An algebraic solution of a problem of Erdős and Purdy

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Abstract

Let P be a set of n points in general position in the plane. Suppose that R is a set of red points disjoint from P such that every line determined by P passes through a point in R. The problem of bounding $|R|$ from below in terms of n was raised by Erdős and Purdy in 1978. The tight bound $|R| \geq n$ for $n \geq 6$ is a consequence of the solution of the Magic Configuration conjecture of Murty. This conjecture of Murty was proved in 2008 by a topological argument. In this paper we provide an algebraic argument showing that $|R| \geq n$ for $n > 6$. On the way we provide a new result about vectors in $\mathbb{R}^2$.

1 Introduction

Let P be a set of n points in the plane. Erdős and Purdy asked the following question in [2]: Assume a set R of points in the plane is disjoint from P and has the property that every line determined by P passes through a point in R. How small can be the cardinality of R in terms of n?

Clearly, if P is contained in a line, then R may consist of just one point. Therefore, the question of Erdős and Purdy is about sets P that are not collinear. The best known lower bound for this question is given in [8], where it is shown that $|R| \geq n/3$.

In [2] Erdős and Purdy considered also the case in which P is in *general position* in the sense that no three points of P are collinear. In this case if n is odd the tight bound $|R| \geq n$ is almost trivial. To observe that this bound is tight let P be the set of vertices of a regular n-gon and let R be the set of n points on the line at infinity that correspond to the directions of the edges (and diagonals) of P.

If n is even, a trivial counting argument shows that $|R|$ must be at least $n-1$. This is because every point in R may be incident to at most $n/2$ lines determined by P. Interestingly enough one can show that $|R| \geq n$ in this case if $n \geq 6$. In this paper we provide an algebraic proof of this result.

Theorem 1. *Let P be a set of $n \geq 6$ points in general position in the plane. Suppose that R is a set of red points disjoint from P such that every line determined by P passes through a point in R. Then $|R| \geq n$.*

Theorem 1 was proved in [4] as a special case of the Magic Configuration conjecture of Murty ([7]). A configuration P of points in the plane is said to be *magic* if it is possible to assign positive weights to the points of P in such a way that the sum of the weights of all points on any given line determined by P is equal to 1. In [7] Murty conjectured that the only magic configurations of n points are those where $n-1$ points are collinear, or P is in general position, or P is a special set of 7 points called the failed Fano configuration (see Figure 1)

The conjecture of Murty about magic configurations was proved in [4]. To see how theorem 1 follows from the theorem about magic configurations in [4], observe that under the contrary assumption we must have $|R| = n-1$ and it follows that every point in R must be incident to precisely $n/2$ lines determined by P. Assign to every point in P a weight of $\frac{1}{4}$ and to every point in R a weight of $\frac{1}{2}$ and

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observe that if R is in general position, then $P \cup R$ is a magic configuration. The only possibility is that $P \cup R$ is the failed Fano configuration of 7 points, or $P \cup R$ consists of three collinear points, two of P and one of R.

If R is not in general position, then we can still conclude the result from in a similar way by considering the pseudo-line version of the dual theorem for lines in the plane. We will not do this here but refer the reader to [4].

In this note we provide a completely different proof of Theorem 1 using algebra. We hope that this new approach will help in providing generalizations and characterization results related to it. Theorem 1 received considerable attention by other people as well. We are aware of an independent proof by Milicevic [6]. Over $\mathbb{F}_p$ this result was proved by Blokhuis, Marino, and Mazzocca [5]. The approach in [5] seems to be very similar at least in the first part of the proof to the approach in this paper.

2 Proof of Theorem 1

It will be more convenient for us to consider the dual theorem using standard duality of points and lines in the plane.

Theorem 2. *Let L be a set of $n \geq 6$ lines in general position in the plane. Suppose that R is a set of red lines, different from the lines in L such that every intersection point of two lines in L is incident to a line in R. Then $|R| \geq n$.*

Notice that because L determines $\binom{n}{2}$ distinct intersection point and every line in R may be incident to at most $\lfloor n/2 \rfloor$ of them, then the theorem is clear if n is odd. Therefore, the main difficulty in the theorem is the case n is even. Notice that there are easy counterexamples for the cases $n = 2$ and $n = 4$ (see Figure 1).

