Points with large α-depth

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Abstract

We show that for any $\epsilon > 0$ there exists an angle $\alpha = \alpha(\epsilon)$ between 0 and π, depending only on ϵ, with the following two properties: (1) For any continuous probability measure in the plane one can find two lines ℓ_1 and ℓ_2, crossing at an angle of (at least) α, such that the measure of each of the two opposite quadrants of angle $\pi - \alpha$, determined by ℓ_1 and ℓ_2, is at least $\frac{1}{2} - \epsilon$. (2) For any set P of n points in general position in the plane one can find two lines ℓ_1 and ℓ_2, crossing at an angle of (at least) α and moreover at a point of P, such that in each of the two opposite quadrants of angle $\pi - \alpha$, determined by ℓ_1 and ℓ_2, there are at least $(\frac{1}{2} - \epsilon)n - 4$ points of P.

1 Introduction

Let P be a set of points in general position in the plane. We would like to find a point $x \in P$ and two lines ℓ_1 and ℓ_2 passing through x that create an angle of α such that each of the two quadrant of angle $\pi - \alpha$, determined by ℓ_1 and ℓ_2, should contain a large portion of the points of P. We are interested in how small should α be to ensure that there exist such $x \in P$ and two portions of points of P whose cardinalities are at least as large as $(\frac{1}{2} - \epsilon)n$, for some $\epsilon > 0$. As we shall see below, for every ϵ there exists such an α that depends only on ϵ and not on the set P.

Similar notions were considered in [BLP]. Brönnimann, Lenchner, and Pach define in [BLP] the notion of an opposite-quadrant depth of a point in a point set. Specifically, let P be a set of n points in general position in the plane. The opposite-quadrant depth of a point x in P is the maximum number $opp(x, P)$ such that the horizontal an vertical lines through x define two opposite closed quadrants each containing at least $opp(x, P)$ points of P.

It is shown in [BLP] that any set P of n points in general position in the plane has a point $p \in P$ such that $opp(p, P) \geq \frac{1}{8}n$. In case the points of P are in convex position there is a point $p \in P$ such that $opp(p, P) \geq \frac{1}{4}n$. Both bounds are shown to be best possible for arbitrarily large values of n. This result can be shown to follow from a very short an elegant argument (see [BFPZ]).

In this paper we would like the two opposite quadrants to contain almost half of the points of P each and we are ready to pay with a larger angle than $\pi/2$ of the two quadrants.

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For a directed line ℓ we denote by $H_L(\ell)$ the open half-plane bounded by ℓ which lies to the left of ℓ. Similarly, $H_R(\ell)$ denotes the open half-plane bounded by ℓ which lies to the right of ℓ.

Let ℓ_1 and ℓ_2 be two directed lines that meet at a single point. The *front wedge* determined by ℓ_1 and ℓ_2 is the set $H_R(\ell_1) \cap H_L(\ell_2)$. We denote this region by $W_F(\ell_1, \ell_2)$. Similarly, the *back wedge* determined by ℓ_1 and ℓ_2 is the set $H_L(\ell_1) \cap H_R(\ell_2)$ which we denote by $W_B(\ell_1, \ell_2)$. In a similar manner we define $W_L(\ell_1, \ell_2) = H_L(\ell_1) \cap H_L(\ell_2)$ and $W_R(\ell_1, \ell_2) = H_R(\ell_1) \cap H_R(\ell_2)$, the *left wedge* and the *right wedge*, respectively, determined by ℓ_1 and ℓ_2 (see Figure 1).

The double-wedge determined by ℓ_1 and ℓ_2, denoted by $W(\ell_1, \ell_2)$, is the pair $W_L(\ell_1, \ell_2)$ and $W_R(\ell_1, \ell_2)$. The point of intersection of ℓ_1 and ℓ_2 is called the *apex* of $W(\ell_1, \ell_2)$. $W(\ell_1, \ell_2)$ is called an α-double-wedge if the angle between ℓ_1 and ℓ_2 equals α (here $0 < \alpha < \pi$). We define the *axis* of $W(\ell_1, \ell_2)$ to be the directed line that passes through the apex of $W(\ell_1, \ell_2)$ and creates an angle of $\frac{\alpha}{2}$ with each of ℓ_1 and ℓ_2 (see Figure 1).

