other. If the circles c, c_1, h do not mutually touch each other, then h intersects c_1 in two points, Q_1 and additional point, say M. Let $g = \overline{c_1}(P_1, M)$ (see Figure 18).

Recall that E is the intersection point of e and e_1 that is not P_1.

Since Q_1 is semi-e_1-connected or that (Q_1, e_1) is dominated by additional weak couple (P', c'), there is a circle $d \in \mathcal{C}$ passing through Q_1 and a point R, which is either E or is lying on $\text{arc}_{e_1}(Q_1E)$. If $d \neq \overline{c_1}(Q_1, Q) = \overline{e_1}(Q_1, E)$, then d intersects $\text{arc}_{c_1}(P_1Q)$, which is empty. Therefore, $d = \overline{c_1}(Q_1, Q) = \overline{e_1}(Q_1, E)$. d intersects g at a point T of degree 2, because every circle through T, other than d, g, would intersect $\text{arc}_{c_1}(Q_1P_1)$ or $\text{arc}_{c_1}(P_1Q)$, that are empty. This is a contradiction.

We conclude that the circles c, c_1, and h mutually touch each other.

By arguments similar to those above, the circle $d = \overline{c_1}(Q_1, Q) = \overline{e_1}(Q_1, E)$ is in $\mathcal{C}$. Moreover, if (P', c') is a weak couple, different from (P_1, c_1), that dominates (Q_1, e_1), then $P' = E$ and $c' = d$ and $(E, d) \longrightarrow_1 (Q_1, e_1)$.

Note that $\deg Q_1 = 4$, since any circle through Q_1, other than c_1, e_1, h, and d, intersects $\mathrm{arc}_c(PQ)$, $\mathrm{arc}_{c_1}(P_1Q)$ or $\mathrm{arc}_{c_1}(Q_1P_1)$, that are empty. Recall that $\deg Q \leq 4$ (we showed this at the beginning of case 2.1), but since $d \in \mathcal{C}$, we have $\deg Q = 4$.

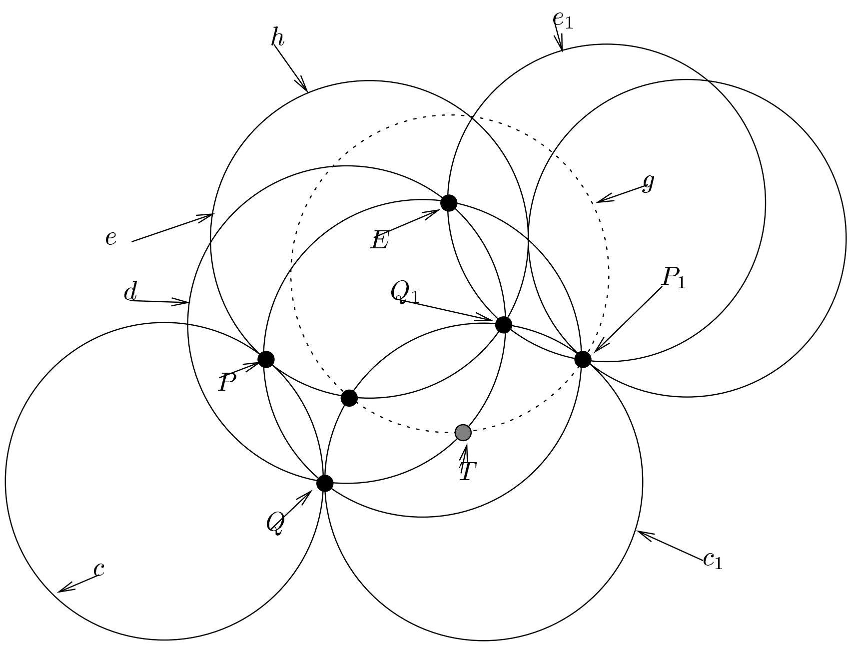

Figure 18: case 2.1(ii), sub-cases a' and b'

We consider two sub-cases: $\deg c_1 \geq 4$ and $\deg c_1 = 3$:

2.1(ii) Sub-case a: $\deg c_1 \geq 4$. Then, $|Val(P_1, c_1)| \leq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. Let S be a point on c_1, other than P_1, Q, and Q_1, then $S \in \mathrm{arc}_{c_1}(QQ_1)$. Let $s = \overline{c_1}(P_1, S)$ (see Figure 19).

s and d intersect at a point T inside $\mathrm{disk}(c_1)$. Any circle through T, other than s or d, will cross c_1 on $\mathrm{arc}_{c1}(Q_1P_1)$ or $\mathrm{arc}_{c_1}(P_1Q)$ that are empty. Therefore, $\deg T = 2$, a contradiction.

2.1(ii) Sub-case b: $\deg c_1 = 3$ (see Figure 19).

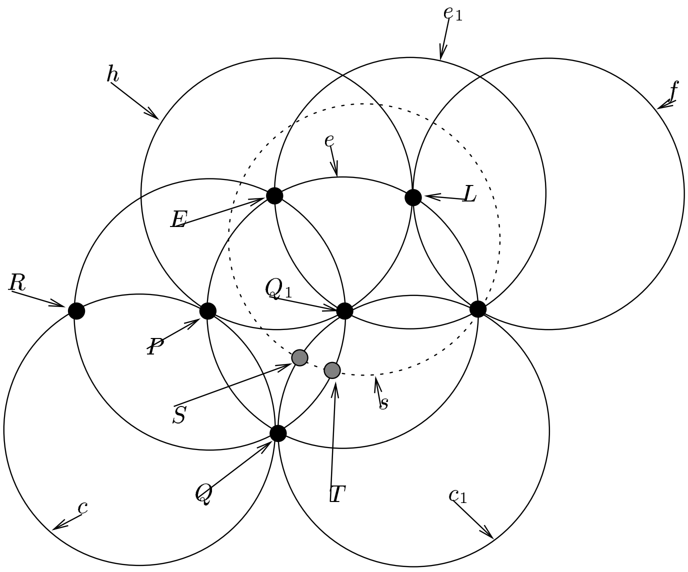

Figure 19: case 2.1 (ii)

We show that in this case $\sum_{P\in c_1} w_2(P,c_1) \geq 1$.

Estimation of $w_b(P_1,c_1)$: Recall that $(P_1,c_1) \longrightarrow_1 (Q,e))$. $\deg e \geq 5$ and $\deg Q = 4$, hence $Val(Q,e) \geq \frac{1}{20}$. Since $(P,c) \longrightarrow_1 (Q,e)$, $w_b(P_1,c_1) = \frac{1}{2} \times Val(Q,e) \geq \frac{1}{40}$.