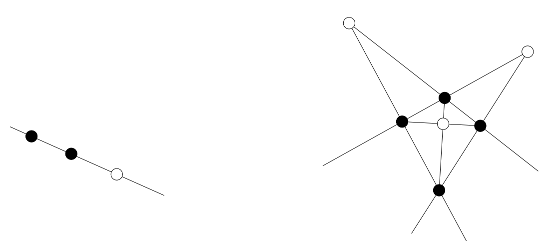

Figure 1: Counterexamples for $n = 2, 4$. The points in P are colored black while the points in R are colored white.

Proof of Theorem 2. Denote the lines in L by $\ell_1, \ldots, \ell_n$. We may assume that no two of the lines in $L \cup R$ are parallel (for example by applying a generic projective transformation). We think of each ℓ_i as a linear polynomial $\ell_i(x, y) = a_i x + b_i y + c_i$, in the variables x and y, whose set of zeroes is the line represented by ℓ_i.

Assume to the contrary that $|R| = n-1$ and denote by $r_1, \ldots, r_{n-1}$ the lines in R, again considered as linear polynomials in the two variables x and y. Specifically, we write $r_i = r_i(x, y) = e_i x + f_i y + g_i$.

With a slight abuse of notation, we denote by

$$R = R(x, y) = r_1(x, y) r_2(x, y) \cdots r_{n-1}(x, y)$$

and observe that the degree of R is $n - 1$. Similarly, we denote

$$P = P(x, y) = \ell_1(x, y) \ell_2(x, y) \cdots \ell_n(x, y).$$

For every $i = 1, \ldots, n$, we denote by $P_i(x, y)$ the polynomial P/ℓ_i that is, the product of all the polynomials $\ell_1, \ldots, \ell_n$ except for ℓ_i. We notice that the degree of every P_i is equal to $n - 1$.

Fix $1 \leq i \leq n$. Consider the polynomial P_i restricted to the line ℓ_i and notice that it vanishes at all intersection points of ℓ_i with the other lines in L. Notice that also the polynomial R restricted to ℓ_i vanishes on the same $n-1$ intersection points. Because both polynomials P_i and R are of degree $n-1$ we conclude that there is a nonzero α_i such that $\alpha_i P_i$ and R are identical if restricted to the line ℓ_i. It follows that ℓ_i is a factor of $\alpha_i P_i - R$. We now observe that ℓ_i must also be a factor of $(\sum_{i=1}^{n} \alpha_i P_i) - R$ (simply because ℓ_i is a factor of every P_j for $j \neq i$). Because this is true for $i = 1, \ldots, n$ and because the degree of $(\sum_{i=1}^{n} \alpha_i P_i) - R$ is smaller than or equal to $n-1$, we conclude that $(\sum_{i=1}^{n} \alpha_i P_i) - R = 0$.

Consider any two distinct lines from L, say ℓ_i and ℓ_j. Let $A = A_{ij}$ denote the intersection point of the two lines ℓ_i and ℓ_j. Let k be the index such that r_k is the line in R passing through A. Consider the polynomial equation

$$\alpha_1 P_1 + \ldots + \alpha_n P_n - R = 0. \tag{1}$$

The partial derivatives (with respect to x and with respect to y) of the left hand side of (1), must be equal to 0, at any point. This is true in particular for the point A. Notice that $\frac{\partial}{\partial y} P_t(A) = 0$ and $\frac{\partial}{\partial x} P_t(A) = 0$ for every t different than i and j. Denote by P_{ij} the polynomial that is the product of all polynomials $\ell_1, \ldots, \ell_n$ except for ℓ_i and ℓ_j. Denote by R_k the polynomial R/r_k.

