A Note on Smaller Fractional Helly Numbers

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Abstract

Let $\mathcal{F}$ be a family of geometric objects in $\mathbb{R}^d$ such that the complexity (number of faces of all dimensions on the boundary) of the union of any m of them is $o(m^k)$. We show that $\mathcal{F}$, as well as $\{F \cap P \mid F \in \mathcal{F}\}$ for any given set $P \in \mathbb{R}^d$, have fractional Helly number at most k. This improves the known bounds for fractional Helly numbers of many families.

1 Introduction

In [8] Matoušek shows that families of sets with bounded dual VC-dimension have a (bounded) fractional Helly number (see definition below). In this paper we draw a similar relation between union complexity and fractional Helly number. We encourage the reader to read through [8] before reading this paper.

The theorem of Helly about convex sets asserts that if $\mathcal{F}$ is a collection of convex sets in $\mathbb{R}^d$ and every $d+1$ sets in $\mathcal{F}$ have a nonempty intersection, then all sets in $\mathcal{F}$ admit a nonempty intersection.

The idea of finding a small number of points that meet all sets of a given family $\mathcal{F}$ has extreme importance in theory and applications. Many different variants of this idea have been studied for many years and are still the focus of an intensive research (see [4] for a good survey). This includes in particular the theory of ϵ-nets and weak ϵ-nets [5].

In this paper we confine ourselves to families $\mathcal{F}$ that consist of subsets of $\mathbb{R}^d$ whose boundary is a $(d-1)$-dimensional surface, and also to the more general case in which we have a (possibly finite) set of points P in $\mathbb{R}^d$ and we identify each set in $F \in \mathcal{F}$ with the subset $F \cap P$. We will be interested in conditions that can guarantee the existence of a small number of points that meet every set in our family. Such a set of points is usually referred to as a *piercing* set, or a *hitting* set. Attempts of finding small piercing sets are the focus of intensive study both in theoretical and applied mathematics [2].

One important tool developed to this end is the fractional Helly number. A set system $\mathcal{F}$ is said to have a fractional Helly number k if for every $\alpha > 0$ there is $\beta > 0$ such that for any collection of n sets in $\mathcal{F}$ in which there are at least $\alpha\binom{n}{k}$ k-tuples that have nonempty

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intersection one can find a point incident to at least βn of the sets. Here β may depend only on α (and the family $\mathcal{F}$) but not on n. Therefore, we consider only infinite families $\mathcal{F}$.

Let d and k be fixed integers and let $P \subset \mathbb{R}^d$ be a fixed set of points. Let $\mathcal{F}$ be a family of geometric objects in $\mathbb{R}^d$. We will assume that each $F \in \mathcal{F}$ is a closed set in $\mathbb{R}^d$ and that its boundary is a surface of dimension $d-1$. We will also assume that $\mathcal{F}$ is in general position in the sense that no point belongs to the boundaries of more than d of the sets in $\mathcal{F}$. Those assumptions on $\mathcal{F}$ are quite natural and apply to most cases that are normally studied in literature

For every $F \in \mathcal{F}$ we denote by F_P the subset of P contained in F, that is, $F_P = F \cap P$. We denote by $\mathcal{F}_P$ the family $\mathcal{F}_P = \{F_P \mid F \in \mathcal{F}\}$. Notice the special case of $P = \mathbb{R}^d$, where we simply have $\mathcal{F}_P = \mathcal{F}$. Denote by $U_{\mathcal{F}}(m)$ the maximum complexity (that is, number of faces of all dimensions) of the boundary of a union of any m members of $\mathcal{F}$ (see [1] for a good survey of the topic of union complexity).

Theorem 1 *If $U_{\mathcal{F}}(m) = o(m^k)$, then $\mathcal{F}_P$ has fractional Helly number at most k.*

In the proof of Theorem 1 we will follow the footsteps of Matoušek in [8] where the fractional Helly number of $\mathcal{F}_P$ is bounded in terms of the dual VC-dimension of $\mathcal{F}$ and more specifically, in terms of the dual shatter function of $\mathcal{F}$. In fact, it is shown in [8] that if the dual shatter function of m sets from $\mathcal{F}$ (that is, the maximum number of cells in the Venn diagram of m sets from $\mathcal{F}$) is $o(m^k)$, then $\mathcal{F}_P$ has fractional Helly number at most k.