Estimation of $w_{a'}(P_1,c_1)+w_{d'}(P_1,c_1)$: Recall that $(P_1,c_1) \longrightarrow_2 (Q_1,e_1)$. $\deg e_1 \geq 5$ and $\deg Q_1 = 4$, hence $Val(Q_1,e_1) \geq \frac{1}{20}$. Recall that if (P',c') is additional weak couple that dominates (Q_1,e_1), then $P' = E$ and $c' = d$ and $(E,d) \longrightarrow_1 (Q_1,e_1)$. $\deg d \geq 5$, hence $|Val(E,d)| \leq \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$.

By Definition 15 part (d'), $w_{a'}(P_1,c_1) + w_{d'}(P_1,c_1) \geq \frac{1}{20} - \frac{1}{30} = \frac{1}{60}$.

Estimation of $w_2(P_1,c_1)$:

$w_2(P_1,c_1) \geq \frac{1}{4} + \frac{1}{60} + \frac{1}{40} = \frac{35}{120}$.

Estimation of $w_e(Q_1,c_1)$: Observe that $\deg Q_1 = 4$ implies that (Q_1,h) is the helper of (Q_1,c_1). Denote the intersection point of d and c by R. Then observe that $\deg h \geq 6$ (the circles $e_1, c_1, c, \overline{c}(P,R)$ intersect h in four points , while e, d intersect h in two more different points). Since $\deg Q_1 = 4$, $w_e(Q_1,c_1) = Val(Q_1,h) \geq \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$.

Estimation of $w_a(Q_1,c_1)+w_d(Q_1,c_1)$: Recall that $(Q_1,c_1) \longrightarrow_1 (P_1,e_1)$. Since $\deg P_1 = 4$, $Val(P_1,e_1) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$, and $w_a(P_1,c_1) + w_d(P_1,c_1) \geq$

$\frac{1}{2} \times Val(P_1, e_1) \geq \frac{1}{40}$.

Estimation of $w_{a'}(Q_1, c_1) + w_c(Q_1, c_1)$: Recall that $(Q_1, c_1) \longrightarrow_2 (Q, d)$. Since $\deg Q = 4$, $Val(Q, d) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$.

If (P', c') is additional weak couple that dominates (Q, d), then $P' = R$ and $c' = c$. Since otherwise, $P' \in \mathrm{arc}_d(QR)$ and hence c' intersects $\mathrm{arc}_{c_1}(P_1Q)$, which is empty. Observe that in this case $(R, c) \longrightarrow_2 (Q, d)$. Hence, we have: $w_{a'}(Q_1, c_1) + w_c(Q_1, c_1) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$.

Estimation of $w_2(Q_1, c_1)$**:**

$w_2(Q_1, c_1) \geq \frac{1}{4} + \frac{1}{40} + \frac{1}{40} + \frac{1}{12} = \frac{46}{120}$.

Estimation of $w_a(Q, c_1) + w_d(Q, c_1)$: Recall that $(Q, c_1) \longrightarrow_1 (Q_1, d)$. Since $\deg Q_1 = 4$, $Val(Q_1, d) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. If (P', c') is additional weak couple that dominates (Q_1, d), then $P' = E$ and $c' = e_1$. Since otherwise, $P' \in \mathrm{arc}_d(EQ_1)$ and hence c' intersects $\mathrm{arc}_{c_1}(P_1Q)$, which is empty. Observe that in this case, $(E, e_1) \longrightarrow_2 (Q_1, d)$. Hence, we have: $w_a(Q, c_1) + w_d(Q, c_1) \geq \frac{1}{20}$.

Estimation of $w_{a'}(Q, c_1) + w_c(Q, c_1)$: Recall that $(Q, c_1) \longrightarrow_2 (P_1, e)$.

Since $\deg P_1 = 4$, $Val(P_1, e) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. Let $f = t(P_1, c_1)$ and denote by L, the intersection point of f and e, that is different from P_1.

If (P', c') is additional weak couple that dominates (P_1, e), then $P' = L$ and $c_1 = f$. Since otherwise, $P' \in \mathrm{arc}_e(LP_1)$ and hence c' intersects $\mathrm{arc}_{c_1}(P_1Q)$, which is empty. Observe that in this case, $(L, f) \longrightarrow_2 (P_1, e)$. Hence, we have: $w_{a'}(Q, c_1) + w_c(Q, c_1) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$.

Estimation of $w_2(Q_1, c_1)$: $w_2(Q, c_1) \geq \frac{1}{4} + \frac{1}{40} + \frac{1}{20} = \frac{39}{120}$.

We conclude that $\sum_{P \in c_1} w_2(P, c_1) \geq \frac{39}{120} + \frac{46}{120} + \frac{35}{120} = 1$.

<u>**Case 2.2:**</u> $(P, c) \longrightarrow_2 (Q, e)$ and $(P_1, c_1) \longrightarrow_2 (Q, e)$.

Most of the arguments in this case are similar to those of case 2.1. Nevertheless, the two cases are not symmetric and hence, for the completeness of the proof we do not skip this case.

Let $Q_1 = \overrightarrow{N}_c P$, $e_1 = \overline{c}(P, Q_1)$. then $(P, c) \longrightarrow_1 (Q_1, e_1)$.

Let $Q_2 = \overrightarrow{N}_{c_1} P_1$, $e_2 = \overline{c_1}(P_1, Q_2)$, then $(P_1, c_1) \longrightarrow_1 (Q_2, e_2)$.

By arguments as in case 2.1, we show that $\deg Q \leq 4$.

If $\deg Q = 4$, let f be the forth circle through Q (beside c, c_1, e) and let F be the intersection point of f and c_1 which is not Q. As in case 2.1, $\mathrm{arc}_{c_1}(FQ)$ is empty.

In the followings we show that (i): (P_1, c_1) is w_2-satisfied and (ii): if (P, c) is not w_2-satisfied, then $\sum_{P' \in c} w_2(P', c) \geq 1$.

2.2(i). We claim that (P_1, c_1) is w_2-satisfied.

In order to prove this we need to consider two sub-cases: $\deg Q = 4$ and $\deg Q = 3$.

2.2(i) Sub-case a. $\deg Q = 4$.