We have

$$\begin{aligned} \frac{\partial}{\partial x} P_i(A) &= a_j P_{ij}(A) \\ \frac{\partial}{\partial x} P_j(A) &= a_i P_{ij}(A) \\ \frac{\partial}{\partial x} R(A) &= e_k R_k(A) \end{aligned}$$

Therefore, taking the partial derivative in the direction of the x-axis of the left hand side of (1) and equating it to 0 we get

$$\alpha_i a_j P_{ij}(A) + \alpha_j a_i P_{ij}(A) - R_k(A) e_k = 0. \tag{2}$$

.

Similarly, by considering the partial derivative in the direction of the y-axis of the left hand side of (1) and equating it to 0 we get

$$\alpha_i b_j P_{ij}(A) + \alpha_j b_i P_{ij}(A) - R_k(A) f_k = 0. \tag{3}$$

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Observe that $P_{ij}(A) \neq 0$ and $R_k(A) \neq 0$. We recall that both α_i and α_j are nonzero.

Dividing both equations (2) and (3) by $\alpha_i \alpha_j P_{ij}(A)$ we get

$$\begin{aligned} \frac{1}{\alpha_j} a_j + \frac{1}{\alpha_i} a_i &= \frac{R_k(A)}{\alpha_i \alpha_j P_{ij}(A)} e_k \\ \frac{1}{\alpha_j} b_j + \frac{1}{\alpha_i} b_i &= \frac{R_k(A)}{\alpha_i \alpha_j P_{ij}(A)} f_k. \end{aligned}$$

This analysis is valid for every i and j. For $i = 1, \ldots, n$ denote by v_i the vector $\frac{1}{\alpha_i}(a_i, b_i)$. For $i = 1, \ldots, n-1$ denote by u_i the vector (e_i, f_i). Observe that because we assume that no two lines among

$\ell_1, \ldots, \ell_n$ and $r_1, \ldots, r_{n-1}$ are parallel, then every pair of vectors from $v_1, \ldots, v_n$ and $u_1, \ldots, u_{n-1}$ are linearly independent.

For every i, j, k such that $i \neq j$ and ℓ_i and ℓ_j meet a point that is incident to r_k, we have that $v_i + v_j$ is a nonzero vector in the linear span of u_k. Moreover, if $j' \neq j$, then $v_i + v_{j'}$ is in the direction of some $u_{k'}$ different from u_k. From this fact alone we will reach a contradiction.

The contradiction will result from the following lemma of independent interest:

Lemma 3. *Let $n > 6$ be even. Let $V = \{v_1, \ldots, v_n\}$ be a set of n vectors in the $\mathbb{R}^2$, every two of which are linearly independent. Assume that for every distinct i, j, k $v_i + v_j$ and $v_i + v_k$ are linearly independent. Then the set of vectors $\{v_i + v_i \mid 1 \leq i < j \leq n\}$ contains at least n vectors every two of which are linearly independent.*

We remark that Lemma 3 is false for $n = 4$ (which is not surprising, as Theorem 1 is false for $n = 4$). To see this just take any four vectors, linearly independent in pairs, $v_1, \ldots, v_4$ whose sum is equal to 0. The case $n = 6$ in Lemma 3 we leave as an open puzzle to the reader (see below).

Proof.

Notice that the set of vectors in $\{v_i + v_j \mid 1 \leq i < j \leq n\}$ contains at least $n-1$ vectors every two of which are linearly independent. This is because for every i the set $V_i = \{v_i + v_j \mid j \neq i, \ \ 1 \leq j \leq n\}$ is such a set of vectors. We need to show that we can add at least one more vector to this list. Assume to the contrary that the $\binom{n}{2}$ sums $v_i + v_j$ where $i \neq j$ one can find at most $n-1$ vectors every two of which are linearly independent. Let $u_1, \ldots, u_{n-1}$ be such a collection of $n-1$ vectors. We notice in particular that for every i the set V_i contain $n-1$ vectors, each of which (or its minus) is parallel to one of $u_1, \ldots, u_{n-1}$. Hence if we couple vectors in $v_1, \ldots, v_n$ whose sum is parallel to say u_j, we will get a perfect matching. This implies that $\sum_{i=1}^{n} v_i$ is a vector parallel to u_j. Because this is true for every j we conclude that $\sum_{i=1}^{n} v_i = 0$.