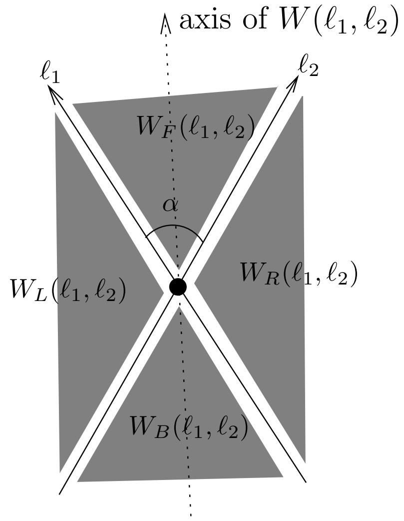


Figure 1: α-double-wedge $W(\ell_1, \ell_2)$

Let P be a finite set of points in the plane. We define the *α-depth* of a point $x \in P$ to be the maximum over all α-double-wedges $W(\ell_1, \ell_2)$ with apex x of the quantity $\min(|P \cap W_L(\ell_1, \ell_2)|, |P \cap W_R(\ell_1, \ell_2)|)$. We denote the α-depth of x by $D_\alpha(x, P)$. In other words the α-depth of $x \in P$ is the maximum number $D_\alpha(x, P)$ such that there exists an α-double-wedge $W(\ell_1, \ell_2)$ with at least $D_\alpha(x, P)$ points of P in each of the regions $W_L(\ell_1, \ell_2)$ and $W_R(\ell_1, \ell_2)$.

It is evident from the definition of $D_\alpha(x, P)$ that $D_\alpha(x, P)$ is always less than $|P|/2$, and that $D_\alpha(x, P) \leq D_{\alpha'}(x, P)$ for every $0 < \alpha' < \alpha < \pi$.

Our main result in this paper is the following theorem.

Theorem 1. *For every $0 < \epsilon$ there exists an angle $\alpha = \alpha(\epsilon)$ strictly between 0 and π, depending only on ϵ, such that any set P of n points in general position in the plane has a point $x \in P$ with $D_\alpha(x, P) \geq (\frac{1}{2} - \epsilon)n - 4$.*

One of the main steps in proving Theorem 1 is the following lemma of independent interest:

Lemma 1. *For every $\epsilon > 0$ there exists $0 < \alpha = \alpha(\epsilon) < \pi$, depending only on ϵ, such that for every continuous probability measure in the plane there exists an α-double-wedge $W(\ell_a, \ell_b)$ with $\mu(W_L(\ell_a, \ell_b)) \geq \frac{1}{2} - \epsilon$ and $\mu(W_R(\ell_a, \ell_b)) \geq \frac{1}{2} - \epsilon$.*

For sets of points in convex position the situation in Theorem 1 is much simpler and one can obtain improved estimates on $\alpha(\epsilon)$ as a function of ϵ:

Theorem 2. *Let $k \geq 1$ be a fixed integer such that $2k+1$ divides n. For any set P of n points in convex position there exists a point $x \in P$ with $D_{\alpha(k)}(x, P) \geq \frac{k}{2k+1}n$ where $\alpha(k) = \frac{\pi}{2k+1}$.*

Theorem 2 is proved in Section 2, Lemma 1 is proved in Section 3, and Theorem 1 is proved in Section 4.

2 The convex case: Proof of Theorem 2

In this section we analyze the situation where the points of P are in convex position, i.e., lie on the boundary of a convex shape in the plane.

Proof of Theorem 2. Let $k \geq 1$ be a fixed integer such that $2k+1$ divides n, the cardinality of P. Let $m = \frac{n}{2k+1}$ and let $p_0, \ldots, p_{n-1}$ be the points of P indexed in a counterclockwise cyclic order according to their convex position. We consider the points $x_i = p_{im}$, for $i = 0, 1, 2, \ldots, 2k$.

We claim that one of the angles $\measuredangle x_{i-k}x_ix_{i+k}$ is greater than or equal to $\frac{\pi}{2k+1}$ (the summation of indices is taken modulo $2k+1$). To see this consider the $(2k+1)$-gon Q whose vertices are $x_0, \ldots, x_{2k}$. The sum of the internal angles of this polygon is $\pi(2k-1)$. For every $i = 0, \ldots, 2k$ consider the $(k+1)$-gon Q_i whose vertices are $x_i, x_{i+1}, \ldots, x_{i+k}$. The sum of the internal angles of each of the polygons Q_i is $\pi(k-1)$. It is not hard to see that if we sum up all the sums of internal angles of all the Q_i's, then this equals k times the sum of the internal angles of Q, minus $\sum_{i=0}^{2k} \measuredangle x_{i-k}x_ix_{i+k}$.

Therefore,

$$\sum_{i=0}^{2k} \measuredangle x_{i-k}x_ix_{i+k} = \pi(2k-1)k - (2k+1)\pi(k-1) = \pi.$$

Hence, there exists $0 \leq i_0 < 2k+1$ such that $\measuredangle x_{i_0-k}x_{i_0}x_{i_0+k} \geq \frac{\pi}{2k+1}$.