Let f and F be defined as above. As in case 2.1, our claim that $\mathrm{arc}_{c_1}(FQ)$ is empty implies that $h := t(P_1, c_1)$ is the helper of (P_1, c_1).

Suppose that $\deg P_1 = x$. $x \geq 4$ because $\deg c_1 \geq 3$. Also, $|Val(P_1, c_1)| = \frac{1}{x-1} - \frac{1}{x}$.

Here we show that (P_1, c_1) is w_2-satisfied:

Estimation of $w_c(P_1, c_1)$: Recall that $(P_1, c_1) \longrightarrow_2 (Q, e)$.

As in case 2.1, $\deg e \geq x + 1$.

Since $\deg Q = 4$, $Val(Q, e) \geq \frac{1}{4} - \frac{1}{x+1}$. Recall also that $(P, c) \longrightarrow_2 (Q, e)$, hence, $w_c(P_1, c_1) = \frac{1}{2} \times Val(Q, e) \geq \frac{1}{2} \times (\frac{1}{4} - \frac{1}{x+1})$

Estimation of $w_e(P_1, c_1)$**:** Since no circle of $\mathcal{C}$ passes through P_1 and one of the two intersection points of f and h, we have $\deg h \geq x + 1$. Therefore, $w_e(P_1, c_1) = Val(P_1, h) \geq \frac{1}{x} - \frac{1}{x+1}$.

Estimation of $w_a(P_1, c_1)+w_b(P_1, c_1)+w_d(P_1, c_1)$**:** Recall that $(P_1, c_1) \longrightarrow_1 (Q_2, e_2)$.

Since $\deg P_1 = x$, $\deg Q_2 \leq x$ and hence, $Val(Q_2, e_2) \geq \frac{1}{x} - \frac{1}{x+1}$.

We assume that (P_1, c_1) is not w_2-satisfied (otherwise, we are done), and so, by Observation 17, $w_a(P_1, c_1)+w_b(P_1, c_1)+w_d(P_1, c_1) \geq \frac{1}{2} \times Val(Q_2, e_2) \geq \frac{1}{2} \times (\frac{1}{x} - \frac{1}{x+1})$.

We are now ready to estimate $w_2(P_1, c_1)$.

$$\begin{aligned} w_2(P_1, c_1) &\geq w_c(P_1, c_1) + w_e(P_1, c_1) + w_a(P_1, c_1) + w_b(P_1, c_1) + w_d(P_1, c_1) \geq \\ &\geq \frac{1}{2} \times (\frac{1}{4} - \frac{1}{x+1}) + \frac{1}{x} - \frac{1}{x+1} + \frac{1}{2} \times (\frac{1}{x} - \frac{1}{x+1}). \end{aligned} \tag{2}$$

The right hand side of (2) is greater than or equal to $\frac{1}{x-1} - \frac{1}{x}$, for every $x \geq 4$.

Hence, $w_2(P_1, c_1) \geq \frac{1}{x-1} = \frac{1}{de\ c_1}$. This implies that (P_1, c_1) is w_2-satisfied.

2.2(i) Sub-case b. $\deg Q = 3$. Our purpose is to prove that (P_1, c_1) is w_2-satisfied. We use similar arguments to those of case 2.1, to show that if this is not the case, then $\deg P_1 \geq 5$.

Let g be a circle through P_1 other than $c_1, t(P_1, c_1), e, e_2$.

Estimation of $w_c(P_1, c_1)$: Recall that $(P_1, c_1) \longrightarrow_2 (Q, e)$. The four circles $c, c_1, t(P, c)$, and $t(P_1, c_1)$, intersect e in four points, while e_2 and g intersect e at least at one more additional point. Hence, $\deg e \geq 5$ and $Val(Q, e) \geq \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$. Since both $(P, c) \longrightarrow_2 (Q, e)$ and $(P_1, c_1) \longrightarrow_2 (Q, e)$, we have $w_c(P_1, c_1) \geq \frac{1}{2} \times \frac{2}{15} = \frac{1}{15}$. Using this and $\deg P_1 \geq 5$, we get: $|Val(P_1, c_1)| \leq \frac{1}{4} - \frac{1}{5} = \frac{1}{20} < \frac{1}{15} \leq w_c(P_1, c_1)$.

We conclude that (P_1, c_1) is w_2-satisfied.

2.2(ii). We claim that if (P, c) is not w_2-satisfied, then $\sum_{P' \in c} w_2(P', c) \geq 1$.

Recall that $h := t(P_1, c_1)$, $Q_1 = \overrightarrow{N}_c P$, $e_1 = \overline{c}(P, Q_1)$ and $(P, c) \longrightarrow_1 (Q_1, e_1)$. E is the intersection point of e and e_1 that is different from P.

If (Q_1, e_1) settles (P, c) and (P, c) dominates (Q_1, e_1) exclusively, then (P, c) is w_2-satisfied. So, assume that (Q_1, e_1) does not settle (P, c) or that (Q_1, e_1) is dominated by additional weak couple (P_2, c_2).

Using arguments as in case 2.1, we can show that the circles c, c_1, and h mutually touch each other and that the circle $d = \overline{c}(Q_1, Q) = \overline{e_1}(Q_1, E)$ is in $\mathcal{C}$.

Moreover, if (P_2, c_2) is a weak couple, different from (P, c), that dominates (Q_1, e_1), then $P_2 = E$ and $c_2 = d$ and $(E, d) \longrightarrow_2 (Q_1, e_1)$.

Note that $\deg Q_1 = 4$, since any circle through Q_1, other than c, e_1, h, and d, would intersect the empty arcs $\mathrm{arc}_{c_1}(QP_1)$, $\mathrm{arc}_c(Q_1 P)$ or $\mathrm{arc}_c(QP)$.

Recall that $\deg Q \leq 4$ (we mentioned this at the beginning of case 2.2), but since $d \in \mathcal{C}$, $\deg Q = 4$.

We consider two sub-cases: $\deg c \geq 4$ and $\deg c = 3$:

2.2(ii) Sub-case a $\deg c \geq 4$. In this case $|Val(P, c)| \leq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. Let S be a point on c, other than P, Q, and Q_1. Then $S \in \mathrm{arc}_c(Q_1 Q)$. Let $s = \overline{c}(P, S)$.

Note that $\deg e_1 \geq 6$ ($t(P, c), c$, and h intersect e_1 in four points , while e and s intersect e_1 in two more different points). Since $\deg Q_1 = 4$, we have $Val(Q_1, e_1) \geq \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$.

