Generalized Thrackles and Geometric Graphs in $\mathbb{R}^3$ with no pair of Strongly Avoiding Edges

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Abstract

We define the notion of a geometric graph in $\mathbb{R}^3$. This is a graph drawn in $\mathbb{R}^3$ with its vertices drawn as points and its edges as straight line segments connecting corresponding points. We call two edges of G *strongly avoiding* if there exists an orthogonal projection of $\mathbb{R}^3$ to a two dimensional plane H such that the projections of the two edges on H are contained in two different rays, respectively, with a common apex that create a non-acute angle. We show that a geometric graph on n vertices in $\mathbb{R}^3$ with no pair of strongly avoiding edges has at most $2n-2$ edges. As a consequence we get a new proof to Vázsonyi's conjecture about the maximum number of diameters in a set of n points in $\mathbb{R}^3$.

1 Introduction

A *topological graph* is a graph drawn in the two dimensional plane (or equivalently on a two dimensional sphere) with its vertices drawn as points (usually in general position) and its edges drawn as Jordan arcs connecting corresponding points. We require that whenever two edges of a topological graph meet at a point, they either properly cross or meet at a common vertex. A *geometric graph* in the plane is a topological graph where the edges are straight line segments.

Definition 1. A topological graph G is called a *generalized thrackle* if every pair of edges of G meet an odd number of times.

The notion of a generalized thrackle was first introduced by Lovász, Pach, and Szegedy in [LPS97], in connection with Conway's thrackle conjecture. It was shown by Cairns and

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Nikolayevsky ([CN00]) that a generalized thrackle on n vertices can have at most $2n-2$ edges.

We relate the notion of a generalized thrackle to the following notion of centrally symmetric S^2-lifting of a graph.

Definition 2. Let G be a graph on a set V of n vertices. A *centrally symmetric S^2-lifting* of G is a bipartite topological graph $\tilde{G}$ embedded on the two dimensional sphere S^2 centered at the origin, together with a correspondence f between the set of vertices of G and the set of vertices of one color class of $\tilde{G}$, with the following properties:

1. The drawing of $\tilde{G}$ is centrally symmetric on S^2, with no two edges crossing.
2. The set of vertices of $\tilde{G}$ consists of the disjoint union of $\{f(v)|v \in V\}$ and $\{-f(v)|v \in V\}$.
3. Two vertices $v_1, v_2 \in V$ are connected by an edge in G if and only if $f(v_1)$ and $-f(v_2)$ are connected by an edge in $\tilde{G}$.

Observe that if a graph G on n vertices has a centrally symmetric S^2-lifting $\tilde{G}$, then G has at most $2n-2$ edges. Indeed, this is because $\tilde{G}$ is a bipartite planar graph on $2n$ vertices that has twice as many edges as G. Therefore, twice the number of edges in G is at most $2(2n)-4 = 2(2n-2)$.

We show the following characterization theorem:

Theorem 1. *Let G be a finite graph. G can be realized as a generalized thrackle if and only if it has a centrally symmetric S^2-lifting.*

Let P be a set of points in $\mathbb{R}^3$. A *diameter* of P is an unordered pair of points of P with maximum distance among all pairs of points in P. The *diameter graph* of P is the graph whose vertices are the points of P and its edges are the diameters of P.

Vázsonyi conjectured (see [E46]) that a set P of n points in $\mathbb{R}^3$ admits at most $2n-2$ distinct diameters. Vázsonyi's conjecture was proved to be tight independently by Grűnbaum ([Gr56]), Heppes ([Hep56]), and Straszewizc ([St57]). All those proofs of Vázsonyi's conjecture use the notion of spherical polytopes, that are different in nature from ordinary polytopes and, for example, are not necessarily 3-connected, as was shown by Kupitz, Martini, and Perles ([KMP]).

In this paper we generalize the notion of a geometric graph to three dimensions an show that in this context Vázsonyi's conjecture is in fact a consequence of a more general result.

A *geometric graph* in $\mathbb{R}^3$ is a graph drawn in $\mathbb{R}^3$ with its vertices drawn as points and its edges drawn as straight line segments connecting corresponding points.

Definition 3. Two *disjoint* segments (edges of a geometric graph) e and f in $\mathbb{R}^3$ are called *strongly avoiding* if there is an orthogonal projection of $\mathbb{R}^3$ onto some two-dimensional plane H such that the projections of e and f on H are contained in two different rays, respectively, with a common apex that create a non-acute angle. (See Figure 1.)

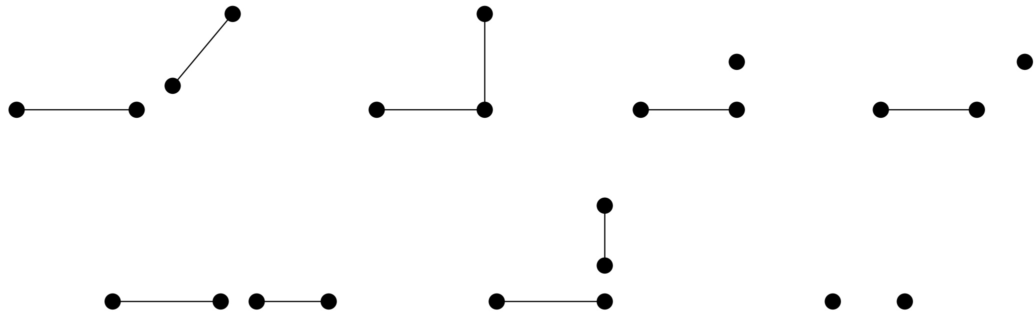

Figure 1: Possible projections of pairs of strongly avoiding edges.

Observe that two parallel segments are strongly avoiding according to Definition 3. This can be seen by projecting the two segments on a plane perpendicular to both. More generally, any two disjoint coplanar segments are strongly avoiding according to Definition 3. Observe also that the property of being strongly avoiding is invariant under isometries of $\mathbb{R}^3$.

The following theorem and its corollary imply Vázsonyi's conjecture, as we will show that no two edges in a diameter graph in $\mathbb{R}^3$ are strongly avoiding.

Theorem 2. *Let G be a geometric graph in $\mathbb{R}^3$. If G has no pair of strongly avoiding edges, then G has a centrally symmetric S^2-lifting.*

Observe that the following corollary is an immediate consequence of Theorem 1 and Theorem 2.

Corollary 1. *Let G be a geometric graph on n vertices in $\mathbb{R}^3$. If G has no pair of strongly avoiding edges, then G has at most $2n-2$ edges and it can be realized as a generalized thrackle.*

For the proof of Theorem 2 we define the following notion of independent interest.

Definition 4. For any three distinct points $x, a, b \in \mathbb{R}^3$ we denote by $L_x(a,b)$ the set of all lines passing through x and a point on the straight line segment whose endpoints are a and b.

