A Note on Caterpillar-Embeddings with no Two Parallel Edges

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Abstract

Let G be a set of n points in general position (i.e., no three points on a line) in the plane, and let C be a caterpillar on n vertices. We show that one can always find a rectilinear embedding of C in the plane such that the vertices of C are the points of G and no two edges of C go to parallel segments. This proves a conjecture of Robert E. Jamison.

1 Introduction

Let G be a set of n points in the plane. A *direction path* for G is a path whose vertices are the points of G and whose edges consist of straight line segments no two of which are parallel (see for example Figure 1). Clearly, a necessary condition for the existence of a direction path for a set G is that the points of G determine at least $n-1$ different directions.

A well known theorem of Ungar ([U82]) asserts that any set of n points, which is not contained in a line, determines at least $n-1$ different directions. In [J87], Jamison used this result to show that any non-collinear set of n points in the plane admits a

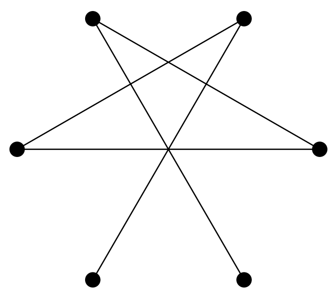

Figure 1: a directed path for the set of vertices of a regular hexagon

direction tree, namely a tree whose vertices are the n points, and every two edges have different directions. In the same paper (and also in [J86]) Jamison conjectured that if G is a set of n points in general position, then not only that it admits a direction tree, but in fact a direction path. On the other hand Jamison ([J86]) constructed arbitrary large non-collinear sets G such that every line determined by G contains at most $|G|/3 + 3$ points, but still G does not admit a direction path.

In this paper we prove a generalization of the conjecture of Jamison. We show that his conjecture is in fact true for any caterpillar, as follows:

Theorem 1.1. *Let C be any given caterpillar on n vertices, and let G be any set of n points in general position in the plane, then there is a rectilinear embedding of C in the plane so that the vertices of C are the points of G and every two edges of C have different directions.*

We note that a special case of Theorem 1.1 where G is the set of vertices of a regular n-gon, was shown by Jamison. There it follows from the equivalence of a tree being a direction tree for the set of vertices of a regular n-gon, and a labeling of the vertices of a tree by the numbers $0, 1, \ldots, n-1$ so that the sums of the labels (modulo n) of every two adjacent vertices of the tree are pairwise different.

2 Caterpillars

In this section we prove Theorem 1.1. We will use duality of points and lines by which Theorem 1.1 will follow from the following theorem on arrangement of lines (in fact everything remains true if we consider arrangements of x-monotone pseudo-lines).

We say that an arrangement of lines $\mathcal{A}$ in the plane is in general position, if no three lines of $\mathcal{A}$ pass through the same point, and no line in $\mathcal{A}$ is vertical. We also require that every two lines in $\mathcal{A}$ cross (that is, no two lines in $\mathcal{A}$ are parallel).

Theorem 2.1. *Let $\mathcal{A}$ be an arrangement of n lines general position in the plane. Let C be any caterpillar on n vertices. Then one can find a correspondence between the lines of $\mathcal{A}$ and the vertices of C so that those intersection points of (pairs of) lines which correspond to adjacent vertices in C, have pairwise different x-coordinates. (See figure 2 for an illustration.)*

It is easy to see how Theorem 1.1 follows from Theorem 2.1. Indeed, let G be a set of n points in general position in the plane. Fix a coordinate system in the plane. By a suitable rotation of the set G we can assume that no two points of G have the same x-coordinates. Apply the points-lines duality which takes a point (A, B) to the lines $y + Ax + B = 0$, and takes a (non-vertical line) $y + Ax + B = 0$ to the point (A, B). Observe that by this duality two parallel lines go to points with the same x-coordinate and vice versa.

The set G is then transfered to an arrangement of lines in general position. If we now apply Theorem 2.1 to this arrangement, we get exactly what we want. Indeed, embed the caterpillar C in the plane by sending the vertices of C to the dual points of their corresponding lines from Theorem 1.1. If l_1, l_2 are two lines in the plane which contain two different embedded edges of C, then the dual of l_1 and l_2 will be two points with different x-coordinates in the dual plane (according to the result in Theorem 1.1). This means that l_1 and l_2 are not parallel.

Instead of proving Theorem 2.1 directly, we will prove a slightly stronger version that will be required for the induction argument. We need a bit of terminology and notation.