In Theorem 1 we take a different approach and bound the fractional Helly number of $\mathcal{F}_P$ in terms of the complexity of the union of objects from $\mathcal{F}$, namely the function $U_{\mathcal{F}}(m)$. In many cases this gives a better bound than the one in [8]. For example, in the case where $\mathcal{F}$ consists of balls $\mathbb{R}^d$, the dual shatter function of m sets from $\mathcal{F}$ is $\theta(m^d)$, while $U_{\mathcal{F}}(m) = \theta(m^{\lceil d/2 \rceil})$ (see [1]). In this case Theorem 1 provides a better upper bound on the fractional Helly number of $\mathcal{F}_P$ already for $d = 2$. The advantage of the bound in [8] is that usually estimating the dual shatter function is much easier than finding the union complexity. Having said this, we remark that in most cases of families $\mathcal{F}$ of geometric objects in $\mathbb{R}^d$ we have that $O(m^d)$ is a trivial upper bound for $U_{\mathcal{F}}(m)$ while the dual shatter function of m sets in $\mathcal{F}$ is usually $\Omega(m^d)$ and so in the "typical" cases the bound in [8] cannot be better than $d+1$ while the bound that Theorem 1 yields is almost always at most $d+1$ and in many, if not most, of the "interesting" cases it is strictly smaller.

The tool of fractional Helly number is essential in the method of Alon and Kleitman [3] who used it, in the form of the fractional Helly theorem of Katchalski and Liu [6], to provide a so called (p, q) theorem for certain families of sets. That is, for certain types of families $\mathcal{F}$ of sets one can show that there exists a number q such that if for some $p \geq q$ the family $\mathcal{F}$ has the so called (p, q)-property, that is, out of every p sets there are q with non-empty intersection, then $\mathcal{F}$ has a piercing set of constant size that depends only on p and q but not on the cardinality of $\mathcal{F}$. In [3] it is shown that a family of convex sets in $\mathbb{R}^d$ has a (p, q) theorem with $q = d+1$. In [8] it is shown that every family of bounded VC-dimension has a (p, q) theorem. This result in [8] in fact uses the Alon-Kleitman method in [3] together with an established fractional Helly number of families with bounded (dual) VC-dimension, and the existence of a bounded ϵ-net for families of bounded VC-dimension [5]. The result in [8] applies to a wide variety of examples and in particular to all families of bounded algebraic description (see [8] for more details).

In practice and also in theory when trying to apply a (p,q)-theorem one need to verify that the family $\mathcal{F}$ of sets in question has the (p,q) property. That is, out of every p sets in $\mathcal{F}$ there are q that have a non-empty intersection. In order to verify this it is highly important to have the number q as small as possible, or else such a verification may be difficult.

As an example consider a family $\mathcal{F}$ of pseudo-discs. This is a family of planar shapes such that their boundaries are either disjoint or cross in at most two points. A family of circular discs is one natural example. Suppose we know that $\mathcal{F}$ does not contain 57 pairwise disjoint sets and we wish to show that $\mathcal{F}$ has a small piercing set of point. The dual shatter function of m pseudo-discs is $\theta(m^2)$ and therefore the result in [8] implies a fractional Helly number of 3 for $\mathcal{F}$. Therefore, if we can show that, for some p, out of every p sets of $\mathcal{F}$ there are 3 with a common point we would be done. This is in fact doable but with some extra effort. Theorem 1 implies a small piercing set immediately. Indeed, the union complexity of m pseudo-discs is $O(m)$ (a well known result from [7]) and therefore by Theorem 1 $\mathcal{F}$ has fractional Helly number at most 2. By the property of $\mathcal{F}$ we know that $\mathcal{F}$ has the $(57,2)$ property and we are done (we only need to recall the well known fact that a family of pseudo-discs has a bounded VC-dimension). For an even more convincing example one can consider a family $\mathcal{F}$ of half-spaces in $\mathbb{R}^3$ and to make it a little more interesting consider $\mathcal{F}_P$ for some (even finite) set of points P.

