On a problem of Felsner about quadrant-depth

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Abstract

Let P be a set of n points in general position in the plane. For every $x \in P$ let $D(x,P)$ be the maximum number such that there are at least $D(x,P)$ points of P in each of two opposite quadrants determined some two perpendicular lines through x. Define $D(P) = \max_{x \in P} D(x,P)$. In this paper we show that $D(P) \geq c|P|$ for every set P in general position in the plane where c is an absolute constant that is strictly greater than $\frac{1}{8}$. This answers a question raised by Stefan Felsner.

1 Introduction

In [BLP], Brönnimann, Lenchner, and Pach define the notion of *opposite-quadrant* depth of a point in a point set. Specifically, let P be a set of n points in general position in the plane. The *opposite-quadrant* depth of a point x in P is the maximum number $opp(x,P)$ such that the horizontal and vertical lines through x define two opposite closed quadrants each containing at least $opp(x,P)$ points of P. The main result established in [BLP] is that any set P of n points in general position in the plane has a point $x \in P$ such that $opp(x,P) \geq \frac{1}{8}|P|$ and that this bound is best possible.

The following theorem from [BFPZ] generalizes the main result in [BLP]:

Theorem 1 ([BFPZ])**.** *Let P be a set of n points in general position in the plane. Then there exists a point $p \in P$ such that the four (closed) quadrants Q_1, Q_2, Q_3, and Q_4, determined by the horizontal and vertical lines through p (where Q_1 and Q_3 are opposite) satisfy:*

$$\min(|Q_1 \cap P|, |Q_3 \cap P|) + \min(|Q_2 \cap P|, |Q_4 \cap P|) \geq \frac{1}{4}(|P| - 1).$$

In an attempt to find an interesting variant of the opposite-quadrant problem, one can consider the following definition. Let P be a set of n points in general position in the plane. For $x \in P$ we define $D(x,P)$ to be the maximum number such that there are at least $D(x,P)$ points of P in each of two opposite quadrants determined by some two perpendicular lines through x. Let $D(P) = \max_{x \in P} D(x,P)$.

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Clearly, for any set P and a point $x \in P$, we have $opp(x, P) \leq D(x, P)$. Therefore, by Theorem 1, $D(P) \geq \frac{1}{8}|P|$ for every set P of points in general position in the plane. The following problem was raised by Stefan Felsner ([F]). Do there exist sets P such that $D(P) = \frac{1}{8}|P|$, or does there exist a constant $c > \frac{1}{8}$ such that $D(P) \geq c|P|$ for every set P ?

We note already that despite the similarity to classical partition theorems in the plane and higher dimensions by perpendicular hyper-planes (see for example [M85] and [W82]), the condition on the perpendicular lines to pass through a point of the set makes the problem completely different and in a way more difficult.

The following simple construction shows that there are sets P for which $D(x, P) \leq \frac{1}{7}n$ for every $x \in P$. Let Q be a regular 7-gon centered at the origin. Very close to each vertex v of Q position $n/7$ points along a line through the origin and v. In order for the resulting set of points to be in general position, slightly perturb the points just to avoid unnecessary collinearities. It is left to the reader to verify that that resulting set P of n points satisfies $D(x, P) \leq n/7 + 1$ for every $x \in P$. See Figure 1 and imagine the points corresponding to each vertex of Q much closer to each other and to the respective vertex of Q.

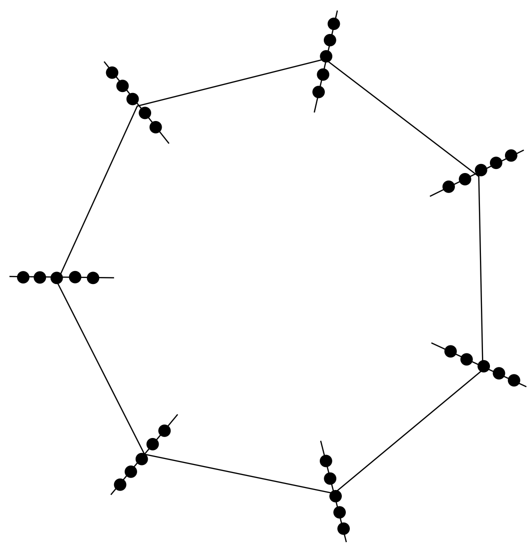

Figure 1: A set P with $D(x, P) \geq n/7$ for every $x \in P$.

