Every simple arrangement of n lines contains an inducing simple n-gon*

Eyal Ackerman† Rom Pinchasi‡ Ludmila Scharf† Marc Scherfenberg†

Abstract

We give a short and simple proof of the existence of an *inducing* simple n-gon for every simple arrangement of n lines in the plane.

1 Introduction

Arrangements of lines in the plane are among the most studied structures in Combinatorial and Computational Geometry (see, e.g., [4, 3]). Every set of straight-line segments S naturally *induces* an arrangement of lines, simply by extending every segment in S into a line. An arrangement of lines is *simple* if every pair of lines intersect, and no three lines intersect at a single point. A polygon (resp., curve) is *simple* if it is non-self-intersecting.

Bose et al. [2] asked the following natural question.

Problem 1. *Does every* simple *arrangement $\mathcal{A}$ of n lines contain a simple n-gon that induces $\mathcal{A}$?*

Figure 1(a) shows a simple arrangement of six lines and a simple hexagon that induces this arrangement.

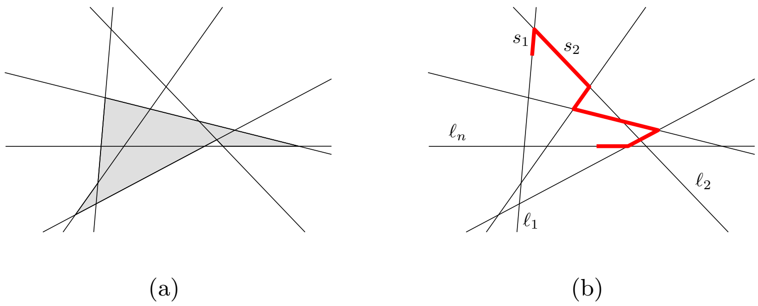


Figure 1: An inducing simple n-gon and n-path.

Several partial results related to Problem 1 were obtained [2, 5], before it was settled in the affirmative in a companion paper [1], along with some other extensions and related problems. [1]

*Research by Eyal Ackerman was supported by a fellowship from the Alexander von Humboldt Foundation. Research by Rom Pinchasi was supported by the Israeli Science Foundation (grant No. 938/06).

†Institute of Computer Science, Freie Universität Berlin, Takustr. 9, 14195 Berlin, Germany. {eyal,scharf,scherfen}@mi.fu-berlin.de.

‡Mathematics Department, Technion—Israel Institute of Technology, Haifa 32000, Israel. room@math.technion.ac.il.

[1]A preliminary version of this proof appeared first in the 25th European Workshop on Computational Geometry [6].

Theorem 2 ([1])**.** *For every simple arrangement $\mathcal{A}$ of n lines in the plane there is a simple n-gon that induces $\mathcal{A}$.*

However, that proof of Theorem 2 in [1] is very long and technical. Here, we provide a short and simple proof of Theorem 2. Our proof is constructive and yields a polynomial-time algorithm for finding an inducing simple n-gon. The proof in [1] results in a faster $O(n \log n)$-time algorithm.

2 A short proof of Theorem 2

Let $\mathcal{A}$ be a simple arrangement of n lines in the plane. We begin by constructing a simple path that visits every line in $\mathcal{A}$ exactly once. This is done in a way similar to the construction of an inducing path in [2]. Consider an arbitrary intersection point of two lines, denote these lines by ℓ_1 and ℓ_2. Walk a short distance on ℓ_1 toward its intersection point with ℓ_2. Remove ℓ_1, and walk on ℓ_2 in a direction that contains at least one intersection point, until reaching the first intersection point. Let ℓ_3 be the other line that determines this intersection point. Remove ℓ_2 and repeat the same process for ℓ_3 and so on and so forth, until reaching a line that has no additional intersection points. Finally, walk a short distance on this line in some direction. See Figure 1(b) for an example.

Since every pair of lines intersect, no line is missed and this process results in a path that induces every line in $\mathcal{A}$. Denote this path by Q. The lines in $\mathcal{A}$ are denoted by $\ell_1, \ell_2, \ldots, \ell_n$ according to the order they were visited by Q. Denote by s_j the segment of ℓ_j on Q. Assume to the contrary that Q is self-intersecting. Then there are two intersecting segments, s_i and s_j, such that $i \leq j-2$. But this is a contradiction to the definition of ℓ_{i+1} as the first line, different from $\ell_1, \ldots, \ell_i$, that we encounter while walking along ℓ_i.

Observe that Q lies in one of the two half-planes determined by ℓ_n. Indeed, otherwise ℓ_n would have crossed Q, contradicting the definition of ℓ_n as the last line in $\mathcal{A}$ we encounter while creating the path Q. We assume without loss of generality that ℓ_n coincides with the x-axis and that Q lies in the half-plane above ℓ_n. For convenience, denote the line ℓ_n by ℓ.