Definition 5. Let S^2 be a two dimensional sphere, centered at the origin o. Let P be a union of distinct pairs of antipodal points on S^2. A labeling of the points of P with labels from an alphabet $\mathcal{A}$ is called a *crossing-free antipodal labeling* if the following three conditions on the labeling are satisfied:

1. Every set of points assigned with the same label is contained in an open hemisphere of S^2, and in particular no two antipodal points are assigned the same label.

2. No two pairs of antipodal points are assigned the same pair of labels.

3. If p_1 and p_2 are assigned a label a and q_1 and q_2 are assigned a label b where $a \neq b$, then $L_o(p_1, p_2) \cap L_o(q_1, q_2)$ is empty unless precisely one point of p_1 and p_2 is antipodal to either q_1 or q_2, in which case $L_o(p_1, p_2) \cap L_o(q_1, q_2)$ consists only of the line through these two antipodal points.

We denote this kind of labeling by $\mathcal{L} = \mathcal{L}(P, \mathcal{A})$. For a crossing-free antipodal labeling $\mathcal{L} = \mathcal{L}(P, \mathcal{A})$ there is a natural graph, denoted by $G_{\mathcal{L}}$, that we associate with it. The underlying vertex set of $G_{\mathcal{L}}$ is $\mathcal{A}$. Two labels $a, b \in \mathcal{A}$ are connected by an edge if there is an antipodal pair of points in P labeled with a and b, respectively. Therefore, the number of edges in $G_{\mathcal{L}}$ is exactly the number of antipodal pairs of points in P.

Lemma 1. *Let S^2 be a two dimensional sphere, centered at the origin. Let P be a union of distinct pairs of antipodal points on S^2. Assume that there is a crossing-free antipodal labeling $\mathcal{L}(P, \mathcal{A})$ of the points of P with an alphabet $\mathcal{A}$. Then $G_{\mathcal{L}}$, the graph associated with $\mathcal{L}$, has a centrally symmetric S^2-lifting.*

We remark that a similar idea for the derivation of Vázsonyi's conjecture through a crossing-free antipodal labeling was carried out independently by Swanepoel ([S]). Swanepoel gives a direct, short, and elegant proof of Vázsonyi's conjecture, without going through the more general result in Theorem 2, and in fact shows that the diameter graph has a centrally symmetric S^2-lifting. For related ideas regarding Vázsonyi's conjecture we refer the reader also to [D00].

2 Proof of Theorem 1

For the proof of Theorem 1 we will need the following easy lemma:

Lemma 2. *Let U be a set of points on a sphere $\mathcal{S}$, centered at the origin, and assume that no two points of U are antipodal. Let U^* denote the set of points antipodal to the points in U. There exists a simple closed curve C on $\mathcal{S}$, centrally symmetric with respect to the origin, that separates the points of U from the points of U^* on $\mathcal{S}$.*

Proof. We prove the lemma by induction on $|U|$. For $|U| = 1$ the lemma is clear and one can take C to be any great circle on $\mathcal{S}$ not passing through the single point in U.

Assume $|U| > 1$. Let $x \in U$ be any point in U. Find an appropriate curve C' for the set $U \setminus \{x\}$. Let $\mathcal{S}_1$ denote the connected component of $\mathcal{S} \setminus C'$ which contains the points of $U \setminus \{x\}$. Let $\mathcal{S}_2$ denote the other component of $\mathcal{S} \setminus C'$. If x happens to be in $\mathcal{S}_1$, then we are done. If x is accidentally on C', then a small modification of C' will do. Otherwise x must be in $\mathcal{S}_2$. Let y and z be any two close point on C' and let C'_{yz} denote the smaller portion of C' between y and z. Let α be a simple curve with endpoints y and z that is contained in $\mathcal{S}_2$ such that the union of α and C'_{yz} is a simple closed curve surrounding x but not any point of U^*. Now define C to be the curve obtained from C' by deleting C'_{yz} as well as $-C'_{yz}$ from C' and adding to it α and $-\alpha$. C is the desired centrally symmetric simple closed curve on $\mathcal{S}$. ∎

Proof of Theorem 1. Let G be a graph on a set of vertices V that has a centrally symmetric S^2-lifting $\tilde{G}$. Let $f: V \longrightarrow S^2$ denote the correspondence between the vertices of G and the vertices of one color class of $\tilde{G}$, as guaranteed by Definition 2. We wish to show that G can be realized as a generalized thrackle. Let U be the set $\{f(v)|v \in V\}$ and let U^* be the set of antipodal points to the points in U, $U^* = \{-u|u \in U\}$. Let C be the centrally symmetric simple closed curve on S^2 guaranteed by Lemma 2 with respect to the sets U and U^*. Let $\mathcal{S}_1$ be the connected component of $S^2 \setminus C$ that contains the points of U. We will draw the graph $G_{\mathcal{L}}$ as a generalized thrackle on the surface $\overline{\mathcal{S}_1}$ obtained from $\mathcal{S}_1$ by joining to it the segments $[x, -x]$ for every $x \in C$ (observe that $\overline{\mathcal{S}_1}$ is homeomorphic to a two dimensional sphere).

The vertices of the drawing will be the points in U. For every $a, b \in V$ that are connected by an edge in G we draw an edge between $f(a)$ and $f(b)$ on $\overline{\mathcal{S}_1}$ in the following way:
Recall that $f(a)$ and $-f(b)$ are connected by an edge in $\tilde{G}$. Denote the curve representing this edge by α. We think of α as a function, $\alpha(t) : [0,1] \longrightarrow S^2$, where $\alpha(0) = f(a)$, $\alpha(1) = f(b)$. $-\alpha$ is the curve representing the edge connecting between $f(b)$ and $-f(a)$ in $\tilde{G}$. By Jordan's theorem α crosses C an odd number of times. Let $0 < t_1 < \ldots < t_{2k+1} < 1$ be all the points where $\alpha(t_i) \in C$. Then draw the edge between $f(a)$ and $f(b)$ as follows: start with $\alpha([0, t_1])$ then add to it the segment $[\alpha(t_1), -\alpha(t_1)]$ and continue with $-\alpha([t_1, t_2])$ add the segment $[-\alpha(t_2), \alpha(t_2)]$ and continue with $\alpha([t_2, t_3])$ and so on. Eventually, the last portion of the curve is $-\alpha([t_{2k+1}, 1])$ ending at $f(b)$. Consider now two distinct edges e_{ab} and e_{cd} of the drawing connecting say $f(a)$ to $f(b)$ and $f(c)$ to $f(d)$ (possibly $\{a, b\} \cap \{c, d\}$ is not empty), respectively. Let α denote the curve representing the edge in $\tilde{G}$ between $f(a)$ and $-f(b)$ on S^2 and let β denote the curve representing the edge in $\tilde{G}$ between $f(c)$ and $-f(d)$ on S^2. e_{ab} is made of portions of α and $-\alpha$ together with an odd number of segments of the form $[x, -x]$ where $x \in C$. Similarly, e_{cd} is made of portions of β and $-\beta$ together with an odd number of segments of the form $[x, -x]$ where $x \in C$. Since no two of the curves α, $-\alpha$, β, and $-\beta$ cross each other, e_{ab} and e_{cd} cross an odd number of times at the origin only.