Consider now the oriented lines $\ell_1 = \overrightarrow{x_{i_0}x_{i_0+k}}$ and $\ell_2 = \overrightarrow{x_{i_0}x_{i_0-k}}$. $W(\ell_1, \ell_2)$ is an α-double-wedge for $\alpha \geq \frac{\pi}{2k+1}$. Moreover $W_L(\ell_1, \ell_2)$ contains the points $p_{i_0m}, p_{i_0m+1}, \ldots, p_{(i_0+k)m}$ a total of $km + 1 > n(\frac{k}{2k+1})$ points of P. Similarly, $W_B(\ell_1, \ell_2)$ contains $n(\frac{k}{2k+1})$ points of P. Therefore, $D_\alpha(x_{i_0}, P) \geq n(\frac{1}{2} - \frac{1}{2(2k+1)})$ for every $0 < \alpha \leq \frac{\pi}{2k+1}$, as required.

Finally, observe that if the points of P are evenly arranged on a circle, then the above estimates cannot be improved. ∎

3 Proof of Lemma 1 and Theorem 1

Proof of Lemma 1: We need to show that for every $\epsilon > 0$ there exists $0 < \alpha = \alpha(\epsilon) < \pi$, depending only on ϵ, such that for every continuous probability measure in the plane, there exists an α-double-wedge $W(\ell_a, \ell_b)$ with $\mu(W_L(\ell_a, \ell_b)) \geq \frac{1}{2} - \epsilon$ and $\mu(W_R(\ell_a, \ell_b)) \geq \frac{1}{2} - \epsilon$.

Let μ be a continuous probability measure in the plane. We say that a line ℓ is a *halving line* with respect to μ if the measure μ of each of the two half-planes bounded by ℓ equals $\frac{1}{2}$. Fix an odd integer $N > 0$ that will be determined later and will depend only on ϵ. For every $k = 0, \ldots, N$ consider the directed line ℓ_k which is a halving line with respect to the measure μ and whose direction is the direction of the vector $(\cos \frac{\pi k}{N}, \sin \frac{\pi k}{N})$. Hence $\ell_0 = -\ell_N$.

If for some $0 \leq k \leq N-1$ there exists a directed line ℓ''_{k+1} parallel to ℓ_{k+1} such that $\mu(W_F(\ell''_{k+1}, \ell_k)) + \mu(W_B(\ell''_{k+1}, \ell_k)) \leq \epsilon$, then we are done. Indeed, let $\ell_a = \ell''_{k+1}$ and let ℓ_b be the directed line ℓ_k. Then $W(\ell_a, \ell_b)$ is an α-double-wedge for $\alpha = \frac{\pi}{N}$ and both $\mu(W_L(\ell_a, \ell_b))$ and $\mu(W_R(\ell_a, \ell_b))$ are greater than or equal to $\frac{1}{2} - \epsilon$ as required.

We assume, therefore, that for no $0 \leq k \leq N-1$ there exists such a line ℓ''_{k+1}.

Claim 1. *For every $0 \leq k \leq N-1$ there exists a line ℓ'_{k+1}, parallel to ℓ_{k+1} such that $\mu(W_F(\ell'_{k+1}, \ell_k)) > \frac{1}{2}\epsilon$ and $\mu(W_B(\ell'_{k+1}, \ell_k)) > \frac{1}{2}\epsilon$.*

Proof. Fix k between 0 and $N-1$. For every $t \in \mathbb{R}$ let $\ell_{k+1}(t)$ be the directed line, parallel to ℓ_{k+1}, that intersects the y-axis at the point $y = t$ (here we use the fact that N is odd and therefore ℓ_{k+1} is never parallel to the y-axis). Recall that by our assumption $\mu(W_F(\ell_{k+1}(t), \ell_k)) + \mu(W_B(\ell_{k+1}(t), \ell_k)) > \epsilon$ for every t.

Denote by $g(t)$ the continuous function $g(t) = \mu(W_F(\ell_{k+1}(t), \ell_k)) - \mu(W_B(\ell_{k+1}(t), \ell_k))$. For large enough t, $g(t)$ is negative because $\mu(W_F(\ell_{k+1}(t), \ell_k)$ goes to 0 as t goes to infinity and for every t we have $\mu(W_F(\ell_{k+1}(t), \ell_k)) + \mu(W_B(\ell_{k+1}(t), \ell_k)) > \epsilon$. Similarly, $g(t)$ is positive for small enough t. Let t_0 be such that $g(t_0) = 0$ and put $\ell'_{k+1} = \ell_{k+1}(t_0)$. Then we have $\mu(W_F(\ell'_{k+1}, \ell_k)) = \mu(W_B(\ell'_{k+1}, \ell_k))$ and $\mu(W_F(\ell'_{k+1}, \ell_k)) + \mu(W_B(\ell'_{k+1}, \ell_k)) > \epsilon$. Hence both $\mu(W_F(\ell'_{k+1}, \ell_k))$ and $\mu(W_B(\ell'_{k+1}, \ell_k))$ are greater than $\frac{1}{2}\epsilon$, as required. ∎

The following observation is very easy to verify. We leave it to the reader to complete the details and just sketch the proof.