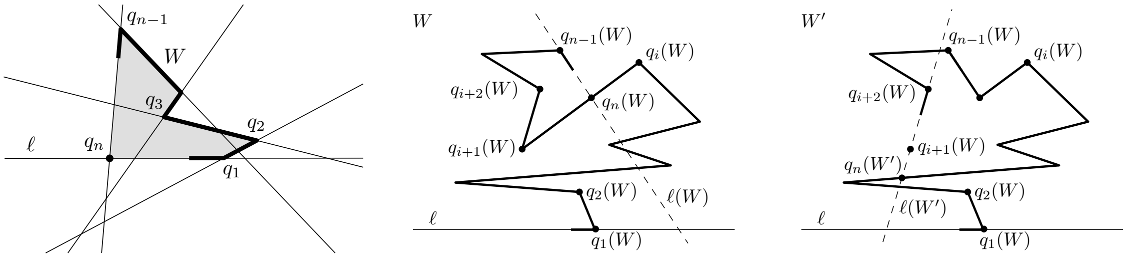
We call a simple inducing n-path of $\mathcal{A}$ *rooted above* ℓ, if it lies in the closed half-plane above ℓ and ℓ includes an extreme segment of the path. As we have just seen, there is at least one simple inducing n-path rooted above ℓ, namely Q. For such a path W we denote by $q_1(W), \ldots, q_{n-1}(W)$ the $n-1$ internal vertices of the path starting from $q_1(W)$ on ℓ. We denote by $\ell(W)$ the line through $q_{n-1}(W)$ that includes the other extreme segment of W. Denote by $\ell^-(W)$ the half-line of $\ell(W)$ that consists of all points with y-coordinates smaller than the y-coordinate of $q_{n-1}(W)$. We denote by $q_n(W)$ the topmost (also first) intersection point of $\ell^-(W)$ with $\ell \cup [q_1(W)q_2(W)] \cup \ldots \cup [q_{n-2}(W)q_{n-1}(W)]$. (Here, $[ab]$ denotes the line segment connecting point a to point b.)

Let $z_1, \ldots, z_m$ denote all the intersection points in $\mathcal{A}$ indexed in any way such that $i < j$ if the y-coordinate of z_i is smaller than the y-coordinate of z_j. We then define for every j, $Y(z_j) = j$.

For a simple inducing n-path W rooted above ℓ, we define $Y(W) = \sum_{i=1}^{n} Y(q_i(W))$. Consider the simple inducing n-path W rooted above ℓ such that $Y(W)$ is minimum. If $q_n(W)$ lies on ℓ, then observe that the vertices $q_1(W), \ldots, q_n(W)$ define a simple inducing closed n-path of $\mathcal{A}$ (see Figure 2(a)). Assume therefore that $q_n(W)$ is the intersection point of $\ell(W)$ with the segment $[q_i(W)q_{i+1}(W)]$ for some $1 \leq i \leq n-2$ (see Figure 2(b)). Then we define W' as the path whose internal vertices are

$$q_1(W), \ldots, q_i(W), q_n(W), q_{n-1}(W) \ldots, q_{i+2}(W),$$

and hence $\ell(W')$ is the line through $q_{i+1}(W)$ and $q_{i+2}(W)$. Observe that W' is a simple inducing n-path rooted above ℓ. We have $Y(W') < Y(W)$ because $q_n(W')$ has a smaller



(a) q_n lies on ℓ (b) q_n lies on W (c) W'

Figure 2: The paths W and W'

y-coordinate than the y-coordinate of $q_{i+1}(W)$ (see Figure 2(c)). We have thus reached a contradiction to the minimality of W.

Remarks:

1. The proof of Theorem 2, presented above, yields an algorithm with running time polynomial in n. This is because $Y(W)$ is always smaller than n^3 and this gives a bound on the number of iterations going from W to W' required to find a simple inducing closed n-path for $\mathcal{A}$. A more involved argument in [1] yields an algorithm that finds a simple induced closed n-path in running time of $O(n \log n)$.

2. Theorem 2 can be extended to a family of arbitrary curves in $\mathbb{R}^3$ as follows: Let $\mathcal{C}$ be a family of n simple curves in $\mathbb{R}^3$, such that every pair of curves in $\mathcal{C}$ intersect at least once and at most finitely many times, and no three of the curves meet at a point. Then $\bigcup_{C \in \mathcal{C}} C$ contains a simple closed path that visits every curve in $\mathcal{C}$ exactly once. A proof of this statement is provided in [1].

References

[1] E. Ackerman, R. Pinchasi, L. Scharf and M. Scherfenberg, On inducing polygons and related problems, preprint.

[2] P. Bose, H. Everett, and S. Wismath, Properties of arrangement graphs, *Int. J. Comput. Geom. Appl.* **13** (2003), 447–462.

[3] S. Felsner, *Geometric Graphs and Arrangements. Some Chapters from Combinatorial Geometry*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Wiesbaden, 2004.

[4] B. Grünbaum, *Arrangements and spreads.* Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 10. American Mathematical Society Providence, R.I., 1972.

[5] L. Scharf and M. Scherfenberg, Inducing polygons of line arrangements, *Proc. 19th Int. Symp. on Algorithms and Computation* (ISAAC), *LNCS* **4059**, 508–520, 2008.

[6] L. Scharf and M. Scherfenberg, Inducing n-gon of an arrangement of lines, *25th European Workshop on Computational Geometry*, Brussels, Belgium, March 2009, 129–132.