Therefore, any two edges in the above drawing of G on $\overline{\mathcal{S}_1}$ cross an odd number of times. This drawing is still not a generalized thrackle because two edges in the drawing, incident to the same vertex, meet an even number of times (odd number of crossings and an additional meeting point at the common vertex). This can be fixed by modifying the drawing in a small neighborhood of each vertex $f(v) \in U$ and adding an extra crossing point to every pair of edges incident to the same vertex. This is illustrated in Figure 2.

We have thus showed that any graph with a centrally symmetric S^2-lifting can be realized as a generalized thrackle. To see the other direction, that any generalized thrackle has a centrally symmetric S^2-lifting we use Conway's doubling technique and a result in [CN00] about the structure of generalized thrackles.

Let G be a generalized thrackle. If G is bipartite, then G is planar. In this case let V_1 and V_2 be the two color classes of G. Let H be a crossing-free drawing of G on the lower hemisphere of S^2. Denote in this drawing the image of each $v \in V_1 \cup V_2$ by $g(v)$. Observe that $-H$ is another crossing-free drawing of G, on the upper hemisphere of S^2. It is now easy to see that $H \cup -H$ is a centrally symmetric S^2-lifting of G. Indeed, define the correspondence f between the vertices of G and those of $H \cup -H$ by $f(v) = g(v)$ for $v \in V_1$ and $f(v) = -g(v)$

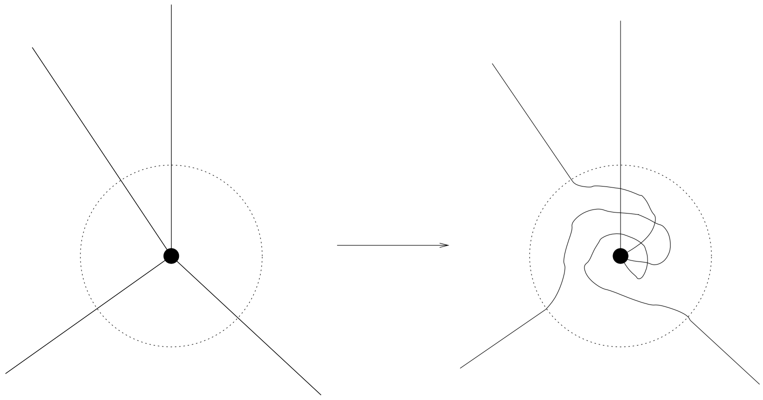

Figure 2: the local change of the drawing near a vertex

for $v \in V_2$.

If G is not bipartite, then let V be the set of vertices of G and let $C = \{v_1, \ldots, v_{2k+1}\}$ be an odd cycle in G, where v_i is a neighbor in G of v_{i+1} for every $1 \leq i < 2k+1$ and v_{2k+1} is a neighbor of v_1. It is not hard to see (for details we refer the reader to [CN00]) that $G \setminus C$ is a bipartite graph. Let A and B denote the two color classes of $G \setminus C$. Following [CN00], perform Conway's doubling technique on the odd cycle C. In this procedure we introduce two vertices x_i, y_i for each vertex v_i, and we define a new graph G' on the set of vertices $A \cup B \cup \{x_1, y_1, \ldots, x_{2k+1}, y_{2k+1}\}$. In this graph the edges between vertices from A and vertices from B are the same as they are in G. We connect a vertex $v \in A$ to x_i in G' if v is connected to v_i in G. We connect a vertex $v \in B$ to y_i in G' if v is connected to v_i in G. In addition we connect x_i to y_{i+1} and y_i to x_{i+1} for every $1 \leq i < 2k+1$, and finally x_{2k+1} is connected to y_1 and y_{2k+1} is connected to x_1. Therefore $x_1, y_2, x_3, \ldots, x_{2k+1}, y_1, x_2, \ldots, y_{2k+1}$ is a cycle C' of length $2(2k+1)$ in G'. As shown in [CN00] (Lemma 2 there) G' is bipartite and can be realized as a generalized thrackle. Hence G' is planar. Moreover, it is shown in [CN00] (Lemma 4 there) that G' has a planar embedding where C' is a face (of size $2(2k+1)$) of the embedding.

Consider a drawing H of G' as a topological crossing-free graph on the lower hemisphere of S^2. Let g denote the correspondence between vertices of G' and the vertices of the drawing H. In this drawing we make sure that $g(x_1), g(y_2), g(x_3), \ldots, g(x_{2k+1}), g(y_1), g(x_2), \ldots, g(y_{2k+1})$ appear in that cyclic order on the equator of S^2 and that for every $1 \leq i \leq 2k+1$ $g(x_i) = -g(y_i)$, and the edges between $g(x_i)$ and $g(y_{i+1})$ and between $g(y_i)$ and $g(x_{i+1})$ are contained in the equator of S^2.

Define $\tilde{G}$ to be $H \cup -H$ and the correspondence f between vertices of G and vertices in the drawing $H \cup -H$ is defined by $f(v) = g(v)$ for $v \in A$, $f(v) = -g(v)$ for $v \in B$, and $f(v_i) = g(y_i)$ for every $1 \leq i \leq 2k+1$.

To see that indeed $\tilde{G}$ is a centrally symmetric S^2-lifting of G observe that $\tilde{G}$ is centrally symmetric by definition. Let u, v be two vertices of G that are neighbors in G, we show that $f(u)$ and $-f(v)$ are neighbors in $\tilde{G}$.

Case 1. $u \in A$ and $v \in B$. Then $f(u) = g(u)$ and $f(v) = -g(v)$. In the drawing H $g(u)$ is connected by an edge to $g(v)$, hence $f(u)$ is connected by an edge in $\tilde{G}$ to $-f(v)$ as required.

Case 2. $u \in A$ and $v = v_i \in C$. In this case u and x_i are connected by an edge in G', $f(u) = g(u)$ and $f(v) = g(y_i) = -g(x_i)$. $g(u)$ and $g(x_i)$ are connected by an edge in H. Therefore $f(u)$ and $-f(v)$ are connected by an edge in $\tilde{G}$.

Case 3. u and v are consecutive vertices in the cycle C, say $u = v_i \in C$ and $v = v_{i+1} \in C$. In this case $f(u) = g(y_i)$ and $f(v) = g(y_{i+1})$. y_i and x_{i+1} are neighbors in G', hence $g(y_i)$ and $g(x_{i+1})$ are connected by an edge in H. Therefore, $f(u) = g(y_i)$ is connected by an edge in $\tilde{G}$ to $-f(v) = -g(y_{i+1}) = g(x_{i+1})$.