The main result in this paper is the existence of a constant c strictly greater than $\frac{1}{8}$ such that $D(P) \geq c|P|$ for every set P.

Theorem 2. *For every finite set P of points in general position in the plane we have $D(P) \geq (\frac{1}{8} + \frac{1}{8 \cdot 39})|P|$.*

2 Proof of Theorem 2

In what follows P will denote a set of n points in the plane. In order to simplify the presentation we will sometimes assume without explicitly saying so that n is divisible by high enough power of 2, and in fact assuming that n is divisible by 4 will be enough.

For a directed line ℓ we denote by $H_L(\ell)$ the open half-plane bounded by ℓ which lies to the left of ℓ. Similarly, $H_R(\ell)$ denotes the open half-plane bounded by ℓ which lies to the right of ℓ.

Let ℓ_1 and ℓ_2 be two directed lines that meet at a single point. The *front wedge* determined by ℓ_1 and ℓ_2 is the set $H_R(\ell_1) \cap H_L(\ell_2)$. We denote this region by $W_F(\ell_1, \ell_2)$. Similarly, the *back wedge* determined by ℓ_1 and ℓ_2 is the set $H_L(\ell_1) \cap H_R(\ell_2)$ which we denote by $W_B(\ell_1, \ell_2)$. In a similar manner we define $W_L(\ell_1, \ell_2) = H_L(\ell_1) \cap H_L(\ell_2)$ and $W_R(\ell_1, \ell_2) = H_R(\ell_1) \cap H_R(\ell_2)$, the *left wedge* and the *right wedge*, respectively, determined by ℓ_1 and ℓ_2 (see Figure 2).

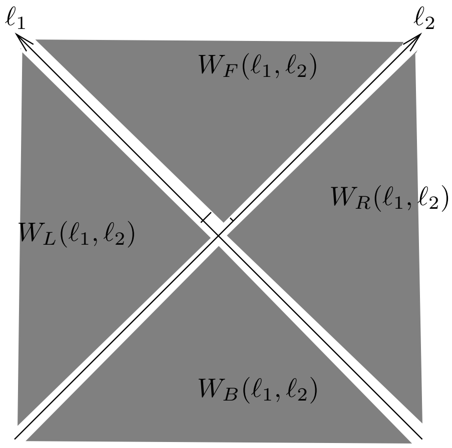

Figure 2: α-double-wedge $W(\ell_1, \ell_2)$

Lemma 1. *Let P be a set of n points in general position in the plane. Let ℓ_1 and ℓ_2 be two perpendicular directed lines and assume that $|H_R(\ell_1) \cap P| \leq \frac{1}{4}n$ and $|H_L(\ell_2) \cap P| \leq \frac{1}{4}n$. Then there exists a point $x \in P$ such that $D(x, P) \geq \frac{1}{8}n + \frac{1}{8}|P \cap W_F(\ell_1, \ell_2)|$.*

Proof. Let $t = |P \cap W_F(\ell_1, \ell_2)|$. Rotate the plane so that ℓ_1 is horizontal and points in the same direction of the negative part of the x-axis, while ℓ_2 is vertical and points in the direction of the positive part of the y-axis.

Let ℓ_1' be a line parallel to ℓ_1 such that $|H_R(\ell_1') \cap P| = \frac{1}{4}n$. Observe that because $|H_R(\ell_1) \cap P| < \frac{1}{4}n$, then ℓ_1' lies to the left of ℓ_1. Let ℓ_1'' be a line parallel to ℓ_1 such that $|H_L(\ell_1'') \cap P| = \frac{1}{4}n$. Let ℓ_2' be a line parallel to ℓ_2 such that $|H_L(\ell_2') \cap P| = \frac{1}{4}n$. Observe that because $|H_L(\ell_2) \cap P| < \frac{1}{4}n$, then ℓ_2' lies to the right of ℓ_2. Let ℓ_2'' be a line parallel to ℓ_2 such that $|H_R(\ell_2'') \cap P| = \frac{1}{4}n$. See Figure 3.