Let Q denote the convex hull of $V \cup -V$. Observe that Q is centrally symmetric. We claim that for every $i = 1, \ldots, n-1$ there are two edges of Q parallel to u_i. To see this fix $1 \leq i \leq n-1$ and consider the vector u_i. Without loss of generality assume that u_i is vertical and orthogonally project Q on the x-axis. Let the segment $[a, b]$ be this projection. We observe that we must have $a = -b$ because Q is centrally symmetric. Moreover, there must be precisely two vertices of Q that are projected to b. Indeed, assume that v_k is projected to b (similarly if $-v_k$ is projected to b). Let v_j be such that $v_k + v_j$ has the same linear span as u_i. Then $-v_j$ must also be projected to b, because the vector $v_k - (-v_j) = v_k + v_j$ is vertical.

It follows that Q has an edge parallel to u_i. Because Q is centrally symmetric it must have two edges parallel to u_i. We conclude that Q has at least $2(n-1)$ edges, and therefore, at least $2(n-1)$ vertices.

We claim that Q has exactly $2(n-1)$ edges. This is to say that it is not possible that all the point in $V \cup (-V)$ are vertices of Q. Indeed, if we assume that Q has $2n$ vertices (notice that this is the contrary assumption, as Q is centrally symmetric), then because n is even and because for every vertex v of Q that is in V, the vertex $-v$ lies in $-V$, it must be that there are two vertices of Q in V consecutive along the boundary of Q (verification of this fact is an easy exercise). Let these two vertices be v and v'. Consider the point $-v_k$ such that the angle $\measuredangle(-v_k)vv'$ is minimum. There must be a point $-v_t$ such that the line connecting $-v_t$ to v' is parallel to the line connecting v and $(-v_k)$. This is impossible because then the angle $\measuredangle(-v_t)vv'$ is smaller than $\measuredangle(-v_k)vv'$, contradicting our assumption.

Having established the fact that Q has precisely $2(n-1)$ vertices, we rename these vertices and denote them by $a_0, \ldots, a_m$, where $m = 2(n-1)$, indexed in correspondence to their clockwise order on the boundary of Q. Without loss of generality we have that $\{a_0, a_2, a_4, \ldots, a_{m-2}\}$ are all in V and $\{a_1, a_3, a_5, \ldots, a_{m-1}\}$ are all in $-V$. We may assume this because as we have seen it is not possible that two consecutive vertices along the boundary of Q will belong to V.

We claim that for every i and $1 \leq k < n/2 - 1$ the line through a_{i-k} and a_{i+1+k} is parallel to the line through a_i and a_{i+1} (here all the indices are taken modulu m, for example when $i-k$ is negative). Once we show this is will follow, because Q is centrally symmetric, that for every i the lines $a_{i-k}a_{i+1+k}$ where $0 \leq k < n-1$ and $k \neq n/2-1$ are pairwise parallel.

We prove this by induction on k. Observe that for every i and $1 \leq k < n/2-1$ the line through a_{i-k} and a_{i+1+k} is parallel to one of the vectors $v_s + v_t$ for $s \neq t$. This is because one of a_{i-k} and a_{i+1+k} belongs to V and the other belongs to $-V$, and we do not have $a_{i-k} = -a_{i+1+k}$. Because we assume that every vector of the form $v_s + v_t$ for $s \neq t$ is parallel to one (in fact two) of the edges of Q, it must be that the line through a_{i-1} and a_{i+2} is parallel to the edge a_ia_{i+1}.