Case 4. $u \in B$ and $v = v_i \in C$. In this case u and y_i are connected by an edge in G', $f(u) = -g(u)$ and $f(v) = g(y_i)$. $g(u)$ and $g(y_i)$ are connected by an edge in H. Hence $-g(u)$ and $-g(y_i)$ are connected by an edge in $-H$. Therefore $f(u) = -g(u)$ and $-f(v) = -g(y_i)$ are connected by an edge in $\tilde{G}$.

All other cases of neighboring vertices in G are symmetric and treated similarly. In exactly the same manner one can show that all neighboring vertices in $\tilde{G}$ are of the form $f(u), -f(v)$ for two neighboring vertices u, v in G. ∎

3 Proof of Lemma 1

For each label $a \in \mathcal{A}$ let T_a denote the union of all geodesics on S^2 connecting pairs of points labeled with a. Observe that each T_a is contained in an open hemisphere of S^2, and hence T_a is disjoint from $-T_a$. By our construction

$$T_a \cup -T_a = S^2 \cap \big(\bigcup_{p,q \in P \text{ labeled } a} L_o(p,q) \big).$$

Therefore, because of the third condition in Definition 5 of a crossing free antipodal labeling, for any two distinct labels a and b, $T_a \cup -T_a$ is disjoint from $T_b \cup -T_b$, unless there is a pair of antipodal points p_a and p_b in P labeled with a and with b, respectively. In the latter case $T_a \cap -T_b = \{p_a\}$ and $T_b \cap -T_a = \{p_b\}$.

It is readily seen that $\{\pm T_a | a \in \mathcal{A}\}$ is a collection of connected sets on S^2 every two of which are either disjoint or touch at a single point.

By choosing a point v_a on each T_a and considering their antipodal points $-v_a$ on S^2, it is not hard to see that one can draw a crossing-free centrally symmetric topological graph $\tilde{G}$ on $\mathcal{S}$ whose vertices are $\{\pm v_a | a \in \mathcal{A}\}$. In this graph v_a is connected by an edge to $-v_b$ if and only if T_a and $-T_b$ touch and this in turn is if and only if there are two antipodal points on P labeled with a and with b, respectively. $\tilde{G}$ is, therefore, a centrally symmetric S^2-lifting of $G_{\mathcal{L}}$. ∎

4 Proof of Theorem 2

We prove Theorem 2 as a direct application of Lemma 1. Let G be a geometric graph on n vertices $v_1, \ldots, v_n$ in $\mathbb{R}^3$, with no pair of strongly avoiding edges. Let $\mathcal{S}$ be a sphere centered at the origin o. For every edge $e = (v_i v_j)$ of G place two antipodal points $p_{e,i}$ and $p_{e,j}$ on $\mathcal{S}$ such that the directed line $\overrightarrow{v_i v_j}$ is parallel to the directed line $\overrightarrow{p_{e,j} p_{e,i}}$. Then assign to $p_{e,i}$ the label $'i'$ and to $p_{e,j}$ assign the label $'j'$. This way we get a set P composed of pairs of antipodal points on $\mathcal{S}$ labeled with labels from $\mathcal{A} = \{1, \ldots, n\}$. Denote this labeling by $\mathcal{L}(P, \mathcal{A})$. Observe that the pairs of antipodal points in P are indeed distinct as there are no two parallel edges in G.

We will show that unless G is a star, $\mathcal{L}(P, \mathcal{A})$ is a crossing-free antipodal labeling according to Definition 5. It will then follow from Lemma 1 that $G_{\mathcal{L}}$ has a centrally symmetric S^2-lifting. However, observe that by the definition of $\mathcal{L}(P, \mathcal{A})$, $G_{\mathcal{L}}$ is isomorphic to the original graph G, and therefore Theorem 2 will follow. In case G is a star, then it clearly can be realized as a generalized thrackle, and hence by Theorem 1 it has a centrally symmetric S^2-lifting.

First observe that no two distinct pairs of antipodal points in P are labeled with the same pair of labels because the labeling of a pair of antipodal points is in one to one correspondence with neighboring vertices in G.

Next we claim that unless G is a star, the set of all points on $\mathcal{S}$ with the same label, say a, is contained in an open hemisphere of $\mathcal{S}$. To see this let e be an edge of G that is not incident to v_a. Let v_b be a vertex of e and project $\mathbb{R}^3$ orthogonally in the direction of the line through v_a and v_b. Under this projection v_a and v_b project to a point z, and the edges incident to v_a as well as e project to segments which have z as an endpoint. Because no pair of edges in G is strongly avoiding, the projection of every edge incident to v_a must create an acute angle with the projection of e. This means that the set of all neighbors of v_a in G is contained in an open half-space bounded by a plane through v_a. This implies that the set of all points on $\mathcal{S}$, labeled with a, is contained in an open hemisphere of $\mathcal{S}$.

It is left to verify the third condition in Definition 5. Assume to the contrary that there are two points $p_{e_1,a}, p_{e_2,a}$ on $\mathcal{S}$ labeled with a and two other points $p_{f_1,b}, p_{f_2,b}$ on $\mathcal{S}$, labeled with b such that $L_o(p_{e_1,a}, p_{e_2,a}) \cap L_o(p_{f_1,b}, p_{f_2,b})$ contains at least one line ℓ_{ab} that without loss of generality does not pass through neither $p_{e_1,a}$ nor through $p_{e_2,a}$.

We will show that one of the edges e_1, e_2 and one of the edges f_1, f_2 are strongly avoiding, contradicting our assumptions.

We need the following easy observation.

Lemma 3. *Let e and f be two disjoint segments in $\mathbb{R}^3$. Assume that there exist two perpendicular planes K_e and K_f such that K_e contains e and misses the relative interior of f, and K_f contains f and misses the relative interior of e, then e and f are strongly avoiding.*

Proof. Let H be a plane perpendicular to both K_e and K_f. The orthogonal projections of e and f on H are contained in two perpendicular rays with a common apex. Therefore, e and f are strongly avoiding. ∎

Recall that the edges e_1 and e_2 are parallel to the segments $op_{e_1,a}$ and $op_{e_2,a}$, respectively. Similarly, the edges f_1 and f_2 are parallel to the segments $op_{f_1,b}$ and $op_{f_2,b}$, respectively. Let f_1' and f_2' be the segments obtained from f_1 and f_2, respectively, by the translation that takes v_b to v_a. Let H be a plane perpendicular to ℓ_{ab}.