Because ℓ_1' lies to the left of ℓ_1 and ℓ_2' lies to the right of ℓ_2, we have $W_F(\ell_1', \ell_2') \supset W_F(\ell_1, \ell_2)$. Hence, $|P \cap W_F(\ell_1', \ell_2')| \geq t$. Let P' denote the set of points of P that are not in $H_R(\ell_1') \cup H_L(\ell_1'') \cup H_L(\ell_2') \cup H_R(\ell_2'')$. It follows that $|P'| \geq t$.

By Theorem 1, there exists a point $x \in P'$ such that the four quadrants Q_1, Q_2, Q_3, and Q_4, determined by the vertical and horizontal lines through x, satisfy

$$\min(|Q_1 \cap P'|, |Q_3 \cap P'|) + \min(|Q_2 \cap P'|, |Q_4 \cap P'|) \geq \frac{1}{4}|P'| \geq \frac{1}{4}(t-1).$$

Let $P'' = P \setminus P'$ and let $s = \min(|Q_1 \cap P''|, |Q_3 \cap P''|)$. Assume without loss of generality that $s = |Q_1 \cap P''|$. Observe that $Q_1 \cup Q_2 \supset H_R(\ell_1')$ and therefore $|(Q_1 \cup Q_2) \cap P''| \geq |H_R(\ell_1') \cap P''| = \frac{1}{4}n$. This implies that $|Q_2 \cap P''| \geq \frac{1}{4}n - s$. Similarly, $Q_1 \cup Q_4 \supset H_R(\ell_2'')$ and therefore $|(Q_1 \cup Q_4) \cap P''| \geq$