As for the induction step, consider the line through a_{i-k} and a_{i+1+k}. By the induction hypothesis the line through a_{i-k} and a_{i+k-1} is parallel to the edge $a_{i-1}a_i$. The line through a_{i-k+2} and a_{i+1+k} is parallel to the edge $a_{i+1}a_{i+2}$. Therefore, the only possible edge of Q that may be parallel to the line through a_{i-k} and a_{i+1+k} is the edge a_ia_{i+1} (or its reflection through O, namely the edge $a_{i+n-1}a_{i+n}$). This completes the induction step.

The next step is to show that the vertices of Q, lie on a quadric. Here we will assume that $2(n-1) > 10$ that is $n > 6$ (leaving the case $n = 6$ open). Let D be a quadric passing through a_0, a_1, a_2, a_3, and a_4. We will show that D must pass through a_5. Then repeating this argument we conclude that all the vertices of Q lie on D.

For every $0 \leq i < j \leq 5$ denote by ℓ_{ij} the line, considered also as polynomial of degree 1 in x and y, through a_i and a_j. The lines ℓ_{03} and ℓ_{12} are parallel and meet at a point A on the line at infinity. The lines ℓ_{14} and ℓ_{05} are parallel and meet at a point B on the line at infinity (here we use the fact that $n > 6$). The lines ℓ_{34} and ℓ_{25} are parallel and meet at a point C on the line at infinity.

Consider the two triples of lines $\ell_{03}, \ell_{14}, \ell_{25}$ and $\ell_{05}, \ell_{12}, \ell_{34}$. These two tirples lines meet at nine points: $a_0, \ldots, a_5$ and A, B, C.

We will use a generalization of Pappus theorem called Chasles theorem [1]. This classical result states that if three lines intersect three other lines in 9 points, then any cubic curve passing through 8 of the intersection points must pass also through the ninth. See Theorem 4.1 in [3] for more details about the history of this result and more references. Let $D(x,y)$ denote the quadric D as a polynomial in x and y. Let ℓ^* denote the line at infinity. Therefore, the polynomial $D(x,y)\ell^*$ passes through all nine points $a_0, \ldots, a_4$ and A, B, C. Therefore, by Chasles theorem, $D(x,y)\ell^*$ passes also through a_5. Because a_5 does not lie on ℓ^* we conclude that D passes through $a_0, \ldots, a_5$, as desired.

Having shown that the points in $V \cup (-V)$ lie on a quadric D we claim that D is an ellipse. Indeed, notice that D and $-D$ intersect in $m = 2(n-1)$ points. For $n > 3$ this is possible only if $D = -D$. This shows that D is not a parabola. If it is a hyperbola, then O must be the center of it but then the points of $V \cup (-V)$ cannot lie in convex position for $n > 3$.

Therefore, D must be an ellipse (as it is also easy to see that it cannot be contained in a union of two lines). By applying a linear transformation, we may assume that D is a circle. Because for every i the edge a_ia_{i+1} is parallel to the line through a_{i-1} and a_{i+2} we conclude that the distance between a_i and a_{i-1} is equal to the distance between a_{i+1} and a_{i+2}. This, together with the fact that Q is centrally symmetric, imply that all the distances a_ia_{i+1} are equal. Hence Q is a regular polygon centered at the origin and consequently V is the set of vertices of a regular $(n-1)$-gon, centered at the origin. Therefore, we have $\sum_{i=1}^{n-1} v_i = 0$. Recall that $\sum_{i=1}^{n} v_i = 0$. From here we reach the desired contradiction that $v_n = 0$, contrary to our assumptions. ∎

We leave it as a puzzle (open puzzle) to the reader to solve the case $n = 6$. We can formulate a more precise question about this case: Suppose $a_0, a_1, \ldots, a_9$ are 10 vertices of a centrally symmetric convex polygon Q, indexed according to their clockwise order on the boundary of Q. Assume that for every $0 \leq i \leq 9$ and the diagonal $a_{i-1}a_{i+2}$ is parallel to a_ia_{i+1} (and therefore also to $a_{i+5}a_{i-4}$ and to $a_{i+4}a_{i-3}$ because Q is centrally symmetric). Does it imply that $a_0, \ldots, a_9$ lie on a quadric (in fact an ellipse)?

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