

# On Some Isoperimetric Inequality In The Universal Covering Space Of The Punctured Plane

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## Abstract

We find the largest  $\epsilon$  for which any simple closed path  $\alpha$  in the universal cover  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  of  $\mathbb{R}^2 \setminus \mathbb{Z}^2$ , equipped with the natural lifted metric from the Euclidean two dimensional plane, satisfies  $L(\alpha) \geq \epsilon A(\alpha)$ . Where  $L(\alpha)$  is the length of  $\alpha$  and  $A(\alpha)$  is the area enclosed by  $\alpha$ . This generalizes a result of Schnell and Gomis, and provides an alternative proof for the same isoperimetric inequality in  $\mathbb{R}^2 \setminus \mathbb{Z}^2$

## 1 Introduction

A classical theorem of Jarnik in number theory asserts that for every embedded closed curve  $\alpha \subset \mathbb{R}^2 \setminus \mathbb{Z}^2$ , with no integer lattice points inside the domain bounded by  $\alpha$ ,  $L(\alpha) \geq A(\alpha)$ , where  $L(\alpha)$  is the length of  $\alpha$  and  $A(\alpha)$  is the area enclosed by  $\alpha$ , (see [Hua],p.123). For a related work on *convex* simple curves in the plane see [B].

We shall now bring a *simple* argument showing the existence of such a linear isoperimetric inequality in a more general setting:

**Theorem 1.1.** *Let  $Z$  be a closed set in  $\mathbb{R}^2$ . Assume that there exists a constant  $M$  such that for every  $x \in \mathbb{R}^2 \setminus Z$ ,  $d(x, Z) \leq M$ , where  $d(x, Z)$  denotes the Euclidean distance from  $x$  to the set  $Z$ . Then there is a linear isoperimetric inequality in  $\mathbb{R}^2 \setminus Z$ . That is, there is an absolute constant  $c_Z > 0$  such that for every simple contractable closed curve  $\alpha$  in  $\mathbb{R}^2 \setminus Z$  we have  $L(\alpha) \geq c_Z A(\alpha)$ .*

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**Proof.** If  $\alpha$  is contained in a disc of radius 1, then the theorem follows from the classic isoperimetric inequality in the plane  $4\pi A(\alpha) \leq L^2(\alpha)$ . Indeed, if  $L(\alpha) \leq 1$ , then  $L(\alpha) \geq L^2(\alpha) \geq 4\pi A(\alpha)$ . And if  $L(\alpha) > 1$ , then  $L(\alpha) > 1 \geq \frac{1}{\pi} A(\alpha)$ .

Let  $x_1, \dots, x_n$  ( $n > 1$ ) be a maximal set of points on  $\alpha$  such that for every  $i \neq j$   $d(x_i, x_j) > 1$ . Then clearly  $L(\alpha) \geq n$ . Observe that any point in  $\alpha$  is at distance at most 1 from some  $x_i$  because of the maximality of  $n$ . Let  $x$  be any point in the region bounded by  $\alpha$ . Let  $z \in Z$  be a point such that  $d(z, x) \leq M$ . The line segment between  $x$  and  $z$  must cross  $\alpha$ , for  $x$  is in the region bounded by  $\alpha$  and  $z$  is not. It follows that  $x$  is at distance at most  $M + 1$  from some point  $x_i$ . Therefore  $A(\alpha) \leq n\pi(M + 1)^2$ . We can now conclude that

$$L(\alpha) \geq n = n\pi(M + 1)^2 \frac{1}{\pi(M + 1)^2} \geq \frac{1}{\pi(M + 1)^2} A(\alpha).$$

□

Taking  $Z = \mathbb{Z}^2$  in Theorem 1.1 we deduce a linear isoperimetric inequality in  $\mathbb{R}^2 \setminus \mathbb{Z}^2$ .

In [ScSe] Schnell and Gomis give a tight (best possible) linear isoperimetric inequality for the relation between the perimeter and the area of a simply connected region in  $\mathbb{R}^2 \setminus \mathbb{Z}^2$ . Their proof is very elegant and relies on Pick's formula in the Euclidean plane.

In this paper we generalize the above result of Schnell and Gomis and show that the same (best possible) linear isoperimetric inequality holds in the more general space  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ , the universal covering of  $\mathbb{R}^2 \setminus \mathbb{Z}^2$  equipped with the natural lifted metric from the two dimensional Euclidean plane. This is a somewhat surprising example for a tight isoperimetric inequality in a base space  $X$  that can be lifted to be the same tight isoperimetric inequality in  $\widetilde{X}$ , the universal covering space of  $X$ . This is not the case for many other spaces such as the torus, the infinite cylinder, and other natural examples.