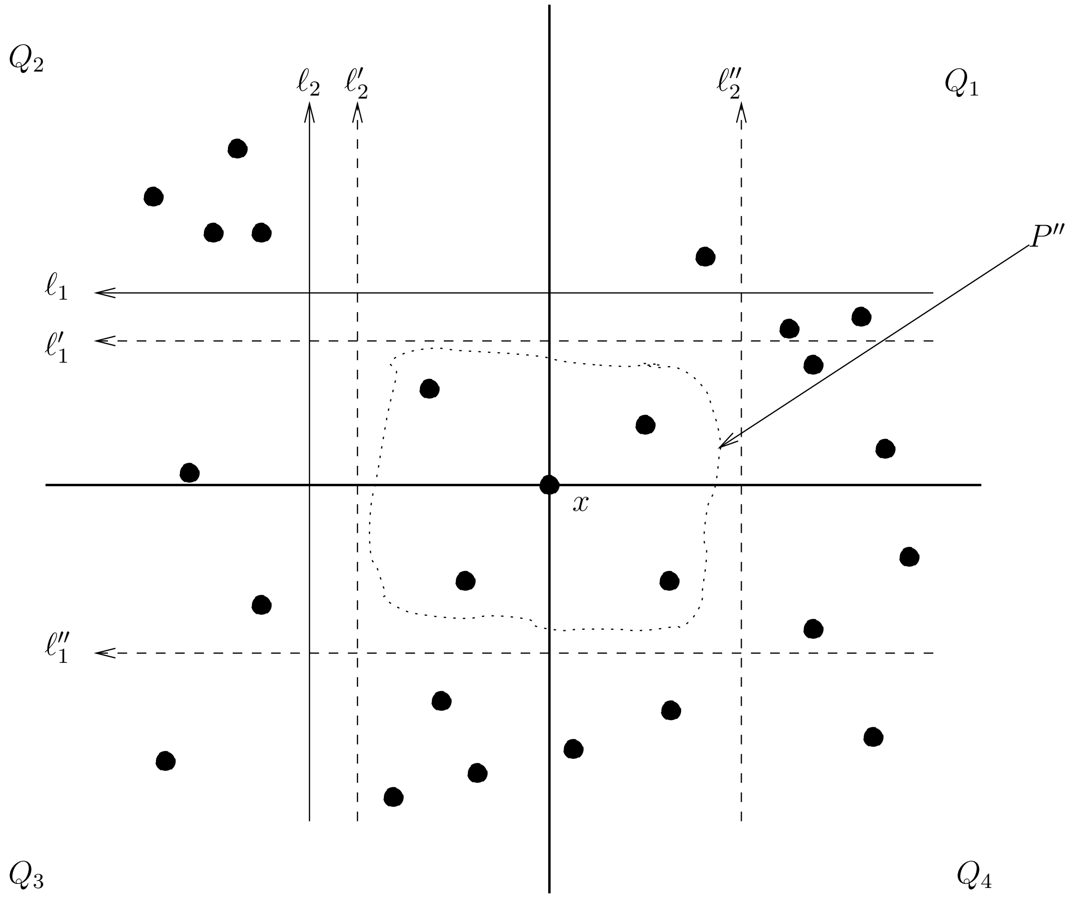

Figure 3: Lemma 1

$|H_R(\ell_2'') \cap P''| = \frac{1}{4}n$. This implies that $|Q_4 \cap P''| \geq \frac{1}{4}n - s$. Hence, $\min(|Q_2 \cap P''|, |Q_4 \cap P''|) \geq \frac{1}{4}n - s$ and we have

$$\min(|Q_1 \cap P''|, |Q_3 \cap P''|) + \min(|Q_2 \cap P''|, |Q_4 \cap P''|) \geq \frac{1}{4}n.$$

We deduce the following:

$$\begin{aligned}
\min(|Q_1 \cap P|, |Q_3 \cap P|) &+ \min(|Q_2 \cap P|, |Q_4 \cap P|) = \\
&= \min(|Q_1 \cap P'| + |Q_1 \cap P''|, |Q_3 \cap P'| + |Q_3 \cap P''|) + \\
&+ \min(|Q_2 \cap P'| + |Q_2 \cap P''|, |Q_4 \cap P'| + |Q_4 \cap P''|) \geq \\
&\geq \min(|Q_1 \cap P'|, |Q_3 \cap P'|) + \min(|Q_1 \cap P''|, |Q_3 \cap P''|) + \\
&+ \min(|Q_2 \cap P'|, |Q_4 \cap P'|) + \min(|Q_2 \cap P''|, |Q_4 \cap P''|) \geq \\
&\geq \frac{1}{4}n + \frac{1}{4}(t-1).
\end{aligned}$$

It follows that either each of Q_1 and Q_3 contains $\frac{1}{8}n + \frac{1}{8}t$ points of P, or each of Q_2 and Q_4 contains $\frac{1}{8}n + \frac{1}{8}t$ points of P, as desired. ∎

Corollary 1. *Let P be a set of n points in general position in the plane. Let ℓ_1 and ℓ_2 be two perpendicular directed lines with $|H_R(\ell_1) \cap P| = a$, $|H_L(\ell_2) \cap P| = b$, and $|W_F(\ell_1, \ell_2) \cap P| = c$. Then there exists a point $x \in P$ such that $D(x, P) \geq \frac{1}{8}n + \frac{1}{8}(c - \max(a - \frac{1}{4}n, 0) - \max(b - \frac{1}{4}n, 0))$.*

Proof. Let ℓ'_1 be a line that is parallel to ℓ_1 such that $|H_R(\ell'_1) \cap P| = \frac{1}{4}n$. Let ℓ'_2 be a line that is parallel to ℓ_2 such that $|H_L(\ell'_2) \cap P| = \frac{1}{4}n$. We have $|P \cap W_F(\ell'_1, \ell'_2)| \geq c - \max(a - \frac{1}{4}n, 0) - \max(b - \frac{1}{4}n, 0)$. The result now follows by applying Lemma 1 to the lines ℓ'_1 and ℓ'_2. ∎

Returning to the proof of Theorem 2, let P be a set of n points in general position in the plane and assume that for every $x \in P$ we have $D(x, P) < (\frac{1}{8} + \epsilon)n$, where $\epsilon > 0$ will be determined later to obtain a contradiction. We will choose ϵ that is not greater than $\frac{1}{8 \cdot 39}$.

