On Inducing Polygons and Related Problems*

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Abstract

Bose et al. [2] asked whether for every simple arrangement $\mathcal{A}$ of n lines in the plane there exists a simple n-gon P that *induces* $\mathcal{A}$ by extending every edge of P into a line. We prove that such a polygon always exists and can be found in $O(n \log n)$ time. In fact, we show that every finite family of curves $\mathcal{C}$ such that every two curves intersect at least once and finitely many times and no three curves intersect at a single point possesses the following Hamiltonian-type property: the union of the curves in $\mathcal{C}$ contains a simple cycle that visits every curve in $\mathcal{C}$ exactly once.

1 Introduction

Arrangements of lines in the plane are among the most studied structures in Combinatorial and Computational Geometry (see, e.g., the monograph by Felsner [5]). Every set of straight-line segments S naturally *induces* an arrangement of lines, simply by extending every segment in S into a line. Bose et al. [2] asked the following natural question.

Problem 1. *Does every* simple *arrangement $\mathcal{A}$ of n lines contain a simple n-gon that induces $\mathcal{A}$?*

An arrangement of lines is *simple* if every pair of lines intersects, and no three lines intersect at a single point. A polygon (resp., curve) is *simple* if it is non-self-intersecting. Figure 1(a) shows a simple arrangement of six lines and a simple hexagon that induces this arrangement.

A few partial results related to Problem 1 were obtained. In [2] it was shown that a simple arrangement $\mathcal{A}$ of n lines contains a subarrangement of $m \geq \sqrt{n-1}+1$ lines that has an inducing simple m-gon, and that $\mathcal{A}$ always has an inducing simple n-path (a polygonal chain consisting of n line segments), which can be constructed in $O(n^2)$ time. Recently, the third and fourth authors [7] showed that an inducing n-path can be constructed in $O(n \log n)$ time, and that there always exists an inducing simple $O(n)$-gon, which can be found in $O(n^2)$ time.

Here we settle Problem 1 on the affirmative.[1]

Theorem 2. *For every simple arrangement $\mathcal{A}$ of $n > 2$ lines in the plane there is a simple n-gon that induces $\mathcal{A}$. Given the set of n lines that form $\mathcal{A}$, such a polygon can be constructed in $O(n \log n)$ time.*

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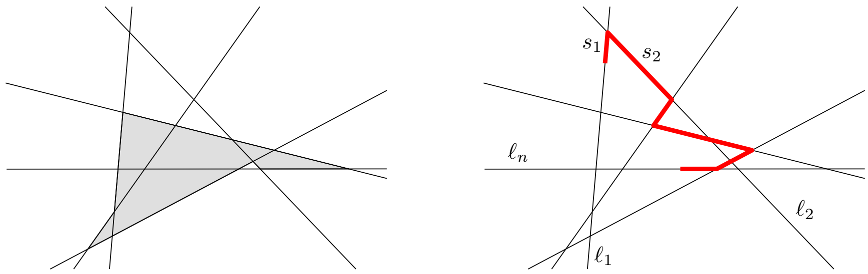
[1] A preliminary version of this proof appeared first in the 25th European Workshop on Computational Geometry [8].



(a) (b)

Figure 1: An inducing simple n-gon and n-path.

A different and much simpler proof of Theorem 2 appears in a companion paper [1]. However, that proof results in an algorithm for finding an inducing simple n-gon whose time complexity is $O(n^4)$.

Note that the first part of Theorem 2 can also be phrased as follows: Every arrangement of lines contains a simple cycle (i.e., a closed curve) that visits every line exactly once. To be more precise, we say that a curve x *visits* another curve y if their intersection contains a point in which they neither cross nor touch. A simple curve visits y *exactly once* if it visits y and their intersection is connected. The first part of Theorem 2 is then equivalent to saying that every simple line arrangement contains a simple (polygonal) cycle that visits every line exactly once. By extending the technique used in [1], we obtain the following generalization of Theorem 2.