Recall that if (P_2, c_2) is additional weak couple that dominates (Q_1, e_1), then $P_2 = E$, $c_2 = d$, and $(E, d) \longrightarrow_2 (Q_1, e_1)$.

Hence, by Definition 15 part (d), $w_a(P, c) + w_d(P, c) \geq |Val(P, c)|$. Namely, (P, c) is w_2-satisfied.

2.2(ii) Sub-case b $\deg c = 3$. We show that in this case, $\sum_{P' \in c} w_2(P', c) \geq 1$.

Estimation of $w_c(P,c)$: Recall that $(P,c) \longrightarrow_2 (Q,e)$. $\deg e \geq 5$ and $\deg Q = 4$, hence $Val(Q,e) \geq \frac{1}{20}$. Since both $(P,c) \longrightarrow_2 (Q,e)$ and $(P_1,c_1) \longrightarrow_2 (Q,e)$, we have $w_c(P,c) = \frac{1}{2} \times Val(Q,e) \geq \frac{1}{40}$.

Estimation of $w_a(P,c) + w_d(P,c)$: Recall that $(P,c) \longrightarrow_1 (Q_1,e_1)$. $\deg e_1 \geq 5$ and $\deg Q_1 = 4$, hence $Val(Q_1,e_1) \geq \frac{1}{20}$.

Recall that if (P_2,c_2) is additional weak couple that dominates (Q_1,e_1), then $P_2 = E$, $c_2 = d$ and $(E,d) \longrightarrow_2 (Q_1,e_1)$. Therefore, by Definition 15, $w_a(P,c) + w_d(P,c) \geq \frac{1}{20}$.

We are now ready to estimate $w_2(P,c)$:

$w_2(P,c) \geq \frac{1}{4} + \frac{1}{40} + \frac{1}{20} = \frac{39}{120}$.

Estimation of $w_e(Q_1,c)$:

Observe that $\deg Q_1 = 4$ implies that (Q_1,h) is the helper of (Q_1,c). Denote the intersection point of d and c_1 by R. Then observe that $\deg h \geq 6$ (the circles e_1, c_1, c, and $\overline{c_1}(P_1,R)$ intersect h in four points , while e and d intersect h in two more different points). Since $\deg Q_1 = 4$, we have $w_e(Q_1,c) = Val(Q_1,h) \geq \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$.

Estimation of $w_{a'}(Q_1,c) + w_{d'}(Q_1,c)$: Recall that $(Q_1,c) \longrightarrow_2 (P,e_1)$.

Since $\deg P_1 = 4$, we have $Val(P_1,e_1) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$.

$\deg c = 3$ and therefore, if (P_2,c_2) is additional weak couple that dominates (P,e_1), then P_2 must be the intersection point of e_1 and $t(P,c)$ that is different from P. We have $c_2 = t(P,c)$, and $(P_2,c_2) \longrightarrow_2 (P,e_1)$ and hence, $w_{a'}(Q_1,c) + w_{d'}(Q_1,c) \geq \frac{1}{2} \times Val(P,e_1) \geq \frac{1}{40}$.

Estimation of $w_a(Q_1,c) + w_b(Q_1,c)$: Recall that $(Q_1,c) \longrightarrow_1 (Q,d)$.

Since $\deg Q = 4$, we have $Val(Q,d) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$.

Therefore, $w_a(Q_1,c) + w_b(Q_1,c) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$

We are now ready to estimate $w_2(Q_1,c)$:

$w_2(Q_1,c) \geq \frac{1}{4} + \frac{1}{40} + \frac{1}{40} + \frac{1}{12} = \frac{46}{120}$.

Finally, we will estimate $w_2(Q,c)$.

Estimation of $w_{a'}(Q,c) + w_{d'}(Q,c)$: Recall that $(Q,c) \longrightarrow_2 (Q_1,d)$.

Since $\deg Q_1 = 4$, we have $Val(Q_1,d) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$.

If (P_2,c_2) is additional weak couple that dominates (Q_1,d), then $P_2 = E$ and $c_2 = e_1$. Indeed, since otherwise $P_2 \in \mathrm{arc}_d(Q_1E)$ and hence c_2 would intersect $\mathrm{arc}_c(QP)$, which is empty.

Observe that in this case, $(E,e_1) \longrightarrow_1 (Q_1,d)$. Since $\deg e_1 \geq 5$, we have $|Val(E,e_1)| \leq \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$. Therefore, $w_a(Q,c_1) + w_d(Q,c_1) \geq \frac{1}{20} - \frac{1}{30} = \frac{1}{60}$.

Estimation of $w_a(Q,c) + w_b(Q,c)$: Recall that $(Q,c) \longrightarrow_1 (P,e)$.

Since $\deg P = 4$, we have $Val(P,e) \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. Therefore $w_a(Q,c) + w_b(Q,c) \geq \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$.

We are now ready to estimate $w_2(Q,c)$: $w_2(Q,c) \geq \frac{1}{4} + \frac{1}{40} + \frac{1}{60} = \frac{35}{120}$.

We conclude that $\sum_{P' \in c} w_2(P',c) \geq \frac{39}{120} + \frac{46}{120} + \frac{35}{120} = 1$.

Case 2.3: $(P,c) \longrightarrow_1 (Q,e)$ and $(P_1,c_1) \longrightarrow_2 (Q,e)$.

Let $Q_1 = \overrightarrow{N}_{c_1} P_1$, $e_1 = \overline{c_1}(P_1,Q_1)$. then $(P_1,c_1) \longrightarrow_1 (Q_1,e_1)$. Let $Q' = \overline{Q}^{c_1}$, $P_1' = \overline{P_1}^{c_1}$, $Q'' = \overline{Q}^{c}$ $P'' = \overline{P}^{c}$.

2.3(i) We claim that (P_1,c_1) is w_2-satisfied.

First, we show that (Q_1,e_1) settles (P_1,c_1). If (Q_1,e_1) doesn't settle (P_1,c_1), then Q_1 is c_1-connected and we get a contradiction by considering all possible locations of Q_1 on c_1:

- $Q_1 \notin \operatorname{arc}_{c_1}(P_1Q')$. Since otherwise, $\overline{c_1}(Q,Q_1)$ intersects $\operatorname{arc}_c(PQ)$, which is empty.
- $Q_1 \neq Q'$. Since otherwise, Q_1 is not c_1-connected.
- $Q_1 \notin \operatorname{arc}_{c_1}(Q'P_1')$. Since otherwise, $\overline{c_1}(Q,Q_1)$ intersects $\operatorname{arc}_c(QP'')$ in a point, say L, but then $\overline{c}(P,L)$ intersects $\operatorname{arc}_{c_1}(QP_1)$ which is empty.
- $Q_1 \neq P_1'$. Since otherwise, Q_1 and P_1 are not c_1-connected.
- $Q_1 \notin \operatorname{arc}_{c_1}(P_1'Q)$. Since otherwise, $\overline{c_1}(P_1,Q_1)$ intersects $\operatorname{arc}_c(PQ)$, which is empty.
- $Q_1 \neq Q$. Since otherwise, $\deg c_1 = 2$.