Fix x in P and consider all pairs of perpendicular directed lines ℓ_1 and ℓ_2 through x. It follows easily by a continuity argument that there are two perpendicular directed lines $\ell_1(x)$ and $\ell_2(x)$ through x such that

$$|P \cap W_L(\ell_1(x), \ell_2(x))| = |P \cap W_R(\ell_1(x), \ell_2(x))|.$$

Let t denote the cardinality $t = |P \cap W_L(\ell_1(x), \ell_2(x))|$. Because $D(x, P) < (\frac{1}{8} + \epsilon)n$, we have $t < (\frac{1}{8} + \epsilon)n$. Moreover, at least one of $|P \cap W_F(\ell_1(x), \ell_2(x))|$ and $|P \cap W_B(\ell_1(x), \ell_2(x))|$ must be smaller than $(\frac{1}{8} + \epsilon)n$. Without loss of generality assume that $s = |P \cap W_B(\ell_1(x), \ell_2(x))| < (\frac{1}{8} + \epsilon)n$.

By Corollary 1, there exists a point $y \in P$ with $D(y, P) \geq \frac{1}{8}n + \frac{1}{8}(s - 2\max(s - \frac{1}{8}n + \epsilon n, 0))$. Therefore, as $D(y, P) < (\frac{1}{8} + \epsilon)n$, we deduce that

$$\frac{1}{8}(s - 2\max(s - \frac{1}{8}n + \epsilon n, 0)) < \epsilon n. \tag{1}$$

If $s > \frac{1}{8}n - \epsilon n$, then we get $\frac{1}{8}(s - 2(s - \frac{1}{8}n + \epsilon n)) < \epsilon n$. This implies that $s > \frac{1}{4}n - 10\epsilon n$. On the other hand we know that $s < \frac{1}{8}n + \epsilon n$. Hence, $\frac{1}{8} < 11\epsilon$ which contradicts for the choice of ϵ.

Therefore, we necessarily have $s < \frac{1}{8}n - \epsilon n$. From (1) we deduce now that $s < 8\epsilon n$.

This implies that $|P \cap W_F(\ell_1(x), \ell_2(x))| > (\frac{3}{4} - 10\epsilon)n$. In particular, for every $x, y \in P$ the interior of $W_F(\ell_1(x), \ell_2(x))$ must intersect with the interior of $W_F(\ell_1(y), \ell_2(y))$, as $(\frac{3}{4} - 10\epsilon)n > \frac{1}{2}n$.

Let $P_1 \subset P$ be the set of all points $x \in P$ such that the vertical line through x cuts through $W_F(\ell_1(x), \ell_2(x))$ (and therefore also through $W_B(\ell_1(x), \ell_2(x))$). The complementary set $P_2 = P \setminus P_1$ consists of all those points $x \in P$ such that the horizontal line through x cuts through $W_F(\ell_1(x), \ell_2(x))$ (and therefore also through $W_B(\ell_1(x), \ell_2(x))$). By suitably rotating the plane we may assume, by continuity arguments, that $|P_1| = |P_2| = \frac{1}{2}n$. Indeed, consider the lines $\ell_1(x)$ and $\ell_2(x)$ fixed for every $x \in P$, and note that we may assume that these each of these lines has a unique direction. Start rotating the plane and observe that the cardinality of P_1 changes by at most one unit at each time. Therefore, we can reach a position where $|P_1| = \frac{1}{2}n$.

We denote the points of P_1 by $x_1, \ldots, x_{\frac{n}{2}}$ according to the decreasing order of their y-coordinates. For every point $x \in P_1$ $W_F(\ell_1(x), \ell_2(x))$ is contained either in the half-plane below the horizontal line through x, or in the half-plane above the horizontal line through x. In the former case we say that x *points downwards* and in the latter case we say that x *points upwards*.