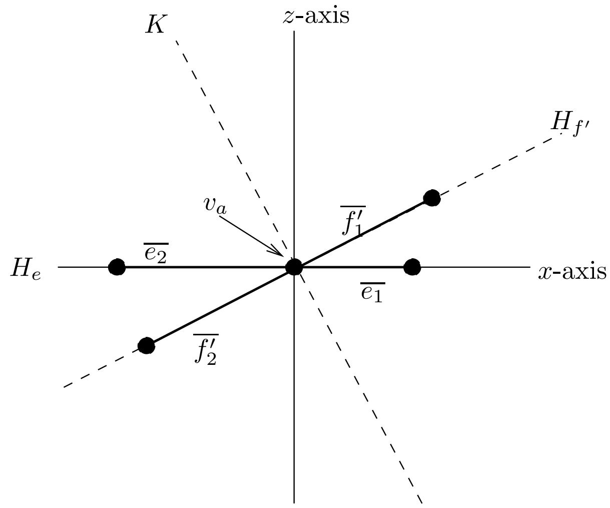

Figure 3: A view from the y-axis direction: the projection on H

Let H_e be the plane containing e_1 and e_2 and let $H_{f'}$ be the plane containing f_1' and f_2'. Both H_e and $H_{f'}$ are perpendicular to H.

Without loss of generality assume that v_a is the origin, H_e is the $(x-y)$-plane, and H is the $(x-z)$-plane. Therefore, the line l_{ab} is parallel to the y-axis. We will assume with loss of generality that the negative part of the y-axis is included the cone, with apex v_a, generated by e_1 and e_2, otherwise reflect $\mathbb{R}^3$ with respect to the $(x-z)$-plane. We will assume that e_1 and f_1' are in the half-space $\{x \geq 0\}$, while e_2 and f_2' are in the half-space $\{x \leq 0\}$. Denote by $\overline{e_1}$, $\overline{e_2}$, $\overline{f_1'}$, and $\overline{f_2'}$ the orthogonal projections on H of the segments e_1, e_2, f_1', and f_2', respectively. Therefore, $\overline{e_1}$ and $\overline{f_2'}$ create a non-acute angle, and so do $\overline{e_2}$ and $\overline{f_1'}$. Without loss of generality we assume that f_1' is in the half-space $\{z \geq 0\}$, and hence f_2' is in the half-space $\{z \leq 0\}$ (otherwise reflect $\mathbb{R}^3$ with respect to the $(x-y)$-plane). Let K be the plane that passes through the origin and is perpendicular to H and $H_{f'}$ (see Figure 3).

We will obtain a contradiction by considering the possible locations of v_b. Without loss of generality we assume that v_b is on or above H_e (for otherwise, rotate at an angle π the entire geometric graph G counterclockwise about the y-axis).

Case 1. v_b is on or below $H_{f'}$. In this case the projections of e_2 and f_1 are contained in two rays with a common apex that create a non-acute angle. Hence e_2 and f_1 are strongly avoiding, a contradiction (see Figure 4).

Case 2. v_b is above $H_{f'}$ and in the half-space $\{x \geq 0\}$. Assume first that the projection on H_e of the line through f_1 misses the relative interior either of e_1 or of e_2. Without loss of generality assume that the projection on H_e of the line through f_1 misses the relative interior of e_1. Let K' be the plane through f_1 perpendicular to H_e. K' misses the relative interior

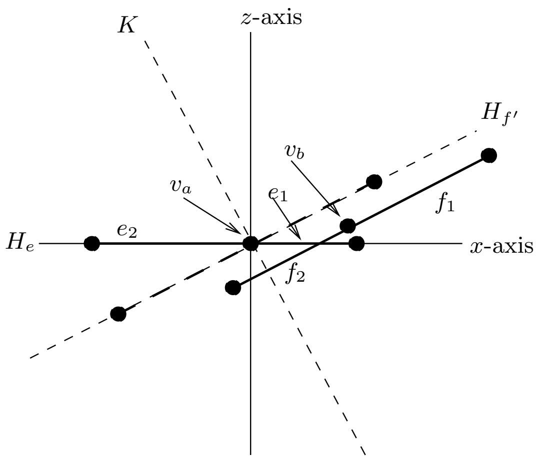

Figure 4: Case 1: v_b is above H_e and below $H_{f'}$. A view from the y-axis direction

of e_1 while H_e misses the relative interior of f_1. It now follows from Lemma 3 that f_1 and e_1 are strongly avoiding, a contradiction.

We will therefore assume that the projection on H_e of the line ℓ through f_1 crosses the relative interiors of both e_1 and e_2. Because the negative part of the y-axis is included in

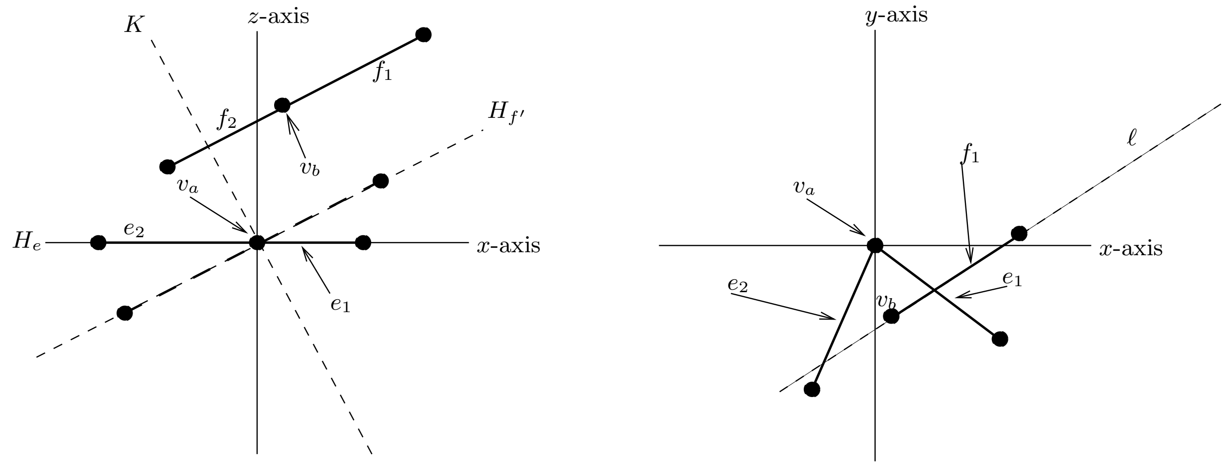

Figure 5: Case 2: On the left - a view from the y-axis direction. On the right - a view from the z-axis direction.

the cone (whose apex is the origin) generated by e_1 and e_2, it follows that the projection of ℓ on H_e must cross the negative part of the y-axis (see Figure 5). Rotate the entire geometric graph G about the z-axis in the counterclockwise direction and keep track of the projections of e_1, e_2, and f_1 on H (we emphsize that throughout the rotation only the geometric graph G is effected while the planes such as H, H_e, and H_f remain fixed). The projection of f_1 on H becomes parallel to the z-axis as soon as the projection of f_1 on H_e is parallel to the y-axis. Since the projection of ℓ on H_e crosses the relative interior of both e_1 and e_2, then before the projection of f_1 on H_e is parallel to the y-axis, the direction of either e_1 or of e_2 must coincide with the direction of the y-axis. At this point the projection of one of e_1 and e_2 on H is a point, the origin. However, the projection of f_1 on H is a segment with positive slope contained in the quadrant $\{x, y \geq 0\}$. It follows that the projection of f_1 and the origin (which is the projection of one of e_1 and e_2 on H) are contained in two different rays with a common apex (the projection of v_b on H) that create a non-acute angle. Hence, f_1 and one of e_1 and e_2 are strongly avoiding, a contradiction.