Observation 1. *Let k and l be two distinct indices from $\{0, \ldots, N-1\}$. Then either one of $W_F(\ell'_{k+1}, \ell_k)$ and $W_B(\ell'_{k+1}, \ell_k)$ is disjoint from both $W_F(\ell'_{l+1}, \ell_l)$ and $W_B(\ell'_{l+1}, \ell_l)$, or one of $W_F(\ell'_{l+1}, \ell_l)$ and $W_B(\ell'_{l+1}, \ell_l)$ is disjoint from both $W_F(\ell'_{k+1}, \ell_k)$ and $W_B(\ell'_{k+1}, \ell_k)$.*

Proof. Without loss of generality assume that $k < l$. Let x denote the intersection point of ℓ_l and ℓ'_{l+1}. The proof proceeds by considering the possible locations of the point x (see Figure 2):

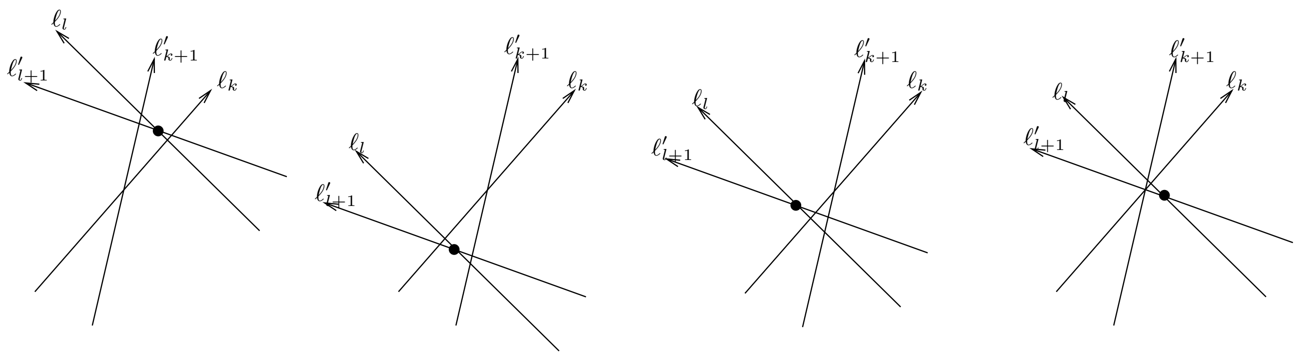
Case 1. $x \in W_F(\ell'_{k+1}, \ell_k)$. In this case $W_B(\ell'_{k+1}, \ell_k)$ is disjoint from both $W_F(\ell'_{l+1}, \ell_l)$ and $W_B(\ell'_{l+1}, \ell_l)$.

Case 2. $x \in W_B(\ell'_{k+1}, \ell_k)$. In this case $W_F(\ell'_{k+1}, \ell_k)$ is disjoint from both $W_F(\ell'_{l+1} \ell_l)$ and $W_B(\ell'_{l+1}, \ell_l)$.

Case 3. $x \in W_L(\ell'_{k+1}, \ell_k)$. In this case $W_F(\ell'_{l+1}, \ell_l)$ is disjoint from both $W_F(\ell'_{k+1}, \ell_k)$ and $W_B(\ell'_{k+1}, \ell_k)$.

Case 4. $x \in W_R(\ell'_{k+1}, \ell_k)$. In this case $W_B(\ell'_{l+1}, \ell_l)$ is disjoint from both $W_F(\ell'_{k+1}, \ell_k)$ and $W_B(\ell'_{k+1}, \ell_k)$. ∎

It follows from Observation 1 that for any collection $I \subset \{0, \ldots, N-1\}$ of indices there is some $i \in I$ and $J \subset I \setminus \{i\}$ with $|J| \geq \frac{1}{2}|I|$ such that for every $j \in J$ one of $W_F(\ell'_{i+1}, \ell_i)$ and $W_B(\ell'_{i+1}, \ell_i)$ is disjoint from both $W_F(\ell'_{j+1}, \ell_j)$ and $W_B(\ell'_{j+1}, \ell_j)$. Hence, one of $W_F(\ell'_{i+1}, \ell_i)$ and $W_B(\ell'_{i+1}, \ell_i)$ is



Case 1. Case 2. Case 3. Case 4.