Theorem 3. *Let $\mathcal{C}$ be a finite family of $n > 2$ simple curves in $\mathbb{R}^3$, such that every pair of curves in $\mathcal{C}$ intersects at least once and at most finitely many times, and no three curves intersect at the same point. Then $\bigcup_{C \in \mathcal{C}} C$ contains a simple cycle that visits every curve exactly once.*

During our quest for a solution to Problem 1, we also proved the following interesting fact.

Theorem 4. *For every simple arrangement $\mathcal{A}$ of n lines in the plane there is an x-monotone n-path that induces $\mathcal{A}$. Such a path can be constructed in $O(n^2)$ time.*

The rest of this paper is organized as follows. Theorem 2 is proved in Section 2. Then we prove its generalization in Section 3. The proof of the existence of an inducing x-monotone n-path appears in Section 4, while Section 5 contains some concluding remarks.

2 Finding an inducing simple n-gon

3 Proof of Theorem 3

Let $\mathcal{C}$ be a family of n simple curves in $\mathbb{R}^3$, such that every pair of curves in $\mathcal{C}$ intersects at least once and at most finitely many times, and no three of the curves meet at a point. We will show that $\bigcup_{C \in \mathcal{C}} C$ contains a simple closed path that visits every curve in $\mathcal{C}$ exactly once.

The proof is a modification of the argument in [1]. We first find a simple path Q that visits every curve in $\mathcal{C}$ exactly once: pick an arbitrary starting point on an arbitrary curve c'. Walk on c' in a direction containing at least one intersection point with a different curve, and remove c' from $\mathcal{C}$ when reaching the first such intersection point q'. Let c'' be the curve intersecting c' at q'. Continue recursively by finding a path that starts at q' and visits every curve in $\mathcal{C} \setminus \{c'\}$ exactly once.

Let c be a curve in $\mathcal{C}$ containing the last segment of Q thus constructed. Observe that c does not meet Q at any point outside the segment of Q contained in c, for otherwise we would have 'stepped' into c earlier.

A simple (oriented) path W that visits every curve in $\mathcal{C}$ exactly once will be called *rooted in* c if c is the first curve visited by W. Clearly, Q is an example for such a path.

For a path W, as above, we denote by $q_1(W), \ldots, q_{n-1}(W)$ the $n-1$ internal vertices of the path starting from $q_1(W)$ on c. For $i = 1, \ldots, n-2$ we denote by $s_i(W)$ the segment of W whose vertices are $q_i(W)$ and $q_{i+1}(W)$, these will be called the *internal segments* of W. We denote by $c(W)$ the curve in $\mathcal{C}$ that passes through $q_{n-1}(W)$ and contains the last segment of W.

Let s be a portion of a curve in $\mathcal{C}$. We define $|s|$ as the number of intersection points of pairs of curves in $\mathcal{C}$ that lie on s. Finally, we define

$$Y(W) = f(|s_1(W)|, \ldots, |s_{n-2}(W)|),$$

where $f(x_1, \ldots, x_{n-2})$ is a strictly monotone increasing function of the lexicographic order of $(x_1, \ldots, x_{n-2})$.[2]

Consider the simple path W that is rooted in c and visits every curve in $\mathcal{C}$ exactly once, such that $Y(W)$ is minimum. Let p be an intersection point of $c(W)$ and c. Let $q_n(W)$ be the intersection point of $c \cup s_1(W) \cup \ldots \cup s_{n-2}(W)$ and the portion of $c(W)$ between $q_{n-1}(W)$ and p that is closest to $q_{n-1}(W)$ along the curve $c(W)$.

If $q_n(W)$ lies on c, then observe that the vertices $q_1(W), \ldots, q_n(W)$ define a simple closed path that visits every curve in $\mathcal{C}$ exactly once. Assume therefore that $q_n(W)$ is an intersection point of $c(W)$ with $s_i(W)$ for some $1 \le i \le n-2$. Let s' denote the portion of $s_i(W)$ delimited by $q_i(W)$ and $q_n(W)$. Let s'' denote the portion of $c(W)$ delimited by $q_n(W)$ and $q_{n-1}(W)$. Then we define W' as the path rooted in c whose internal segments are

$$s_1(W), \ldots, s_{i-1}(W), s', s'', s_{n-2}(W), s_{n-3}(W) \ldots, s_{i+2}(W),$$

and $c(W')$ is the curve containing the segment $s_{i+1}(W)$.