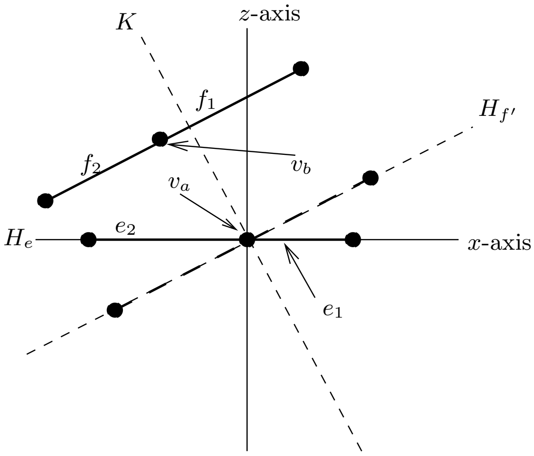

Figure 6: Case 3. The projection on H: a view from the y-axis direction.

Case 3. v_b is below K (but above H_e). See Figure 6. This case can be resolved by switching the role of e_1 and e_2 with the role of f_1 and f_2 and then referring to the above Case 2. To see this, rotate $\mathbb{R}^3$ about the y-axis until $H_{f'}$ coincides with the $(x - y)$-plane, and f_1' is contained in $\{x \geq 0\}$. Then reflect $\mathbb{R}^3$ with respect to the $(y - z)$-plane.

Case 4. v_b is above K and in the half-space $\{x \leq 0\}$ (see Figure 7).

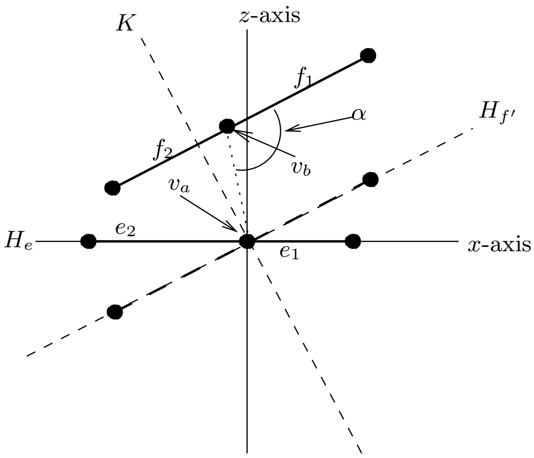

Figure 7: Case 4. The projection on H: a view from the y-axis direction.

Case 4(i). The angle between either e_1 or e_2 and the negative part of the y-axis is non-acute. See Figure 8.

If the projection on H_e of the line through f_1 misses the relative interior either of e_1 or of e_2, then we act as in Case 2: Without loss of generality assume that the projection on H_e of the line through f_1 misses the relative interior of e_1. Let K' be the plane through f_1 perpendicular to H_e. K' misses the relative interior of e_1 while H_e misses the relative interior of f_1. It now follows from Lemma 3 that f_1 and e_1 are strongly avoiding, a contradiction.

Hence, assume that the projection on H_e of the line ℓ through f_1 crosses the relative interiors of both e_1 and e_2. In this case, since the negative part of the y-axis is included in the cone generated by e_1 and e_2, the projection of ℓ on H_e must cross the negative part of the y-axis.

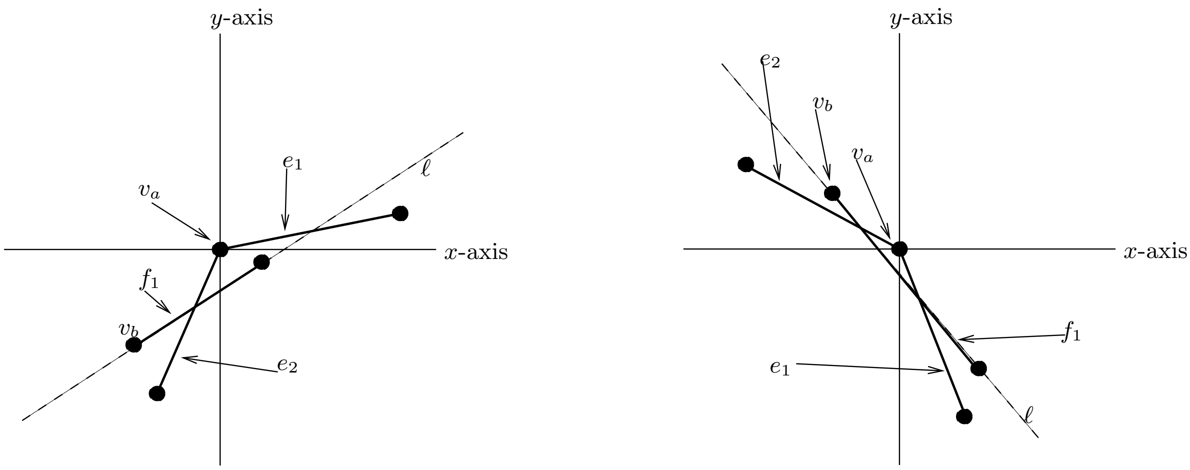

Figure 8: Case 4(i). Either e_1 or e_2 creates a non-acute angle with the negative part of the y-axis. A view from the z-axis direction.

If e_1 creates a non-acute angle with the negative part of the y-axis, then the slope of the projection of ℓ on H_e must be positive and v_b must belong to the half-space $\{y \leq 0\}$. Let v_c denote the vertex of f_1 other than v_b.

Rotate the entire geometric graph G about the z-axis in the counterclockwise direction and keep track of the projections of e_1, e_2, and f_1 on H. In particular define α to be the angle $\measuredangle \overline{v_c}\,\overline{v_b} o$, where $\overline{v_c}$ and $\overline{v_b}$ are the projections of v_c and v_b, respectively, on H, and keep track of the size of α throughout the rotation. At the beginning α is non-acute. During the rotation α increases as long as it is not equal to π.