Figure 2: Observation 1

disjoint from both $W_F(\ell'_{j+1}, \ell_j)$ and $W_B(\ell'_{j+1}, \ell_j)$ for at least $\frac{1}{2}|J|$ indices $j \in J$. We conclude that for any collection $I \subset \{0, \ldots, N-1\}$ of indices there is some $i \in I$ and $J \subset I \setminus \{i\}$ with $|J| \geq \frac{1}{4}|I|$ such that one of $W_F(\ell'_{i+1}, \ell_i)$ and $W_B(\ell'_{i+1}, \ell_i)$ is disjoint from both $W_F(\ell'_{j+1}, \ell_j)$ and $W_B(\ell'_{j+1}, \ell_j)$ for every $j \in J$.

It now follows that there are at least $\lfloor \log_4 N \rfloor$ sets, each equals $W_B(\ell'_{k+1}, \ell_k)$ or $W_F(\ell'_{k+1}, \ell_k)$ for some k and every two of the sets are disjoint. Hence, we get a contradiction if $\lfloor \log_4 N \rfloor > \frac{2}{\epsilon}$ because the measure μ of each of these sets is at least $\frac{1}{2}\epsilon$.

Taking $N > 4^{\frac{2}{\epsilon}}$ yields the required α-double-wedge $W(\ell_a, \ell_b)$ with $\alpha = \frac{\pi}{N}$. ∎

Remark: The dependence of $\alpha(\epsilon)$ on ϵ in Lemma 1 can be shown to be polynomial rather than exponential. This is because the sets $W_B(\ell'_{k+1}, \ell_k)$ and $W_F(\ell'_{k+1}, \ell_k)$ are semi-algebraic of bounded complexity and hence (see [APPRS05]) there exists an absolute constant $\delta > 0$ such one can find N^δ such sets every two of which are disjoint, or every two of which intersect. Using Observation 1, one can show that the former case always happens. See [APPRS05] for more details on intersection patterns of semi-algebraic sets.

One direct way to show a polynomial dependency of $\alpha(\epsilon)$ in ϵ is as follows.

Lemma 2. *Given m distinct wedges $W(l_1, l'_1), \ldots, W(l_m, l'_m)$, one can always find a subset I of $\{1, \ldots, m\}$ with $|I| \geq m^{\frac{1}{3}}$ such that either for every $i \neq j \in I$ the interiors of $W_F(l_i, l'_i)$ and $W_F(l_j, l'_j)$ intersect, or for every $i \neq j \in I$ the interiors of $W_F(l_i, l'_i)$ and $W_F(l_j, l'_j)$ are disjoint.*

Proof. Without loss of generality we may assume that at least half of the front wedges $W_F(l_i, l'_i)$ contain a ray that point rightwards, that is, in the direction of a vector $(\cos\theta, \sin\theta)$ for some $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We concentrate only on these front wedges and define two partial orders $<_1$ and $<_2$ on this set. We say that $W_F(l_i, l'_i) <_1 W_F(l_j, l'_j)$ if the following two conditions are satisfied:

1. The apex of $W_F(l_i, l'_i)$ is to the left of (that is, has a smaller x-coordinate than that of) the apex of $W_F(l_j, l'_j)$.
2. $W_F(l_j, l'_j)$ lies *above* $W_F(l_i, l'_i)$. That is, the intersection of $W_F(l_j, l'_j)$ with any vertical line lie above the intersection of that vertical line with $W_F(l_i, l'_i)$.

Similarly, we say that $W_F(l_i, l'_i) <_2 W_F(l_j, l'_j)$ if the apex of $W_F(l_i, l'_i)$ is to the right of the apex of $W_F(l_j, l'_j)$ and $W_F(l_j, l'_j)$ lies *above* $W_F(l_i, l'_i)$ (see Figure 3).