Observe that W' is a simple path rooted on c that visits every curve in $\mathcal{C}$ exactly once. It immediately follows that $Y(W') < Y(W)$, because $s_j(W') = s_j(W)$ for $j = 1, \ldots, i-1$ while it is easy to see that $|s_i(W')| < |s_i(W)|$ as $s_i(W') = s' \subset s_i(W)$ and $q_{i+1}(W)$ is an intersection point in $s_i(W) \setminus s_i(W')$. We have thus reached a contradiction to the minimality of W.

Remarks. (1) Because Theorem 3 is stated in $\mathbb{R}^3$, geometry actually does not play any role here. We may conclude the same result for "combinatorial curves" that "intersect" finitely many times, as long as there is a total order on the set of intersection points in each curve.
(2) The result in Theorem 3 is valid also if the curves in $\mathcal{C}$ are not simple and have self-crossings. In this case we repeat the proof and ignore self-intersections of curves. Finally, when obtaining the resulting closed path we observe that self-intersections of the closed path result only from loops in the path. These loops can easily be canceled.

[2]For example, $f(11, 0, 6, \ldots) > f(6, 9, 5, \ldots) > f(6, 9, 4, \ldots)$.

4 x-monotone inducing n-path: Proof of Theorem 4

In this section we show that every simple arrangement of n lines contains an inducing x-monotone n-path. Since the path is x-monotone, it is clearly simple. Suppose first that n is an even number. We sort the lines according to their slopes, and denote by A the set of the first $n/2$ lines in this order, and by B the rest of the lines. Initially, all the lines are unmarked. Pick the leftmost intersection point of two unmarked lines, one from A and one from B, then mark these lines. Continue to pick a total of $n/2$ points $p_1, p_2, \ldots, p_{n/2}$ this way. We will construct an x-monotone n-path through $p_1, p_2, \ldots, p_{n/2}$.

Denote the lines that intersect at p_i by $a_i \in A$ and $b_i \in B$, $i = 1, 2, \ldots, n/2$. First, pick arbitrarily one of the lines that intersect at p_1, say a_1, walk a short distance on a_1 from a point left of p_1 to p_1, then walk a short distance on b_1 rightwards. Assume that we have built an x-monotone $2i$-path that goes a short distance rightwards beyond p_i and induces the lines $a_1, \ldots, a_i$ and $b_1, \ldots, b_i$. We will show how to extend it into an x-monotone $2(i+1)$-path that goes a short distance rightwards beyond p_{i+1} and induces the lines $a_1, \ldots, a_{i+1}$ and $b_1, \ldots, b_{i+1}$.

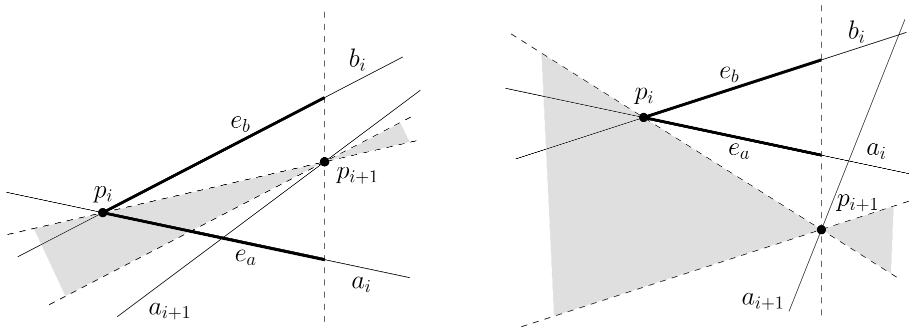
Observation 5. *The intersection of a_{i+1} (resp., b_{i+1}) and b_i (resp., a_i) is to the right of p_i.*

Proof. Otherwise this intersection point would be picked instead of p_i. □

Consider the triangle with a vertex at p_i, an edge on the vertical line through p_{i+1}, an edge e_a on a_i, and an edge e_b on b_i (see Figure 2).