Observe that x_1 must point downwards. This is because if x_1 points upwards, then it is impossible for $W_F(\ell_1(x_1), \ell_2(x_1))$ to contain $(\frac{3}{4} - 10\epsilon)n$ points of P, as it does not contain any point of P_1. Here we use the assumption that $10\epsilon < \frac{1}{4}$, and so $(\frac{3}{4} - 10\epsilon)n > \frac{1}{2}n$. Similarly, $x_{\frac{n}{2}}$ must point upwards. Therefore, there exists an index $1 \leq k < \frac{n}{2}$, such that x_k points downwards but

x_{k+1} points upwards. Because x_k points downwards, $W_F(\ell_1(x_k), \ell_2(x_k))$ cannot contain any of the points $x_1, \ldots, x_k$. Therefore, $W_F(\ell_1(x_k), \ell_2(x_k))$ contains at most $n-k$ points of P. On the other hand we know that $W_F(\ell_1(x_k), \ell_2(x_k))$ contains more than $\frac{3}{4}n - 10\epsilon n$ points of P. Hence, $k < (\frac{1}{4} + 10\epsilon)n$. Similarly, x_{k+1} points upwards, and therefore $W_F(\ell_1(x_{k+1}), \ell_2(x_{k+1}))$ contains at most $n - (\frac{n}{2} - k) = \frac{n}{2} + k$ points of P. Hence, $k \geq (\frac{1}{4} - 10\epsilon)n$.

Let m be the horizontal line through x_k. We already observed that $|P \cap W_B(\ell_1(x_k), \ell_2(x_k))| \leq 8\epsilon n$. On the other hand, the half-plane bounded above m contains k points of P_1. It follows that there are at least $k - 8\epsilon n \geq (\frac{1}{4} - 18\epsilon)n$ points of P_1 above m but outside of $W_B(\ell_1(x_k), \ell_2(x_k))$. Note that the area above m but outside of $W_B(\ell_1(x_k), \ell_2(x_k))$ is included in the union of $W_L(\ell_1(x_k), \ell_2(x_k))$ and $W_R(\ell_1(x_k), \ell_2(x_k))$, and each of $W_L(\ell_1(x_k), \ell_2(x_k))$ and $W_R(\ell_1(x_k), \ell_2(x_k))$ contains less than $(\frac{1}{8} + \epsilon)n$ points of P. Denote by A_L the area above m that is contained in $W_L(\ell_1(x_k), \ell_2(x_k))$. Denote by A_R the area above m that is contained in $W_R(\ell_1(x_k), \ell_2(x_k))$. Therefore, there are more than $(\frac{1}{8} - 19\epsilon)n$ points of P_1 in each of A_L and A_R.

In a similar way one defines m' to be the horizontal line through x_{k+1} and let B_L denote the area below m' that is contained in $W_L(\ell_1(x_{k+1}), \ell_2(x_{k+1}))$, and let B_R denote the area below m' that is contained in $W_R(\ell_1(x_{k+1}), \ell_2(x_{k+1}))$. Then one shows that there are at least $(\frac{1}{8} - 19\epsilon)n$ points of P_1 in each of B_L and B_R. See Figure 4.

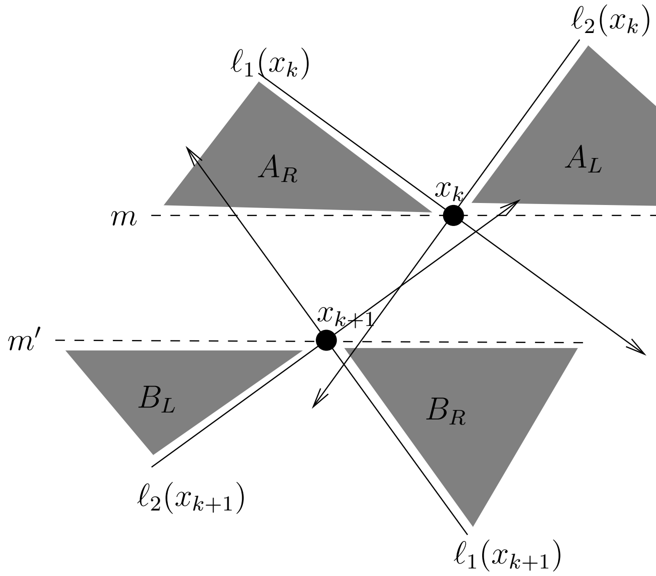

Figure 4: If $x_{k+1} \in W_R(\ell_1(x_k), \ell_2(x_k))$, then $B_L \cap W_F(\ell_1(x_k), \ell_2(x_k)) = \emptyset$