Note that since P_1 is always c_1-connected, the fifth argument above implies that $\operatorname{arc}_{c_1}(P_1'Q)$ is empty. We conclude that (Q_1,e_1) settles (P_1,c_1). Therefore, If (Q_1,e_1) is dominated exclusively by (P_1,c_1), or if (Q_1,e_1) is counterclockwise dominated by additional couple, then (P_1,c_1) is w_2-satisfied.

We claim that (Q_1,e_1) is not clockwise dominated by any couple, except for (P_1,c_1). To see this, assume there is a weak couple (P_2,c_2), different from (P_1,c_1), such that $(P_2,c_2) \longrightarrow_1 (P_1,c_1)$. Then, $c_2 = t(Q_1,c_1)$ and P_2 is the intersection point of the circles c_2, e_1 that is different from Q_1. Moreover, $t(P_2,c_2) \in \mathcal{C}$ (see Figure 20).

Note that $\operatorname{arc}_{c_1}(Q_1P_1')$ is empty. Since otherwise, let $L \in \operatorname{arc}_{c_1}(Q_1P_1')$. Since $Q_1 \in \operatorname{arc}_{c_1}(PP')$, $\overline{c_1}(P_1,L)$ intersects $\operatorname{arc}_{c_2}(P_2Q_1)$, which is empty.

Since both $\operatorname{arc}_{c_1}(Q_1P_1')$ and $\operatorname{arc}_{c_1}(P_1'Q)$ are empty (and $P_1' \notin \mathcal{P}$), $\deg c_1 = 3$.

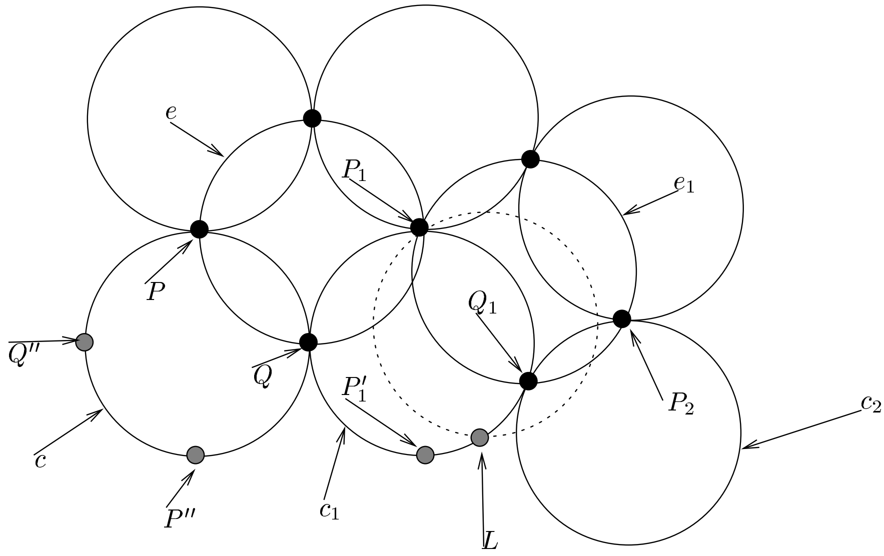

Figure 20: case 2.3 (i)

We have shown that Q_1 is not c_1-connected, hence $\overline{c_1}(Q_1, Q) \notin \mathcal{C}$, which implies that $\deg Q_1 = 3$ and $\deg Q = 3$.

Since $t(P_2, c_2), t(P_1, c_1), t(P, c) \in \mathcal{C}$, and $\mathrm{arc}_{c_1}(Q_1 Q)$ is empty, if we take the circles c_1, e_1, e out of $\mathcal{C}$, we are left with a connected family of circles with $n - 3$ circles. This family of circles doesn't have Q_1, P_1, Q as intersection points, a contradiction to the minimality of $\mathcal{C}$. Conclude that (P_1, c_1) is w_2-satisfied.

2.3(ii) We claim that (P, c) is w_2-satisfied.

If (Q, e) settles (P, c), then since $(P, c) \longrightarrow_1 (Q, e)$ and $(P_1, c_1) \longrightarrow_2 (Q, e)$, (P, c) is w_2-satisfied. Otherwise, (Q, e) doesn't settle (P, c) and hence by Lemma 18, Q is c-connected.

Recall that $\deg c \geq 3$. Let $L \neq Q, P$ be a point on c.

Using the c-connectivity of P and Q, we get a contradiction by considering all possible locations of L on c (see Figure 21):

- $L \notin \mathrm{arc}_c(Q'' P)$. Since otherwise, $\overline{c}(Q, L)$ intersects $\mathrm{arc}_{c_1}(Q P_1)$, which is empty.
- $L \neq Q''$. Since otherwise, Q is not c-connected.

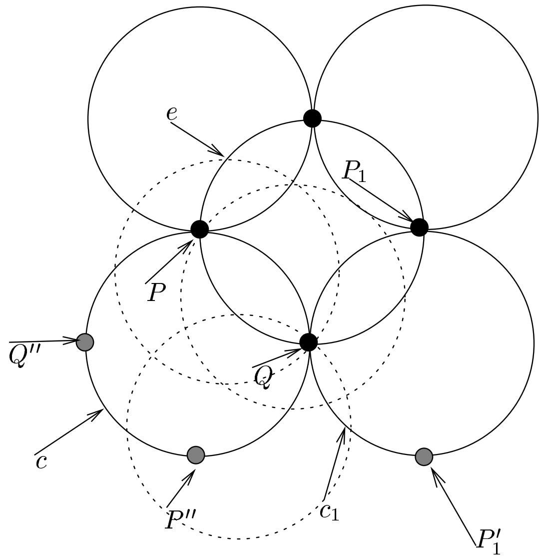

Figure 21: case 2.3 (ii)

- $L \notin \text{arc}_c(P''Q'')$. Since otherwise, $\overline{c}(Q, L)$ intersects $\text{arc}_{c_1}(P_1'Q)$ in a point, say M, but then $\overline{c_1}(P_1, M)$ intersects $\text{arc}_c(PQ)$ which is empty.
- $L \neq P''$. Since otherwise, P is not c-connected.
- $L \notin \text{arc}_c(QP'')$. Since otherwise, $\overline{c}(P, L)$ intersects $\text{arc}_{c_1}(QP_1)$, which is empty.