Since the projection of ℓ on H_e crosses the relative interior of both e_1 and e_2, then before the projection of f_1 on H_e is parallel to the y-axis, the direction of e_2 must coincide with the direction of the y-axis. At this point the projection of e_2 on H is a point, the origin. If the value of α has not reached yet π, then α is non-acute. This means that the projections of f_1 and of e_2 on H are contained in two rays with a common apex (the projection of v_b on H) that create a non-acute angle, a contradiction. If the value of α has reached π at some point before, then at that point the projection of f_1 on H has a positive slope and is collinear with the origin. This implies that the projections of f_1 and e_2 on H are contained in two different rays with a common apex (the origin) that create a non-acute angle, a contradiction.

Consider now the case in which e_2 creates a non-acute angle with the negative part of the y-axis. If v_b belongs to the half-space $\{y \geq 0\}$, then Rotate the entire geometric graph G about the z-axis in the clockwise direction and repeat the analysis in the previous argument. Now, before the projection of f_1 on H is parallel to the y-axis, the direction of e_1 must coincide with the direction of the y-axis. At this point the projection of e_1 on H is a point, the origin. If the value of α has not reached yet π, then α is non-acute. This means that the projections of f_1 and of e_1 on H are contained in two rays with a common apex (the projection of v_b on H) that create a non-acute angle, a contradiction. If the value of α has reached π at some point before, then at that point the projection of f_1 on H has a positive slope and is collinear with the origin. This implies that the projections of f_1 and e_2 on H are contained in two different rays with a common apex (the origin) that create a non-acute angle, a contradiction.

If v_b belongs to the half-space $\{y \leq 0\}$, then rotate the entire geometric graph G about the z-axis until e_2 is contained in the negative part of the y-axis. At this point f_1 is contained in the quadrant $\{x, y \geq 0\}$ and its projection on H has a positive slope. This implies that the projections of f_1 and e_2 on H are contained in two rays with a common apex (the projection of v_b on H) that create a non-acute angle, a contradiction.

Case 4(ii) The angle between the negative part of the y-axis and both e_1 and e_2 is acute.

If the projection on H_e of the line through f_1 misses the relative interior either of e_1 or of e_2, then we act as before: Without loss of generality assume that the projection on H_e of the line through f_1 misses the relative interior of e_1. Let K' be the plane through f_1 perpendicular to H_e. K' misses the relative interior of e_1 while H_e misses the relative interior of f_1. It now follows from Lemma 3 that f_1 and e_1 are strongly avoiding, a contradiction.

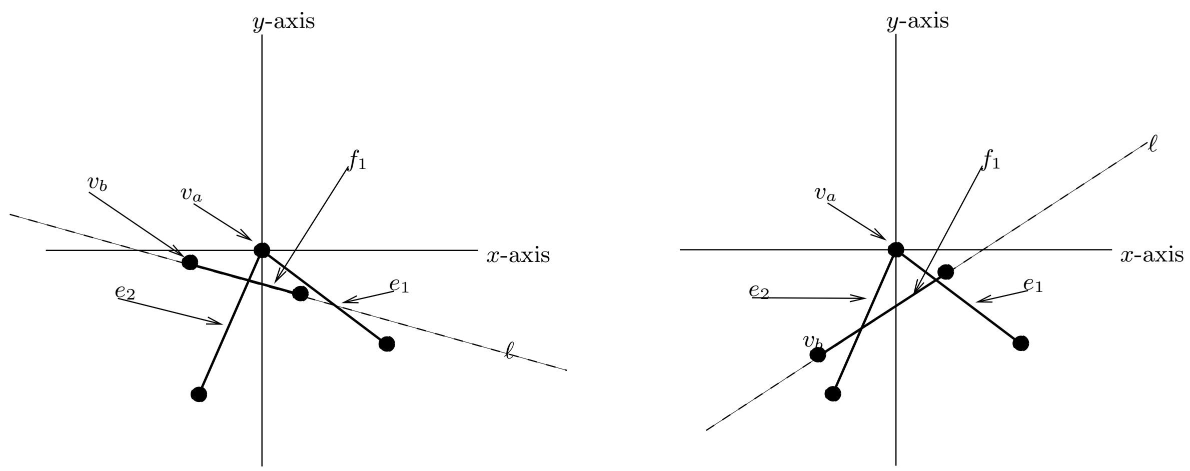

Figure 9: Case 4(ii). Both e_1 and e_2 create an acute angle with the negative part of the y-axis. A view from the z-axis direction.

Hence, assume that the projection on H_e of the line ℓ through f_1 crosses the relative interiors of both e_1 and e_2. It follows that the projection on H_e of the line ℓ through f_1 crosses the negative part of the y-axis (see Figure 9). If the slope of the projection of ℓ on H_e is non-negative we repeat the argument in Case 4(i) (in the sub-case of Case 4(i) where the projection of ℓ on H_e has a non-negative slope) and rotate the entire geometric graph G about the z-axis in the counterclockwise direction, to obtain a contradiction by showing that f_1 and e_2 are strongly avoiding.

If the slope of the projection of ℓ on H_e is negative (see Figure 9), then we will aim to show that the projection of the line through e_1 on H_f misses the relative interior of either f_1 or of f_2. Hence, if K' is the plane through e_1 perpendicular to H_f, then K' misses the relative interior of either f_1 or of f_2. H_f misses the relative interior of e_1, since e_1 is below H_f. Lemma 3 now implies that e_1 and one of f_1 and f_2 are strongly avoiding.

To see that indeed the projection of the line through e_1 on H_f misses the relative interior of either f_1 or of f_2, consider f_1' and f_2' again. Because the slope of the projection of ℓ on H_e is negative, it follows that f_1' is contained in the half-space $\{y \leq 0\}$ (see Figure 9).

Let H' be the plane $\{y = -1\}$. Let E_1 be the intersection point of the line through e_1 with H'. Let F_1' be the intersection of of the line through f_1' with H'. Denote by O the point $(0, -1, 0)$ on H', that is, the intersection of the y-axis with H' (see Figure 10). The angle $\measuredangle F_1'OE_1$ is acute, because the angle created by $\overline{f_1'}$ and $\overline{e_1}$ is acute.

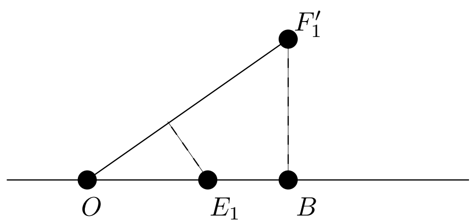

Figure 10: Case 4(ii). The plane H', a view from the y-axis direction.

Because f_1 is parallel to ℓ and because the projection of ℓ on H_e intersects the relative interior of both e_1 and e_2, it follows that the projection of f_1' on H_e is not contained in the cone generated by e_1 and e_2. Therefore, as $\measuredangle F_1'OE_1$ is acute, the projection B of F_1' on the line through O and E_1 is such that E_1 belongs to the segment $[O, B]$. But then the projection of E_1 on the line through O and F_1' belongs to the segment OF_1'. If f_2' creates an acute angle with the negative part of the y-axis, then let F_2' denote the intersection point of the line through f_2 with H'. O belongs to the segment $[F_1', F_2']$ on H' and therefore also E_1 belongs to this segment. This means that the projection on $H_{f'}$ of the line through e_1 belongs to the cone generated by f_1' and f_2'. Hence the projection on H_f of the line through e_1 cannot intersect the relative interiors of both f_1 and f_2.