Let C be a caterpillar. We say that C has type (a, b) where a and b are positive integers, if as a bipartite graph, C has its two color classes with cardinalities a and b respectively. We then write $V(C) = (V_1, V_2)$ for the set of vertices of C. V_1 and V_1 are the two color classes of C as a bipartite graph, and of course $|V_1| = a$ and $|V_2| = b$. The vertices of C are divided into two kinds, the *leaves*, and those vertices which are not leaves which we call *links*. In a caterpillar the vertices which are links form a path which we call *the spine*. A vertex of C is called a *head*, if it is either an extreme link of the spine, or if it is a leaf connected to an extreme link of the spine.

We say that an arrangement $\mathcal{A}$ of lines is of *type* (a, b), if it consists of precisely a lines with positive slopes and b lines with negative slopes. We denote $\mathcal{A} = (L^+, L^-)$ where L^+ is the set of lines with positive slopes and L^- is the set of lines with negative slopes.

We are now ready to state the stronger version of Theorem 2.1. (The statement of the theorem is a bit long but very easy to understand.)

Theorem 2.2. *Let C be a caterpillar of type (a, b). $V(C) = (V_1, V_2)$. We assume that V_1 contains a vertex v which is a head of C. Let $\mathcal{A} = (L^+, L^-)$ be an arrangement of lines of type (a, b) in general position in the plane. Assume that there is a point p on precisely one of the lines in L^+ such that every line from L^- passes above p and every line in L^+ passes through or below p. Then one can find a one to one correspondence between the lines in L^+ and the vertices in V_1 and a one to one correspondence between the lines in L^- and the vertices in V_2, such that the following is true:*

1. *The x-coordinates of the intersection points of pairs of lines which correspond to adjacent vertices in C are pairwise different and strictly greater than the x-coordinate of p.*

2. *If l is any line with negative slope that passes through p then the x-coordinate of the intersection point of l with the line l_v which corresponds to v is smaller than the x-coordinate of the intersection point of any two lines $l_1, l_2 \in \mathcal{A} \setminus \{l_v\}$ which correspond to two adjacent vertices in C.*

Theorem 2.1 is indeed an easy consequence of Theorem 2.2, since by a suitable affine transformation which does not change the x-coordinates of the points one can take any set of $a + b$ lines in general position to an arrangement of type (a, b). Then just find a right point p and apply Theorem 2.2.

Proof of Theorem 2.2. We start with an easy observation.

Claim 2.3. *Let p' be any intersection point of a line from L^+ with a line from L^-. Then the x-coordinate of p' is strictly larger than that of p.*

Proof. Indeed, if $l^+ \in L^+$ and $l^- \in L^-$, then we know that l^- passes above p and l^+ passes through or below p. Since the slope of l^+ is larger than that of l^-, l^+ is always below l^-, when we are to the left of p. In particular, l^+ and l^- cannot intersect to the left of p nor at a point with the same x-coordinate as p. ■

If $a = 1$, this means that v is the only link in C, and therefore C is a star. We denote by l_v the only line in L^+, and assign it to the vertex v. We arbitrarily assign the lines in L^- to the b vertices in V_2. Clearly, the intersection points of l_v with the lines in L^- have pairwise different x-coordinates (since $\mathcal{A}$ is in general position and thus no three lines of it pass through the same point). Moreover, by Claim 2.3, all these intersection points have their x-coordinates greater than that of p. Assertion (2) in the theorem is in this case void, since one of every two adjacent vertices must be v. Therefore we are done.

Thus we may assume that $a > 1$ and hence there is more than one link in C. Let $m \geq 1$ denote the degree of v in C. v has $m - 1$ neighbors which are leaves and another neighbor which is a link.

For every line $l \in L^+$, let $z_1^l, \dots z_m^l$ denote the x-coordinates of the m leftmost intersection points of l with lines from L^-. Let l_v be such that $z_m^{l_v} = min_{l \in L^+}\{z_m^l\}$. Let p_v denote that (intersection) point on l_v whose x-coordinate is $z_m^{l_v}$. Let $l_1^-, \dots l_m^-$ denote the lines from L^- which intersect l_v at the points with x-coordinates $z_1^{l_v}, \dots, z_{m-1}^{l_v}$ respectively.

Claim 2.4. **(1)** *Every line of $L^- \setminus \{l_1^-, \dots, l_{m-1}^-\}$ passes through or above p_v.*
(2) *Every line of $L^+ \setminus \{l_v\}$ passes below p_v.*

Proof. **(1)** is obvious, since if $l^- \in L^- \setminus \{l_1^-, \dots, l_{m-1}^-\}$ passes below p_v, then there are m lines from L^-, namely $l_1^-, \dots, l_{m-1}^-$ and l^- which intersect l_v at points with x-coordinates smaller than that of p_v contradicting the definition of p_v.