We conclude that (P, c) is w_2-satisfied as required.

Case 2.4: $(P, c) \longrightarrow_2 (Q, e)$ and $(P_1, c_1) \longrightarrow_1 (Q, e)$.

Let $Q_1 = \overleftarrow{N}_{c_1} P_1$, $e_1 = \overline{c_1}(P_1, Q_1)$. then $(P_1, c_1) \longrightarrow_2 (Q_1, e_1)$.

Let $Q_2 = \overrightarrow{N}_c P$, $e_2 = \overline{c}(P, Q_2)$, then $(P, c) \longrightarrow_1 (Q_2, e_2)$. (See Figure 22).

Note that $\deg Q = 3$, for every circle through Q other than c, c_1, e intersects $\text{arc}_c(QP)$ or $\text{arc}_{c_1}(P_1Q)$.

Also, $\deg e \geq 6$, because $c, c_1, t(P, c), t(P_1, c_1)$ intersect e in four points, while $\overline{c}(P, Q_2)$ and $\overline{c_1}(P_1, Q_1)$ intersect e in two more different points. To see that these two points are indeed different, let L be the touching point of $t(P, c)$ and $t(P_1, c_1)$ (which also lies on e). Then, $\overline{c}(P, Q_2)$ intersects $\text{arc}_e(LP_1)$ and $\overline{c_1}(P_1, Q_1)$ intersects $\text{arc}_e(PL)$. The two arcs are disjoint.

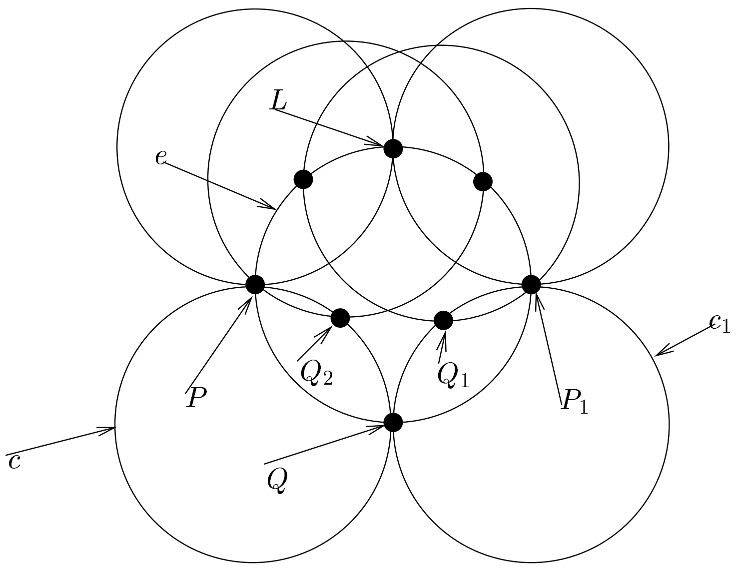

Figure 22: case 2.4

Hence, $Val(Q,e) \geq \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$. $\deg c_1, \deg c \geq 3$ implies that $|Val(P_1, c_1)| \leq \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ and $|Val(P,c)| \leq \frac{1}{12}$.

Since $(P,c) \longrightarrow_2 (Q,e)$, $(P_1,c_1) \longrightarrow_1 (Q,e)$, we have:

$w_d(P_1,c_1) = |Val(P_1,c_1)|$

$w_{d'}(P,c) = Val(Q,e) - |Val(P_1,c_1)| \geq \frac{1}{6} - \frac{1}{12} = \frac{1}{12} \geq |Val(P,c)|$

and hence, both (P,c) and (P_1,c_1) are w_2-satisfied

∎

Proposition 20 shows that if a circle c is not problematic, then its w_2-weight (i.e. $\sum_{P\in c} w_2(P,c)$) is at least one.

Proposition 21 bellow, completes the proof of Theorem 3. It shows that if c is a problematic circle with $P, Q \in \mathcal{P}$ the two intersection points on it, and $d = \overline{c}(P,Q)$, then we can transfer the extra w_2-weight from d in the amount of $\sum_{P\in d} w_2(P,d) - 1 \geq \frac{1}{3}$ to c to make up for the deficiency of w_2-weight on c and at the same time keep the total w_2 weight of d to greater than or equal to 1 (in fact equal to 1). It also shows that the extra w_2-weight on such d is "used" exclusively by such c.

Both propositions imply that we can distribute the initial weight (which is equal to $|\mathcal{P}|$), such that the total weight on every circle is at least one.

So, we can use the same argument as in the proof of Theorem 1 to conclude Theorem 3.

Proposition 21. *Let c be a problematic circle with $P, Q \in \mathcal{P}$ the two intersection points on it, and let $d = \overline{c}(P, Q)$ with $\deg d$ equals to 4 or 5. Then,*

- **a.** *All the couples (X, d) with $P, Q \neq X \in d$, are independent, and (P, d), (Q, d) are dominated only by (Q, c), (P, c), respectively. Also, none of the couples (X, d) with $X \in d$, is the helper of a weak couple.*

- **b.** $\sum_{P \in d} \frac{1}{\deg P} \geq 1\frac{1}{3}$.

- **c.** *c is the only problematic circle crossing d at two points.*

Proof: suppose w.l.o.g. that the clockwise distance from P to Q is less than π. Let $A := \overline{Q}^d$ be the intersection point of $t(P, c)$ with d which is not P, and $B := \overline{P}^d$, the intersection point of $t(Q, c)$ with d which is not Q. Denote $c' = t(P, c)$, $c'' = t(Q, c)$. We consider the two cases: $\deg d = 4$ and $\deg d = 5$.

Case a. deg $d = 4$. Since $\deg A \geq 3$ and $\deg B \geq 3$, either $\overline{d}(A, B) \in \mathcal{C}$, or $t(A, d), t(B, d) \in \mathcal{C}$ or both. In the first case (see Figure 23), we take c and d out of $\mathcal{C}$. The new collection of circles is connected, has $n - 2$ circles and does not have P, Q as intersection points, a contradiction to the minimality of $\mathcal{C}$.