Claim 1. *x_{k+1} must be contained in the open interior of $W_F(\ell_1(x_k), \ell_2(x_k))$.*

Proof. Assume to the contrary that x_{k+1} is not contained in the interior of $W_F(\ell_1(x_k), \ell_2(x_k))$. Then without loss of generality x_{k+1} is contained in $W_R(\ell_1(x_k), \ell_2(x_k))$. Because $W_F(\ell_1(x_k), \ell_2(x_k))$ intersects with the interior of $W_F(\ell_1(x_{k+1}), \ell_2(x_{k+1}))$, $\ell_2(x_{k+1})$ must lie below x_k (see Figure 4). Now it is easy to see the B_L is disjoint from $W_F(\ell_1(x_k), \ell_2(x_k))$. This is impossible because B_L contains more than $(\frac{1}{8} - 19\epsilon)n$ points of P_1 that together with the k points of P_1 above m sum up to more than $(\frac{3}{8} - 29\epsilon)n$ points of P_1 non of which lies in $W_F(\ell_1(x_k), \ell_2(x_k))$ which by itself contains at least $(\frac{3}{4} - 10\epsilon)n$ points of P. We thus get a total of $(\frac{9}{8} - 39\epsilon)n > n$ points in P which contradicts the choice of $\epsilon < \frac{1}{8 \cdot 39}$. ∎

In a symmetric manner one can show the following:

Claim 2. *x_k must be contained in the open interior of $W_F(\ell_1(x_{k+1}), \ell_2(x_{k+1}))$.*

Next we argue symmetrically about P_2. Specifically, we denote the points of P_2 by $y_1, \ldots, y_{\frac{n}{2}}$ according to the increasing order of their x-coordinates. For any point $y \in P_2$, $W_F(\ell_1(y), \ell_2(y))$ is contained either in the half-plane to the left of the horizontal line through y, or in the half-plane to the right of the horizontal line through y. In the former case we say that *y points to the left* and in the latter case we say that *y points to the right*.

Here too, there exists an index $1 \leq j < \frac{n}{2}$, such that y_j points to the right but y_{j+1} points to the left. Just like in the case of the index k, one can show that $(\frac{1}{4} - 10\epsilon)n \leq j < (\frac{1}{4} + 10\epsilon)n$. Let w be the vertical line through y_j. Denote by C_L the area to the left of w that is contained in $W_L(\ell_1(y_j), \ell_2(y_j))$. Denote by C_R the area to the left of w that is contained in $W_R(\ell_1(y_j), \ell_2(y_j))$. Then one can show that there are at least $(\frac{1}{8} - 19\epsilon)n$ points of P_2 in each of C_L and C_R.

In a similar way one defines w' to be the vertical line through y_{j+1} and let D_L denote the area to the right of w' that is contained in $W_L(\ell_1(y_{j+1}), \ell_2(y_{j+1}))$, and let D_R denote the area to the right of w' that is contained in $W_R(\ell_1(y_{j+1}), \ell_2(y_{j+1}))$. Then there are at least $(\frac{1}{8} - 19\epsilon)n$ points of P_2 in each of D_L and D_R.

Similar to Claims 1 and 2, one can show that y_{j+1} must be contained in the open interior of $W_F(\ell_1(y_j), \ell_2(y_j))$ and that y_j must be contained in the open interior of $W_F(\ell_1(y_{j+1}), \ell_2(y_{j+1}))$.