To prove **(2)**, let $l^+ \in L^+ \setminus \{l_v\}$. l^+ cannot pass through p_v since l_v and a l_m^- pass through it. If l^+ passes above p_v, then the lines $l_1^-, \dots, l_m^-$ intersect l^+ at points with x-coordinates smaller than that of p_v. This would mean $z_m^{l^+} < z_m^{l_v}$ contradicting the definition of l_v. ■

Claim 2.5. *Let l be any line with negative slope through p. Let q be the intersection point of l with l_v. Then the x-coordinate of q is greater than or equal to the x-coordinate of p and is strictly smaller than x-coordinate of p_v.*

Proof. If $q = p$, then there is nothing to prove since we know, from Claim 2.3, that p_v has its x-coordinate strictly larger than that of p.

From now on we assume that $q \neq p$. In this case l_v passes below p (rather than through p). Just like in the proof of Claim 2.3, since l passes through p and l_v passes below p and the slope of l is smaller then that of l_v, then q must be to the right of p, that is, its x-coordinate is greater than that of p.

We will now show that the x-coordinate of q is strictly smaller than that of p_v. Let l_p^+ denote the line from L^+ which passes through p. We know that $l_p^+ \neq l_v$. Assume to the contrary that q (which is the intersection point of l with l_v) is either p_v or its x-coordinate is larger than the x-coordinate of p_v. Then, since l has negative slope and l_v has positive slope, l passes through or above p_v (which lies on l_v). Observe that l_p^+ is above l when we are to the right of p so in particular l_p^+ passes above p_v (p_v is to the right of p and l passes through or above p_v). This is a contradiction to part (2) of Claim 2.4. ■

We will now conclude the proof of the theorem by using induction (on the size of the caterpillar C). We assign l_v to the vertex v and arbitrarily assign $l_1^-, \ldots, l_{m-1}^-$ to the $m-1$ neighbors of v which are leaves.

Let C' be the caterpillar of type $(a-1, b-m+1)$ obtained from C by removing the vertex v and its $m-1$ neighbors which are leaves. Denote $V(C') = (V_1', V_2')$. Recall that by our assumption V_2' contains a head of C' (namely, the neighbor of v which is a link in C). Define $L'^+ = L^+ \setminus \{l_v\}$ and $L'^- = L^- \setminus \{l_1^-, \ldots, l_{m-1}^-\}$. Clearly, $|L'^+| = a-1$ and $|L'^-| = b-m+1$. Observe that there is a line in L'^-, namely l_m^- which passes through p_v. By Claim 2.4, every line from L'^+ passes below p_v and every line from L'^- passes through or above p_v.

Now apply the induction hypothesis on the smaller caterpillar C' by interchanging the roles of C_1' and C_2', so that now there is a head of C' in the right color class. As the arrangement of lines we take $L'^+ \cup L'^-$ reflected about the x-axis. Therefore, the reflection of L'^+ will serve as the set of lines with negative slopes and the reflection of L'^- will serve as the set of lines with positive slopes. We take the point p_v to play the role of p in the statement of the theorem. Let v' be the (only) neighbor of v which is a link in C. $v' \in V_2'$ is a head of C'. We take v' to play the role of v in the statement of the theorem.

It is easy to see that all the conditions of the theorem are satisfied by the new caterpillar C' and the new reflected arrangement of lines. We can thus find a correspondence between the lines of L'^+ and the vertices of V_1' and between the lines of L'^- and the vertices of V_2'. By this correspondence all intersection points between lines which correspond to adjacent vertices in C' have pairwise different x-coordinates, and they all lie to the right of p_v so that they have different x-coordinates than those intersection points between l_v and $l_1^-, \ldots l_{m-1}^-$.

We have to consider only one more intersection point between the line l_v and the line l_v' which corresponds to v' (recall that v and v' are neighbors in C). Denote that intersection point by r. We want to show that the x-coordinate of r is different than the x-coordinate of the intersection point of any two lines which correspond to adjacent vertices in C. Clearly, the x-coordinate of r is different than those of the intersection points of l_v with any of the lines $l_1^-, \ldots, l_{m-1}^-$.