Therefore, we assume the latter case (see Figure 24), that is, $t(A, d), t(B, d) \in \mathcal{C}$ and $\overline{d}(A, B) \notin \mathcal{C}$. Then, $\deg A = \deg P = \deg Q = \deg B = 3$. Hence, $\sum_{P \in d} \frac{1}{\deg P} = 1\frac{1}{3}$.

Moreover, none of the couples (X, d) where $X \in \{A, P, Q, B\}$, is the helper of a weak couple, because (X, d) being a helper of somebody implies $\deg X \geq 4$.

We show here that the couples (A, d), (B, d) are independent, and (P, d), (Q, d) are dominated only by (Q, c), (P, c), respectively:

- (A, d) is independent. If not, then (P, c') dominates (A, d). This is impossible, since $t(A, d)$ intersects c' in the point $\overline{P}^{c'}$, which implies that P is not c'-connected and hence, (P, c') is not weak.

- (P, d) is not dominated by (A, c'). Otherwise, $\overline{d}(A, B) = t(A, c') \in C$, a contradiction.

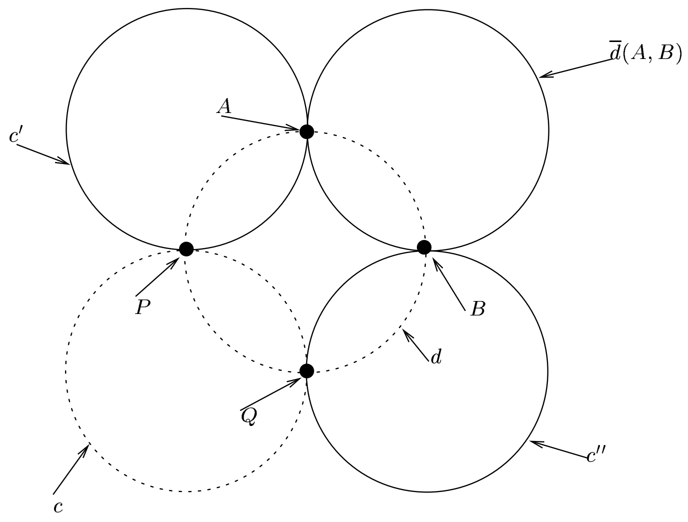

Figure 23: c is problematic, $\deg d = 4$ and $\overline{d}(A,B) \in \mathcal{C}$

- (Q,d) is not dominated by (B,c''), since otherwise $\overline{d}(A,B) = t(B,c'') \in C$, a contradiction.
- (B,d) is independent. Indeed, only (Q,c'') can dominate (B,d), however, (Q,c'') is not weak.

It is also easy to see that none of the circles c' and c'' is problematic, as both contain at least 3 intersection points each. This proves part (c) of Proposition 21.

Case b. deg $d = 5$. Let L be the fifth point of d. If $\overline{d}(L,A), \overline{d}(L,B) \in \mathcal{C}$ or $\overline{d}(A,B) \in \mathcal{C}$, then as in the former case, we get a contradiction to the minimality of $\mathcal{C}$ by considering $\mathcal{C}' = \mathcal{C} \setminus \{c,d\}$. Since $\deg c = 2$, $\overline{d}(L,P), \overline{d}(L,Q) \notin \mathcal{C}$. But, $\deg L \geq 3$ and so, either $\overline{d}(L,A) \in \mathcal{C}$, or $\overline{d}(L,B) \in \mathcal{C}$.

W.l.o.g. assume that $e := \overline{d}(L,A) \in \mathcal{C}$, $\overline{d}(L,B) \notin \mathcal{C}$. Also, $\overline{d}(A,B) \notin \mathcal{C}$, $\overline{d}(L,P), \overline{d}(L,Q) \notin \mathcal{C}$ and in addition, $t(B,d), t(L,d) \in \mathcal{C}$, because $\deg B \geq 3$ and $\deg L \geq 3$. (See Figure 25.)

Then, $\deg A = 3$ or 4, $\deg L = \deg P = \deg Q = \deg B = 3$. Hence, $\sum_{P \in d} \frac{1}{\deg P} > 1\frac{1}{3}$.

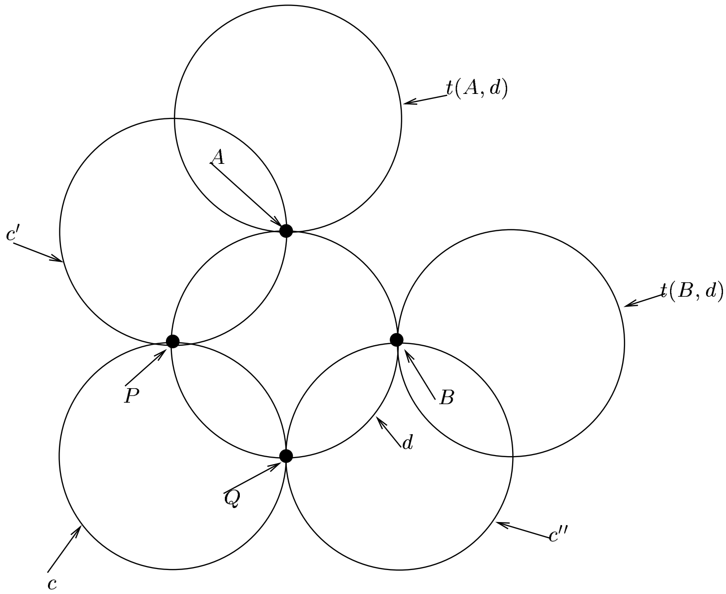

Figure 24: c is problematic, $\deg d = 4$ and $\overline{d}(A,B)$ is not in $\mathcal{C}$

We claim that none of the couples (X,d) with $X \in d$, is the helper of a weak couple. Indeed, the only candidate of being a helper is (A,d) (in the case $\deg A = 4$). Suppose that (A,d) is the helper of $(A, t(A,d))$, then e intersects c' in a point that lies in the interior of disk(d). Hence, $L \in \text{arc}_d(BQ)$. Then, we take c, d out of $\mathcal{C}$. The new collection is connected, and by the same argument, as in the case $\deg d = 4$, we get a contradiction to the minimality of $\mathcal{C}$.