2 The proof

Proof of Theorem 1. We follow a similar argument of Matoušek [8]. Let $\alpha > 0$ be given and assume that for a given subfamily of $\mathcal{F}_P$ of cardinality n it is known that at least $\alpha\binom{n}{k}$ k-tuples of sets from that subfamily of $\mathcal{F}_P$ intersect. With a slight abuse of notation we will assume that $\mathcal{F}_P$ is equal to that subfamily of cardinality n. We will show the existence of $\beta > 0$, independent of n, such that there exists a point in P incident to at least βn sets in $\mathcal{F}_P$.

Let m be large enough so that $U_{\mathcal{F}}(m/2) < \frac{1}{4}\alpha\frac{1}{2^{3d+3k}}\binom{m}{k}$. We take $\beta = \frac{1}{2m}$ and assume that n is large enough so that $\beta n > m$. Suppose that no point in P is incident to βn of the sets in $\mathcal{F}_P$. Consider all the pairs (A,B) where A is an m-tuple of sets in $\mathcal{F}$ and $B \subset A$ is a k-tuple of sets in A. We call the pair (A,B) *good* if there is a point in P incident to every set in B but to no set in $A \setminus B$. We first bound from below the probability that a pair (A,B) as above, chosen uniformly at random among all such pairs, is good. This part of the argument is identical (verbatim) to that in [8]. Pick a k-tuple B of sets from $\mathcal{F}$. We bound from below the number of subsets A of $\mathcal{F}$ of cardinality m such that (A,B) is good. With probability of at least α we know that $\cap_{F\in B}F_P \neq \emptyset$. Assume that B is such a k-tuple that $\cap_{F\in B}F_P \neq \emptyset$ and let $x \in \cap_{F\in B}F_P$. We choose a family C of $m-k$ sets at random from $\mathcal{F}\setminus B$. We now compute the probability that x does not belong to any of these $m-k$ sets. The point x belongs to at most βn of the sets in $\mathcal{F}$ and so with probability of at least $\frac{\binom{(1-\beta)n}{m-k}}{\binom{n-k}{m-k}}$ the pair (A,B) is good, where $A = C \cup B$. It will be more convenient for us to bound this probability from below as follows

$$\frac{\binom{(1-\beta)n}{m-k}}{\binom{n-k}{m-k}} = \prod_{i=0}^{m-k-1}\frac{(1-\beta)n-i}{n-k-i} \geq (\frac{(1-\beta)n-m}{n})^m = (1-\beta-\frac{m}{n})^m.$$

Because $m \leq \beta n$ and $\beta = \frac{1}{2m}$, the above expression is at least $(1-2\beta)^m = (1-\frac{1}{m})^m \geq \frac{1}{4}$. Therefore the probability that a random pair (A, B) is good is at least $\frac{1}{4}\alpha$.

To get a contradiction, we now bound from above the probability that a pair (A, B) is good. Fix a family A of m sets from $\mathcal{F}$. We bound from above the number of distinct subsets $B \subset A$ of cardinality k such that (A, B) is good. Suppose B is a k-tuple of sets in A such that (A, B) is good. Consider the set

$$W_B = (\bigcap_{F \in B} F) \cap (\bigcap_{F \in (A \setminus B)} \overline{\mathbb{R}^d \setminus F}),$$

where by $\overline{\mathbb{R}^d \setminus F}$ we mean the closure of $\mathbb{R}^d \setminus F$.

By the assumption that (A, B) is good, W_B is not empty. Let x be a face of smallest dimension of the boundary of W_B. Because x is a face of smallest dimension on the boundary of W_B, then it is equal to the intersection of the boundaries of all the sets, from $B \cup \{\overline{\mathbb{R}^d \setminus F} \mid F \in A \setminus B\}$, containing it. That is, x is equal to the intersection of at most d (because no point belongs to more than d) boundaries of sets from $B \cup \{\overline{\mathbb{R}^d \setminus F} \mid F \in A \setminus B\}$ which is the same as being equal to the intersection of the boundaries at most d sets from A. Moreover, x is contained in all the k sets in B and in at most d of the sets in $A \setminus B$ (in case x is contained in a set from $A \setminus B$, then it is in fact contained in the boundary of this set). Altogether, x is contained in at most $d + k$ sets from A.

