A one-page solution of a problem of Erdős and Purdy

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Abstract

The following theorem was conjectured by Erdős and Purdy: Let P be a set of $n > 4$ points in general position in the plane. Suppose that R is a set of points disjoint from P such that every line determined by P passes through a point in R. Then $|R| \geq n$.

In this paper we give a very elegant and elementary proof of this, being a very good candidate for the "book proof" of this conjecture.

1 Introduction

A beautiful result of Motzkin [7], Rabin, and Chakerian [3] states that any set of non-collinear red and blue points in the plane determines a monochromatic line. Grünbaum and Motzkin [5] initiated the study of biased coloring, that is, coloring of the points such that no purely blue line is determined. The intuition behind this study is that if the number of blue points is much larger than the number of red points, then, unless the set of blue points is collinear, the set of blue and red points should determine a monochromatic blue line.

The same problem was independently considered by Erdős and Purdy [4] who stated it in a slightly different way.

Let P be a set of n points in the plane. Erdős and Purdy asked the following question in [4]: Assume a set R of points in the plane is disjoint from P and has the property that every line determined by P passes through a point in R. How small can be the cardinality of R in terms of n?

Clearly, if P is contained in a line, then R may consist of just one point. Therefore, the question of Erdős and Purdy is about sets P that are not collinear. The best known lower bound for this question is given in [9], where it is shown that $|R| \geq n/3$.

In [4] Erdős and Purdy considered the same problem in the case where the set P is in *general position* in the sense that no three points of P are collinear. In this case if n is odd the tight bound $|R| \geq n$ is almost trivial because every point in R may be incident to at most $\frac{n-1}{2}$ of the $\binom{n}{2}$ lines determined by P. To observe that this bound is tight, let P be the set of vertices of a regular n-gon and let R be the set of n points on the line at infinity that correspond to the directions of the edges (and diagonals) of P. This construction is valid also when n is even.

If n is even, a trivial counting argument shows that $|R|$ must be at least $n-1$. This is because every point in R may be incident to at most $n/2$ lines determined by P. This trivial lower bound for $|R|$ is in fact sharp in the cases $n = 2$ and $n = 4$, as can be seen in Figure 1. Is the bound $|R| \geq n-1$ sharp also for larger values of n?

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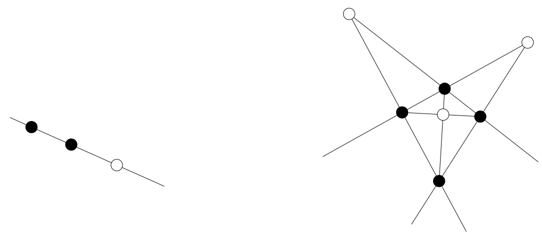

Figure 1: Constructions with $|R| = n-1$ for $n = 2, 4$. The points in P are colored black while the points in R are colored white.

The following conjecture is attributed to Erdős and Purdy [4].

Conjecture 1 (EP78)**.** *Let P is a set of n points in general position in the plane, where $n > 4$ is even. Assume R is another set of points disjoint from P such that every line through two points of P contains a point from R. Then $|R| \geq n$.*

Conjecture 1 was first proved in [1] (see Theorem 8 there), as a special case of the solution of the Magic Configurations conjecture of Murty [8]. The proof in [1] contains a topological argument based on Euler's formula for planar maps and the discharging method. An elementary (and long) proof of Conjecture 1 was given by Milićević in [6]. An algebraic proof of Conjecture 1 is given in [10]. Conjecture 1 was considered also over $\mathbb{F}_p$ and there it was proved by Blokhuis, Marino, and Mazzocca [2].

In this paper we provide a short, elementary, and completely different proof of Conjecture 1. This one-page proof is a good candidate for being the "book proof" of this result.

2 The Proof

Under the contrary assumption we must have $|R| = n-1$. Then it must be that every point in R is incident to precisely $n/2$ lines determined by P and every line determined by P is incident to precisely one point in R. We would like to reach a contradiction.