If we take the line l in the statement of the theorem to be the reflection of l_v

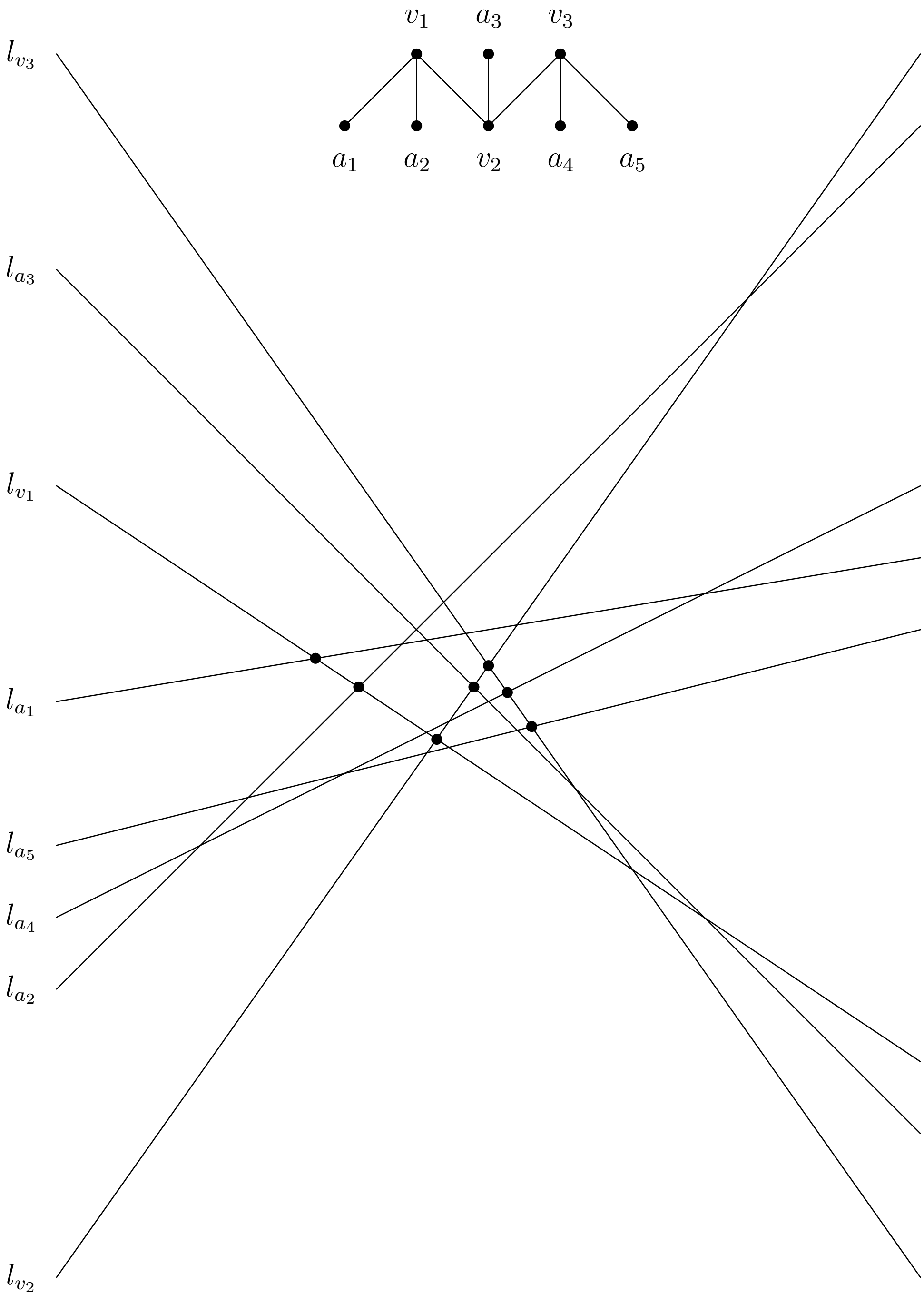

Figure 2: Embedding a caterpillar in an arrangement of lines

about the x-axis (recall that l_v passes through p_v), then it follows from the induction hypothesis that r is to the left of any intersection point of two lines $l_1, l_2 \in \mathcal{A} \setminus \{l_v, l_{v'}\}$ which correspond to adjacent vertices in C. This implies that, the x-coordinate of r is different from every x-coordinate of two lines wich correspond to adjacent vertices in C as required.

Finally, we have to show the validity of asseration (2) in the theorem. Let l be any line with negative slope through p. Observe that every two lines $l_1, l_2 \in \mathcal{A} \setminus \{l_v\}$ which correspond to two adjacent vertices, must be in $L'^{+} \cup L'^{-}$. By the indeuction hypothesis, the intersection point of any two lines $l^{+} \in L'^{+}$ and $l^{-} \in L'^{-}$ which correspond to adjecent vertices in C' is to the right of p_v. However, the intersection point of l and l_v is, by Claim 2.5, to the left or p_v. (Figure 2 shows the resulting embedding of a caterpillar, following from this inductive proof.) ■

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