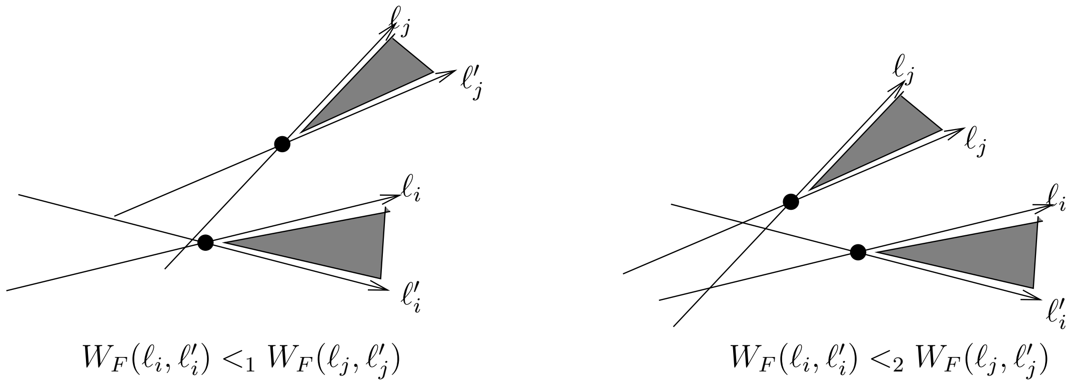


Figure 3: definition of $<_1$ and $<_2$

We leave it to reader to verify that both relations $<_1$ and $<_2$ are indeed partial orders, and that any two disjoint front wedges in our collection are comparable by (exactly) one of these partial order relations.

It now follows from Dillworth's theorem, by a standard argument that either there is a chain of size at least $(\frac{1}{2}m)^{1/3}$ with respect to one of the relations $<_1$ and to $<_2$, or there are $(\frac{1}{2}m)^{1/3}$ front wedge no two of which are comparable by any of the partial orders $<_1$ and $<_2$. In the first case we obtain $(\frac{1}{2}m)^{1/3}$ front wedges with pairwise disjoint interiors. In the second case we obtain $(\frac{1}{2}m)^{1/3}$ front wedges with pairwise intersecting interiors. ∎

Returning to the proof of Lemma 1, it follows now from Lemma 2 that one can find a subset $I \subset \{0, \ldots, N-1\}$ of cardinality of at least $(\frac{1}{2}N)^{1/3}$ such that either for every $i \neq j \in I$ the interiors of $W_F(\ell'_{i+1}, \ell_i)$ and $W_F(\ell'_{j+1}, \ell_j)$ are disjoint, or for every $i \neq j \in I$ the interiors of $W_F(\ell'_{i+1}, \ell_i)$ and $W_F(\ell'_{j+1}, \ell_j)$ intersect. In the former case, we obtain a contradiction when $(\frac{1}{2})N^{1/3}\frac{\epsilon}{2} > 1$ because the measure of each $W_F(\ell'_{i+1}, \ell_i)$ is at least $\frac{\epsilon}{2}$. In the latter case, if follows from Observation 1 that for every $i \neq j \in I$ the interiors of $W_B(\ell'_{i+1}, \ell_i)$ and $W_B(\ell'_{j+1}, \ell_j)$ are disjoint. We thus get the desired contradiction once again as soon as $(\frac{1}{2})N^{1/3}\frac{\epsilon}{2} > 1$.

Hence, taking $N > 2(\frac{2}{\epsilon})^3$ yields the required α-double-wedge $W(\ell_a, \ell_b)$ with $\alpha = \frac{\pi}{N}$.

4 Proof of Theorem 1

In this section we prove Theorem 1. We will need the following theorem of independent interest:

Theorem 3. *Let P be a set of n points in the plane. For every $\epsilon > 0$ there exists $0 < \alpha(\epsilon) < \pi$, depending only on ϵ, such that for every $0 < \alpha \leq \alpha(\epsilon)$ there is an α-double-wedge $W(\ell_a, \ell_b)$, whose apex belongs to P, with $|P \cap W_L(\ell_a, \ell_b)| > (1 - \epsilon)n$. Moreover, one can find such a double-wedge whose axis creates an angle of at most $\frac{16}{\epsilon}\alpha$ with the positive part of the y-axis.*

Proof. We take $\alpha(\epsilon) = \frac{1}{16}\epsilon\pi$. Let α be any angle such that $0 < \alpha \leq \alpha(\epsilon)$. Let $x_1, \ldots, x_k$ denote the vertices of $\mathrm{conv}(P)$ arranged in a counterclockwise order on the boundary of $\mathrm{conv}(P)$. Without loss of generality we assume that x_1 is the rightmost point in P.

Let ℓ_1 denote the directed line through x_1 in the direction of the positive part of the y-axis. Denote $i_1 = 1$. Assume that $i_1, \ldots, i_j$ have already been defined and so are $\ell_0, \ldots, \ell_j$. If $i_j = k$ we stop, otherwise we define i_{j+1} to be the maximum index such that $i_j < i_{j+1} \leq k$ and $|W_F(\overrightarrow{x_{i_j} x_{i_{j+1}}}, \ell_j) \cap P| \leq \frac{1}{2}\epsilon n$. We define $\ell_{j+1} = \overrightarrow{x_{i_j} x_{i_{j+1}}}$. Such an index i_{j+1} necessarily exists because $|W_F(\overrightarrow{x_{i_j} x_{i_j+1}}, \ell_j) \cap P| = 0$.