We apply a projective transformation to the plane such that the convex hull of P becomes a triangle. This can always be done for example by taking the line at infinity to be a line whose dual point lies in a triangular face of the arrangement of lines dual to the points in P. As a result of this, observe that no point of R lies outside the convex hull of P. Indeed, such a point r should be incident to two lines supporting the convex hull of P. Each of these supporting lines must contain precisely two points of P. This is impossible if the convex hull of P is a triangle.

We regard the points in the plane as complex numbers. For every point p in the plane we denote by p_x the x-coordinate of p.

For every point $p \in P$ define $f(p) = \frac{\prod_{r \in R}(p-r)}{\prod_{p' \in P \setminus \{p\}}(p-p')}$. Notice that $f(p)$ is in fact a real number for every $p \in P$. This is because for every $p' \in P \setminus \{p\}$ there is a unique $r \in R$ such that p, p', and r are collinear. Then $\frac{p-r}{p-p'}$ is real. Moreover, this expression, and therefore also $f(p)$, is invariant under rotations of the plane.

Observe that if the vertical line through p does not contain any other point in $P \cup R$, then $f(p) = \frac{\prod_{r \in R}(p_x - r_x)}{\prod_{p' \in P \setminus \{p\}}(p_x - p'_x)}$. This is because for every point $p' \in P \setminus \{p\}$ there is a unique point $r \in R$ such that p, p', and r are collinear. Hence, $\frac{p-r}{p-p'} = \frac{p_x - r_x}{p_x - p'_x}$.

The crucial observation that follows from here is the following. Let p and q be any two points in P and let r be the unique point in R collinear with p and q. Then

$$\frac{f(p)}{f(q)} = -\frac{p-r}{q-r}. \qquad (1)$$

Indeed, rotate the plane so that the x-coordinates of p, q, and r are equal (or nearly equal and take the limit) and use the fact that $f(p) = \frac{\prod_{r' \in R}(p_x - r'_x)}{\prod_{p' \in P \setminus \{p\}}(p_x - p'_x)}$ and $f(q) = \frac{\prod_{r' \in R}(q_x - r'_x)}{\prod_{p' \in P \setminus \{q\}}(q_x - p'_x)}$.

Notice in particular that $\frac{f(p)}{f(q)} > 0$ iff the unique point $r \in R$ that is collinear with p and q lies in the straight line segment delimited by p and q.

Consider the complete graph G whose vertices are the points in P. For every two points $p, p' \in P$ we color the edge connecting p and p' red if the unique point in R collinear with p and p' lies outside the straight line segment connecting p and p', that is if $\frac{f(p)}{f(p')} < 0$. Otherwise we color this edge green, in which case $\frac{f(p)}{f(p')} > 0$.

The vertices of G can naturally be partitioned into two sets: $V_1 = \{p \in P \mid f(p) > 0\}$ and $V_2 = \{p \in P \mid f(p) < 0\}$.

Notice that the edges connecting two vertices of V_1 or two vertices of V_2 are green and the edges connecting a vertex of V_1 and a vertex of V_2 are red.

Recall that the convex hull of P is a triangle and no point of R lies outside the convex hull of P. Consequently, all the three edges of the convex hull of P are green. Without loss of generality assume that the three vertices of the convex hull of P belong to V_1. We claim that $|V_1| = \frac{n}{2} + 1$.

To see this let $a, b \in V_1$ be two vertices of the convex hull of P. Let $c \in R$ be the point in R collinear with a and b. The point c must be between a and b on the boundary of the convex hull of P. Therefore, all the other $\frac{n}{2} - 1$ pairs of points of P that are collinear with c must form red edges. Consequently, precisely one point of every such pair belongs to V_1. Together with a and b, we obtain $|V_1| = \frac{n}{2} + 1$.

Consider now any triangulation of the convex hull of P using all the vertices of V_1. This triangulation contains precisely $3|V_1| - 6$ edges all of which correspond to green edges in the graph G. Each edge in the triangulation contains a unique point in R. Therefore, $|R| \geq 3|V_1| - 6 = 3\frac{n}{2} - 3$. This expression is greater than or equal to n for every $n \geq 6$, as desired. ∎

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