Observation 6. *e_a (resp., e_b) is crossed by a_{i+1} or b_{i+1}.*

Proof. We consider two cases based on whether p_{i+1} is inside the wedge determined by a_i and b_i. Suppose that it is. Then a_{i+1} (resp., b_{i+1}) must cross either e_a and e_b. Suppose, w.l.o.g., that they both cross e_a (otherwise, we can reflect everything with respect to the x-axis). See Figure 2(a). Then a_{i+1} must have a larger slope than b_i, otherwise it will cross



(a) p_{i+1} is inside the wedge determined by a_i and b_i.

(b) p_{i+1} is outside the wedge determined by a_i and b_i.

Figure 2: An illustration for the proof of Observation 6. a_{i+1} cannot be in the shaded region.

b_i to the left of p_i, contradicting Observation 5. This is of course impossible.

Suppose that p_{i+1} is outside the wedge determined by a_i and b_i. We can assume, w.l.o.g., that it is below the wedge, for otherwise we can reflect everything with respect to the x-axis. If a_{i+1} does not cross both e_a and e_b, then it must have a larger slope than b_i, or cross b_i to the left of p_i, which is impossible. See Figure 2(b) for an illustration. □

Now, suppose that the path built so far goes a short distance rightwards beyond p_i on e_a (resp., e_b). Then by Observation 6 there is a line $\ell \in \{a_{i+1}, b_{i+1}\}$ that crosses e_a (resp., e_b). Walk on e_a (resp., e_b) until the intersection point with ℓ, then walk on ℓ until p_{i+1}, and finally walk a short distance rightwards on the other line in $\{a_{i+1}, b_{i+1}\}$. The new path is an x-monotone $2(i+1)$-path that goes a short distance rightwards beyond p_{i+1} and induces the lines $a_1, \ldots, a_{i+1}$ and $b_1, \ldots, b_{i+1}$.

It remains to consider the case that n is an odd number. Let ℓ be the line with the median slope. Create a new line ℓ' that is a slightly rotated copy of ℓ such that its slope is slightly smaller than the slope of ℓ, and their intersection point is the leftmost intersection point in the arrangement $\mathcal{A} \cup \{\ell'\}$. Now continue as before, while choosing ℓ' as the first induced line. Finally, remove the segment of ℓ' from the constructed path.

Time complexity. An inducing x-monotone n-path can be found in $O(n^2)$ time as follows. First we construct the arrangement of lines. This can be done in $O(n^2)$ time [3]. Then we find the sets A and B in $O(n \log n)$ time. For every line in A we find its leftmost intersection point with a line from B. The first vertex of the path is the leftmost point among these points. The two lines defining this minimum point are removed from the arrangement while updating the minimum leftmost points for the other lines. This can be done in $O(n)$ time. The process of finding the next leftmost intersection point between a line from A and a line from B (among the remaining lines), removing the corresponding lines, and making appropriate updates is then repeated $O(n)$ times.

5 Concluding remarks

We have proved that every simple arrangement of n lines contains an inducing simple n-gon, and that such a polygon can be constructed in $O(n \log n)$ time. We believe that this time complexity is the best possible, but leave it as an open question.

An inducing simple polygon need not be unique. It would be interesting to determine the maximum and minimum number of inducing simple n-gons of an arrangement of n lines. Figure 3 shows an arrangement with exponentially many inducing simple n-gons.

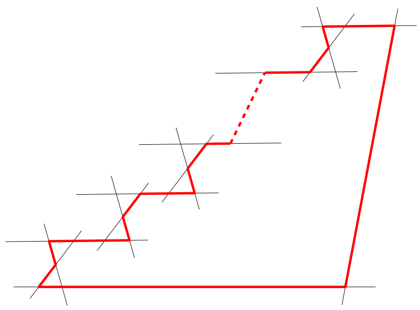


Figure 3: n lines with exponentially many inducing simple n-gons. At every "step" of the "stairs" one can "climb" either from left or from right.

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