We show here that all the couples (X,d) with $P, Q \neq X \in d$, are independent, and (P,d), (Q,d) are dominated only by (Q,c), (P,c), respectively:

- (B,d) is independent. Otherwise, (Q,c'') dominates (B,d). This is impossible, since $t(B,d)$ intersects c'' at the point $\overline{Q}^{c''}$, which implies that (Q,c'') is not weak.

- (Q,d) is not dominated by (B,c''). Otherwise, $\overline{d}(A,B) = t(B,c'') \in C$, a contradiction.

- (P,d) is not dominated by (A,c'). Otherwise, $\overline{d}(A,B) = t(A,c') \in C$, a contradiction.

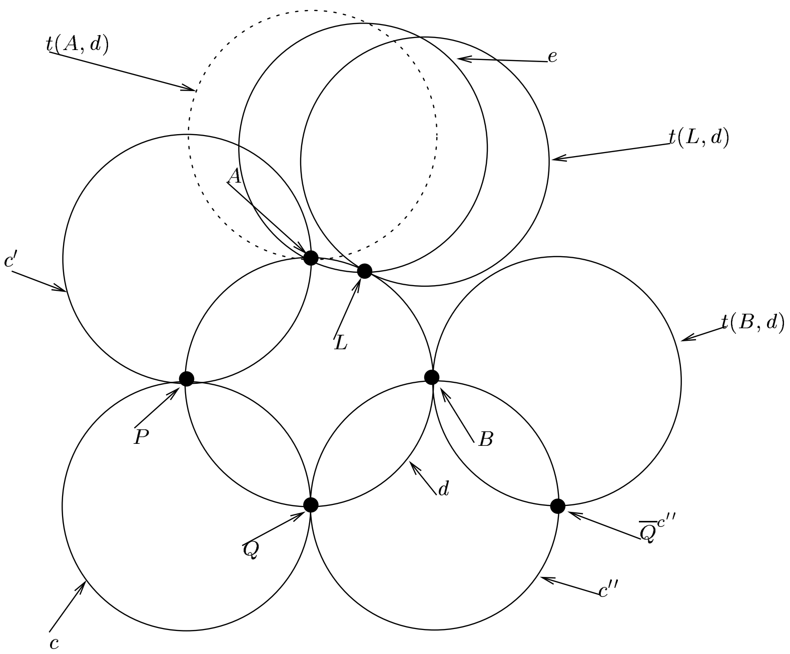

Figure 25: c is problematic and $\deg d = 5$

- (L, d) is independent. Otherwise, (A, e) dominates (L, d), where $e = \overline{d}(A, L)$. Then $t(A, e)$ intersects d at point $\overline{L}^d$, which is different than A, B, Q, P, a contradiction to $\deg d = 5$.
- (A, d) is independent. Otherwise, there are two cases:
  1. (L, e) dominates (A, d). Then, $t(L, e)$ intersects c in two points, one of them is Q and the other one is different from P (otherwise there will be three different circles $c, d, t(L, e)$ passing through the same two points, P, Q), a contradiction to $\deg c = 2$.
  2. (P, c') dominates (A, d). Let M be the intersection point of e and c', which is not A (there is such point because $L \neq B$ and so $e \neq t(A, c')$). Since (P, c') is weak, P is c'-connected, but $\overline{c'}(P, M)$ intersects c in a point , different from P, Q. A contradiction to $\deg c = 2$.

It is also easy to see that none of the circles c', c'', and e are problematic, as each of these circles contains at least 3 intersection points each. This proves part (c) of Proposition 21. ∎

4 A counter-example to a different conjecture of Bezdek

While working out the argument in this paper we realized a counter-example to a conjecture made by A. Bezdek. Bezdek conjectured ([4]) that any finite connected collection of at least two unit circles in the plane determines an intersection point through which at most 3 circles pass. Figure 26 shows a configuration of 13 unit circles where through each intersection point at least 4 circles pass.

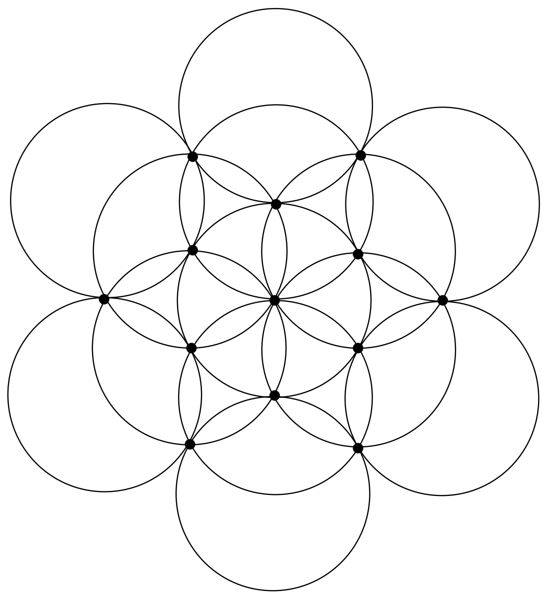

Figure 26: a configuration of unit circles where through every intersection point at least 4 circles pass

It is interesting to note that it is an easy consequence of Euler's formula that any connected collection of at least two unit circles in the plane determines an intersection point through which at most 5 circles pass. On the other hand it is shown in [6] that any finite collection of at least five pairwise intersection unit circles in the plane determines an intersection point through which exactly two circles pass, as was conjecture by A. Bezdek. For a general collection of pairwise intersecting unit circles it is shown in [1] that any big enough collection of pairwise intersecting circles in the plane not all pass through same pair of point and not all touch at the same point, determine an intersection point through which at most 3 circles pass.

References

[1] N. Alon, H. Last, R. Pinchasi, and M. Sharir, *On the complexity of arrangements of circles in the plane*, Discrete and Computational Geometry, **26** (2001), 465–492

[2] A. Bezdek, *On the density of dual circle coverings*, Geometriae Dedicata **33** (1990), 227–238

[3] A. Bezdek, *On the intersection points of unit circles*, Amererican Mathematical Monthly **99** (1992), 779–780

[4] A. Bezdek, *Incidence problems for points and unit circles*, in Paul Erdős and His Mathematics (A. Sali, M. Simonovits and V.T. Sós, Eds.), Budapest (1999), 33–36

[5] K. Bezdek and R. Connelly, *Intersection points*, Annales Univ. Sci. Budapest, Sect. Math **31** (1988), 115–127

[6] R. Pinchasi, *Gallai-Sylvester Theorem for Pairwise Intersecting Unit Circles*, Discrete and Computational Geometry, **28** (2002), 607–624