Assume that we stop after t steps. The indices $i_1 = 1, \ldots, i_t = k$ are defined as well as the directed lines $\ell_1, \ldots, \ell_t$. We have $|W_F(\ell_j, \ell_{j+1}) \cap P| \leq \frac{1}{2}\epsilon n$ for every $j = 0, \ldots, t-1$. Moreover, because of the maximality of each of the indices i_j we have $|W_F(\overrightarrow{x_{i_j} x_{i_{j+2}}}, \ell_j) \cap P| > \frac{1}{2}\epsilon n$ for every $j = 1, \ldots, t-2$.

Note that t cannot exceed $\frac{4}{\epsilon}$. This is because the sets $W_F(\overrightarrow{x_{i_1} x_{i_3}}, \ell_1), W_F(\overrightarrow{x_{i_3} x_{i_5}}, \ell_3), W_F(\overrightarrow{x_{i_5} x_{i_7}}, \ell_5), \ldots$ are pairwise disjoint while each contains at least $\frac{1}{2}\epsilon n$ points of P.

Case 1. For every $1 \leq j < t$, the angle between ℓ_j and ℓ_{j+1} is smaller than α. In this case because $t \leq \frac{4}{\epsilon}$, ℓ_t passes through x_k and creates an angle of at most $\beta = \frac{4}{\epsilon}\alpha$ with the positive part of y-axis. Because $\alpha \leq \alpha(\epsilon) = \frac{1}{16}\epsilon\pi$, we have $\beta \leq \frac{\pi}{4}$.

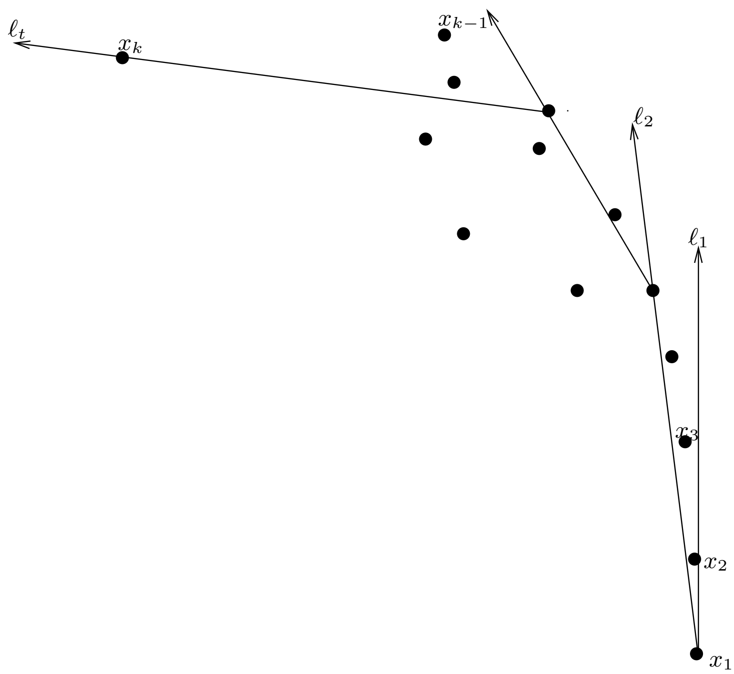


Figure 4: Theorem 3

The points of P to the right of ℓ_t are all contained in $W_F(\ell_t, \ell_{t-1})$ (see Figure 4). Hence all but at most $\frac{1}{2}\epsilon n$ points of P are to the left of ℓ_t. Let ℓ_a be the line obtained from ℓ_t by rotating it around x_k counterclockwise at an angle of α. Let ℓ_b be the line ℓ_t. Then $W(\ell_a, \ell_b)$ is an α-wedge whose apex is x_k. Moreover, $|P \cap W_L(\ell_a, \ell_b)| \geq (1 - \frac{\epsilon}{2})n$, and the axis of $W_F(\ell_a, \ell_b)$ creates an angle of at most $\frac{16}{\epsilon}\alpha$ with the positive part of the y-axis, as required.