We need a final key observation to finish the proof of Theorem 2:

Claim 3. *y_j must be contained in the open interior of $W_F(\ell_1(x_k), \ell_2(x_k))$.*

Proof. Let u be the vertical line through x_k. We first show that y_j must lie to the left of u. Indeed, recall that y_j points to the right. This means that if y_j lies to the right of u, then $W_F(\ell_1(y_j), \ell_2(y_j))$ is disjoint from A_R. This is a contradiction because $W_F(\ell_1(y_j), \ell_2(y_j))$ contains more than $(\frac{3}{4} - 10\epsilon)n$ points of P, A_R contains more than $(\frac{1}{8} - 19\epsilon)n$ points of P_1 and to the left of u there are at least $j \geq (\frac{1}{4} - 10\epsilon)n$ points of P_2. Altogether this sums up to more than $(\frac{9}{8} - 39\epsilon)n \geq n$ which is an absurdity. Here we use the assumption that $\epsilon \leq \frac{1}{8 \cdot 39}$.

Therefore, if y_j is not contained in the interior of $W_F(\ell_1(x_k), \ell_2(x_k))$, then it must lie to the left of u and necessarily $W_F(\ell_1(x_k), \ell_2(x_k))$ is disjoint from C_L (see Figure 5). Again we obtain a contradiction for the choice of $\epsilon < \frac{1}{8 \cdot 39}$. ∎

Similar to Claim 3, one can show that both y_j and y_{j+1} must lie in the open interior of both $W_F(\ell_1(x_k), \ell_2(x_k))$ and $W_F(\ell_1(x_{k+1}), \ell_2(x_{k+1}))$. Also, both x_k and x_{k+1} must lie in the open interior of both $W_F(\ell_1(y_j), \ell_2(y_j))$ and $W_F(\ell_1(y_{j+1}), \ell_2(y_{j+1}))$.

We are now ready to complete the proof of Theorem 2. We claim that the four points x_k, x_{k+1}, y_j, and y_{j+1} must be in convex position. Indeed, this follows because each of these points lies on the boundary of a convex region that contains the three others. Here we mean that x_k lies on the boundary of $W_F(\ell_1(x_k), \ell_2(x_k))$ which contains x_{k+1}, y_j, and y_{j+1} and similarly we argue about each of x_{k+1}, y_j, and y_{j+1}.

Next, consider the convex quadrangle Q whose vertices are x_k, x_{k+1}, y_j, and y_{j+1}. The angle at each of the vertices must be acute. Indeed, consider x_k for example. We know that x_{k+1}, y_j, and y_{j+1} all lie in the *interior* of $W_F(\ell_1(x_k), \ell_2(x_k))$. This implies that the inner angle of Q at x_k must be acute. This is a contradiction because at least one of the four inner angles of any convex quadrangle must be non-acute. ∎

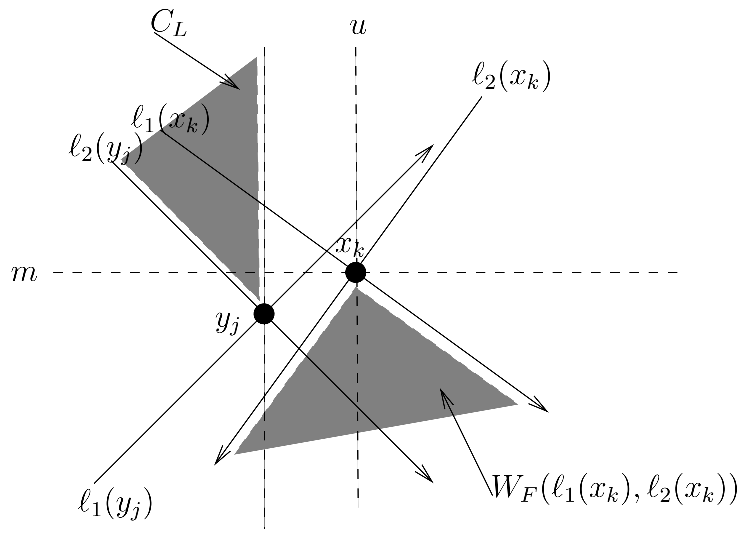

Figure 5: If $y_j \notin W_F(\ell_1(x_k), \ell_2(x_k))$, then $C_L \cap W_F(\ell_1(x_k), \ell_2(x_k)) = \emptyset$

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