If f_2' creates a non-acute angle with the negative part of the y-axis, then we switch the roles of f_1 and f_2 with the roles of e_1 and e_2 and conclude by Case 4(i). Indeed, if the negative part of the y-axis is contained in the cone generated by f_1' and f_2', then rotate $\mathbb{R}^3$ about the y-axis until $H_{f'}$ coincides with the $(x - y)$-plane and f_1' is contained in the half-plane $\{x \geq 0\}$. Then reflect $\mathbb{R}^3$ with respect to the $(x - y)$-plane. Observe that now f_2'

creates a non-acute angle with the negative part of the y-axis. We can now apply Case 4(i) to conclude that e_1 and one of f_1 and f_2 are strongly avoiding to get a contradiction.

If the negative part of the y-axis is contained in the cone generated by f'_1 and f'_2, then rotate $\mathbb{R}^3$ about the y-axis until $H_{f'}$ coincides with the $(x-y)$-plane and f'_1 is contained in the half-plane $\{x \geq 0\}$. Then reflect $\mathbb{R}^3$ with respect to the $(x-y)$-plane, and reflect $\mathbb{R}^3$ again with respect to the $(x-z)$-plane Observe that now f'_1 creates a non-acute angle with the negative part of the y-axis. Hence by Case 4(i), e_1 and one of f_1 and f_2 are strongly avoiding, a contradiction. ∎

5 Vázsonyi's conjecture

Theorem 2 implies very easily Vázsonyi's conjecture about the maximum number of diameters in a set of n points in $\mathbb{R}^3$. Consider the diameter graph G on a set P of n points in $\mathbb{R}^3$, that is, the geometric graph whose vertices are the points of P where two points are connected by an edge if they determine a diameter of the set. We will show that G satisfies the conditions of Theorem 2. This will imply that the number of diameters in P is at most $2n-2$.

We need to show that no two edges of G are strongly avoiding. Assume to the contrary that $e=(v_1,v_2)$ and $f=(v_3,v_4)$ are two strongly avoiding edges of G. Let H be a plane such that the orthogonal projections of e and f on H are contained in two rays, with a common apex o, that create an a non-acute angle.

Without loss of generality assume that H is the $(x-y)$-plane and that o is the origin. Let u be the unit vector in H in the direction of the ray containing the projection of e on H. Let w be the unit vector in H in the direction of the ray containing the projection of f on H. Let z be the unit vector in the direction of the z-axis, perpendicular to H.

Writing v_1, v_2, v_3, and v_4 in terms of u, w, z, there are nonnegative numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and real numbers c_1, c_2, c_3, and c_4, such that

$$\begin{aligned} v_1 &= \lambda_1 u + c_1 z \\ v_2 &= \lambda_2 u + c_2 z \\ v_3 &= \lambda_3 w + c_3 z \\ v_4 &= \lambda_4 w + c_4 z \end{aligned}$$

Considering the squared mutual distances between v_1, v_2, v_3, v_4 and the fact that the scalar product of u and w is non-positive (since the angle between u and w is non-acute), we obtain the following equations and inequalities.

$$\begin{aligned}
(\lambda_1-\lambda_2)^2+(c_1-c_2)^2 &= 1 \qquad (1)\\
(\lambda_3-\lambda_4)^2+(c_3-c_4)^2 &= 1 \qquad (2)\\
\lambda_1^2+\lambda_3^2+(c_1-c_3)^2 &\le 1 \qquad (3)\\
\lambda_1^2+\lambda_4^2+(c_1-c_4)^2 &\le 1 \qquad (4)\\
\lambda_2^2+\lambda_3^2+(c_2-c_3)^2 &\le 1 \qquad (5)\\
\lambda_2^2+\lambda_4^2+(c_2-c_4)^2 &\le 1 \qquad (6)
\end{aligned}$$

Assume without loss of generality that $\lambda_1 \ge \lambda_2$ and $\lambda_3 \ge \lambda_4$. From (1) and (3) it follows that

$$(c_1-c_2)^2 \ge (c_1-c_3)^2, \qquad (7)$$

with equality only if $\lambda_2 = \lambda_3 = \lambda_4 = 0$.

From (1) and (4) it follows that

$$(c_1-c_2)^2 \ge (c_1-c_4)^2 \qquad (8)$$

with equality only if $\lambda_2 = \lambda_4 = 0$.

From (2)and (3) it follows that

$$(c_3-c_4)^2 \ge (c_3-c_1)^2 \qquad (9)$$

with equality only if $\lambda_4 = \lambda_1 = \lambda_2 = 0$.

From (2) and (5) it follows that

$$(c_3-c_4)^2 \ge (c_3-c_2)^2 \qquad (10)$$

with equality only if $\lambda_2 = \lambda_4 = 0$.

Summing (1), (2) and deducting the sum of (3) and (6) we deduce:

$$(c_1-c_4)(c_2-c_3) \le 0 \qquad (11)$$

with equality only if $\lambda_2 = \lambda_4 = 0$.

Summing (1), (2) and deducting the sum of (4) and (5) we deduce:

$$(c_1-c_3)(c_2-c_4) \le 0 \qquad (12)$$

with equality only if $\lambda_2 = \lambda_4 = 0$.

If $c_2 = c_4$, we get equality in (12) and hence $\lambda_2 = \lambda_4 = 0$. Therefore, $v_2 = v_4$ a contradiction. Hence we assume without loss of generality that $c_2 > c_4$.

If $c_4 > c_1$, then $c_2 > c_4 > c_1$. This is impossible when (9) and (10) are valid.

If $c_4 < c_1$, then from (11) $c_2 \leq c_3$. If $c_2 = c_3$, then we have equality in (7) and therefore $\lambda_2 = \lambda_3 = 0$. It follows that $v_2 = v_3$ a contradiction. Therefore, if $c_4 < c_1$ we get $c_2 < c_3$. Hence $c_3 > c_2 > c_4$. This is impossible when (7) and (8) are valid.

If $c_1 = c_4$, then we get equality in (9). Therefore, $\lambda_1 = \lambda_4 = 0$ and hence $v_1 = v_4$, a contradiction. ∎

As an immediate corollary of Theorem 2 and Theorem 1 and the fact that there are no two avoiding edges in the diameter graph in $\mathbb{R}^3$, we obtain the following:

Corollary 2. *Let P be a finite set of points in $\mathbb{R}^3$ and let G be the diameter graph of P. Then G can be realized as a generalized thrackle.*

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