Case 2. There exists a minimal $1 \leq j < t$, such that the angle between ℓ_j and ℓ_{j+1} is greater than or equal to α. In this case too we have $j < t \leq \frac{4}{\epsilon}$. Therefore both ℓ_j and ℓ_{j+1} create an angle of at most $\frac{4}{\epsilon}\alpha$ with the positive part of the y-axis. Let $\ell_a = \ell_{j+1}$ and let ℓ_b be the directed line ℓ_j. The apex of $W(\ell_a, \ell_b)$ is x_{j+1} and $W_L(\ell_a, \ell_b)$ is an α-wedge. $W_L(\ell_a, \ell_b)$ contains all the points of

P that are not to the right of ℓ_{j+1} and not to to the right of ℓ_j. However, all the point of P which are to the right of ℓ_{j+1} are contained in $W_F(\ell_{j+1}, \ell_j)$ and by the construction of the line ℓ_{j+1} there are at most $\frac{1}{2}\epsilon n$ such points. Similarly, all the point of P which are to the right of ℓ_j are contained in $W_F(\ell_j, \ell_{j-1})$ and by the construction of the line ℓ_j there are at most $\frac{1}{2}\epsilon n$ such points. Hence $|P \cap W_L(\ell_a, \ell_b)| \geq (1-\epsilon)n$ as required. ∎

We are now ready to proceed to the proof of the main theorem:

Proof of Theorem 1. We need to show that for every $\epsilon > 0$ there exists $0 < \alpha < \pi$, depending only on ϵ with the following property: For any set P of n points in general position in the plane one can find an α-double-wedge $W(\ell_a, \ell_b)$ with an apex in P for which $|P \cap W_L(\ell_a, \ell_b)| > (\frac{1}{2} - \epsilon)n - 4$ and $|P \cap W_R(\ell_a, \ell_b)| > (\frac{1}{2} - \epsilon)n - 4$.

Replace each point x of P with a small disc centered at x. We assume that those discs are so small that no line meets more than two of them. By applying Lemma 1 to the resulting uniform probability measure on the union of all these discs, we deduce the existence of an α_1-double-wedge $W(\ell_1, \ell_2)$ with the following properties:

- Each of $W_L(\ell_1, \ell_2)$ and $W_R(\ell_1, \ell_2)$ contains at least $(\frac{1}{2} - \frac{\epsilon}{2})n - 4$ points of P.
- $0 < \alpha_1 < \pi$ and α_1 depends only on ϵ.

Without loss of generality assume that the axis of $W(\ell_1, \ell_2)$ is in the direction of the positive part of the y-axis. Let $P' = P \cap W_L(\ell_1, \ell_2)$ and denote $m = |P'|$. Recall that $m \geq (\frac{1}{2} - \frac{\epsilon}{2})n - 4$.

We apply Theorem 3 to the set P' and conclude that there exists α_2 depending only on ϵ such that for every $0 < \alpha < \alpha_2$ there is an α-double-wedge $W(\ell_a, \ell_b)$ whose apex belongs to P' such that $|W_L(\ell_a, \ell_b) \cap P'| > (1 - \frac{\epsilon}{2})m$ and the axis of $W(\ell_a, \ell_b)$ creates an angle of at most $\frac{16}{\frac{\epsilon}{2}}\alpha = \frac{32}{\epsilon}\alpha$ with the positive part of the y-axis.

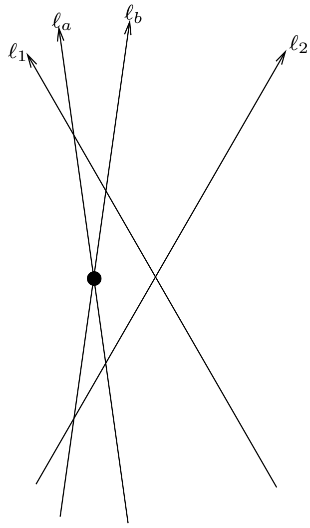


Figure 5: Theorem 1

Take $\alpha = \min(\alpha_2, \frac{1}{200}\epsilon\alpha_1)$. Observe that α depends only on ϵ. Because $\alpha \leq \frac{1}{200}\epsilon\alpha_1$, each of ℓ_a

and ℓ_b creates an angle of at most $\frac{32}{200}\alpha_1$ with the positive part of the y-axis, which is also the axis of $W(\ell_1, \ell_2)$.

Since the apex of $W(\ell_a, \ell_b)$ is located in $W_L(\ell_1, \ell_2)$ and because each of ℓ_a and ℓ_b creates an angle smaller than $\frac{1}{2}\alpha_1$ with the positive part of the y-axis, it follows that $W_R(\ell_a, \ell_b) \supset W_R(\ell_1, \ell_2)$ (see Figure 5).

Hence $|P \cap W_R(\ell_a, \ell_b)| \geq (\frac{1}{2} - \frac{\epsilon}{2})n - 4$. On the other hand we have:

$$|P \cap W_L(\ell_a, \ell_b)| \geq |P' \cap W_L(\ell_a, \ell_b)| \geq (1 - \frac{\epsilon}{2})m \geq (1 - \frac{\epsilon}{2})((\frac{1}{2} - \frac{\epsilon}{2})n - 4) \geq (\frac{1}{2} - \epsilon)n - 4.$$

This proves the theorem. ∎

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