We charge the face x to B. Notice that x may be charged to more than one k-tuple, B, of sets from A.

Claim 2 *The number of all those faces x that are charged to subsets $B \subset A$ of cardinality k is bounded from above by $\frac{1}{4}\alpha\frac{1}{2^{d+k}}\binom{m}{k}$.*

Proof. Suppose that x is a face that is charged to some k-tuple of sets B. Pick $\frac{m}{2}$ sets from A at random with uniform probability. Let us bound from below the probability that x is a face of the boundary of the union of all sets of $\mathcal{F}$ in A that were picked. As we have seen already, x is equal to the intersection of r (boundaries of) sets from A, where $1 \leq r \leq d$. In addition x is contained in only q of the sets in A, where $1 \leq q \leq d + k$.

Therefore, the probability that x is a face of the boundary of the union of $m/2$ random sets from A is at least $\frac{\binom{m-q}{m/2-r}}{\binom{m}{m/2}}$. This stands for the case where we choose the r sets such that the intersection of their boundaries is equal to x and we do not choose any of the other $q - r$ sets containing x. Since $1 \leq q \leq k + d$ and $1 \leq r \leq d$ we can bound this probability from below as follows:

$$\frac{\binom{m-q}{m/2-r}}{\binom{m}{m/2}} \geq \frac{\binom{m-d-k}{m/2-r}}{\binom{m}{m/2}} \geq \frac{\binom{m-d-k}{m/2}}{\binom{m}{m/2}} \geq (\frac{m/2-d-k}{m-d-k})^{d+k} \geq \frac{1}{4^{d+k}}.$$

In the last inequality, we assume that m is large enough with respect to $d + k$, say $m > 3(d + k)$. Therefore, with probability of at least $\frac{1}{4^{d+k}}$ the face x is a face of the boundary of the union of all sets that were picked. The number of those sets that are picked is $m/2$ and hence the number of faces on the boundary of their union is at most $U_{\mathcal{F}}(m/2) \leq \frac{1}{4}\alpha\frac{1}{2^{3d+3k}}\binom{m}{k}$. Let N denote the number of all faces x that are charged to some subset B of A of cardinality k. Then the expected number of faces that we see on the boundary of the union of a random family of $m/2$ sets from A is at least $\frac{1}{4^{d+k}}N$. On the other hand the maximum possible number of faces that we may see on the boundary of the union of any $m/2$ sets from A is at most $U_{\mathcal{F}}(m/2) \leq \frac{1}{4}\alpha\frac{1}{2^{3d+3k}}\binom{m}{k}$. Therefore, $\frac{1}{4^{d+k}}N < \frac{1}{4}\alpha\frac{1}{2^{3d+3k}}\binom{m}{k}$. From here we conclude the requested bound $N < \frac{1}{4}\alpha\frac{1}{2^{d+k}}\binom{m}{k}$. ∎

If a face x is charged to some $B \subset A$, where B is a k-tuple of sets from A, then from all the sets in A the face x may be contained in at most $d+k$ sets. Consequently, x may be charged to at most $\binom{d+k}{k} \leq 2^{d+k}$ subsets B of A. Because every subset B of A such that (A, B) is good is charged with some face x, we have that the number of subsets B of A of cardinality k such that (A, B) is good is smaller than $2^{d+k}\frac{1}{4}\alpha\frac{1}{2^{d+k}}\binom{m}{k} \leq \frac{1}{4}\alpha\binom{m}{k}$. Therefore, the probability that a random pair (A, B) is good is smaller than $\frac{\frac{1}{4}\alpha\binom{m}{k}}{\binom{m}{k}} = \frac{1}{4}\alpha$. This is a contradiction, as we saw earlier that this probability is greater than or equal to $\frac{1}{4}\alpha$. ∎

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