It follows from a general theorem of Bonk and Eremenko ([BE]), that contractible closed curves satisfy a linear isoperimetric inequality in  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ . Polterovich and Sikorav ([PS 1]) showed that for a generalized definition of the area of a closed contractible curve  $\beta \subseteq \mathbb{R}^2 \setminus \mathbb{Z}^2$ ,  $(1 + \sqrt{2})L(\beta) \geq A(\beta)$ . In fact, using methods in [PS 1], one can show that for an embedded closed curve  $\beta \subseteq \mathbb{R}^2 \setminus \mathbb{Z}^2$ ,  $(1 + \sqrt{2})L(\beta) \geq A(\beta)$  ([PS 2]). In this paper we find the tight linear isoperimetric inequality in  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ . More specifically, we define a constant that we denote by  $\epsilon$  and prove, in Section 4, the following main result of this paper:

**Theorem 1.2.** *Let  $\alpha : S^1 \rightarrow \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  be a simple closed curve. Then  $L(\alpha) \geq \epsilon A(\alpha)$ . The constant  $\epsilon$  is best possible.*

The constant  $\epsilon$  is defined in Section 3. There it will also be shown that  $\epsilon$  can be obtained implicitly by the equations:

$$\epsilon = \frac{\frac{\pi - \alpha}{\sin \alpha}}{\frac{\pi - \alpha}{4 \sin^2 \alpha} + \frac{\cos \alpha}{4 \sin \alpha} + \frac{1}{2}}, \quad \sin \alpha = \frac{\epsilon}{2}, \quad \frac{\pi}{2} \leq \alpha \leq \pi.$$

An approximated solution to this system is  $\epsilon \approx 1.71579$ .

We note that since the constant  $\epsilon$  is the same constant found by Schnell and Gomis, the fact that  $\epsilon$  cannot be replaced by a larger constant in the statement of Theorem 1.2 follows from the tightness of the result in [ScSe].

## 2 A Description of $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$

**Definition 2.1.** A basic square  $S$  in  $\mathbb{R}^2 \setminus \mathbb{Z}^2$  is a closed square of area one in  $\mathbb{R}^2$ , centered at a point  $(m + \frac{1}{2}, n + \frac{1}{2})$ , where  $n, m \in \mathbb{Z}$ , with all whose four vertices excluded.

Consider the grid  $G \subset \mathbb{R}^2 \setminus \mathbb{Z}^2$  consisting of horizontal and vertical lines through the points  $(m + \frac{1}{2}, n + \frac{1}{2})$ , where  $n, m \in \mathbb{Z}$ . The universal cover  $\widetilde{G}$  of  $(G, \text{Euclidean})$  is an infinite tree for which the degree of every vertex is four and the length of every edge is one. Let  $P : \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2} \rightarrow \mathbb{R}^2 \setminus \mathbb{Z}^2$ , be the covering map.  $G$  is a deformation retract of  $\mathbb{R}^2 \setminus \mathbb{Z}^2$ , hence  $P^{-1}(G)$  is a deformation retract of  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ . Therefore,  $\pi_1(P^{-1}(G)) = 1$  which implies that  $P^{-1}(G)$  can be identified with  $\widetilde{G}$ . Let  $S$  be a basic square in  $\mathbb{R}^2 \setminus \mathbb{Z}^2$ . Then the contractability of  $S$  implies that any connected component of  $P^{-1}(S)$  can be identified with  $S$ . Hence,  $P^{-1}(S)$ , the lifting of every basic square  $S$  in  $\mathbb{R}^2 \setminus \mathbb{Z}^2$  centered at a fixed point  $(m_0 + \frac{1}{2}, n_0 + \frac{1}{2})$ , is a set of infinitely many copies of  $S$  centered at the points of  $P^{-1}(m_0 + \frac{1}{2}, n_0 + \frac{1}{2})$ . One can think of the universal covering space  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  as a thick 4-regular tree of width one, that is, the tree  $\widetilde{G}$  with infinitely many basic squares centered at its vertices.

**Definition 2.2.** A fundamental square in  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  is a connected component of  $P^{-1}(S)$ , where  $S$  is a basic square in  $\mathbb{R}^2 \setminus \mathbb{Z}^2$ .

An edge in  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  is a boundary edge of a fundamental square in  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ .

We now show a fundamental property of closed embedded curves in  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ .

**Lemma 2.3.** Let  $\beta : [0, 1] \rightarrow \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  be an oriented simple closed curve. Let  $\beta(t_1)$  and  $\beta(t_2)$  ( $t_1 < t_2$ ) be two intersection points of  $\beta$  with an edge  $e$  in  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  such that there is no  $t_1 < c < t_2$  with  $\beta(c) \in e$ . Assume further that  $t_1$  is a point where  $\beta$  exits a fundamental square  $S$  with respect to the orientation of  $\beta$ . Then there is no  $t_1 < c < t_2$  such that  $\beta(c)$  intersects  $S$ .

In other words. If  $\beta$  exits a fundamental square  $S$  through an edge  $e$ , the next time it intersects  $S$  is through the same edge  $e$ .

**Proof.** Assume not. Then there is  $c$  ( $t_1 < c < t_2$ ), such that  $\beta(c) \in S$ . We can assume that  $\beta(c)$  is an intersection point with an edge  $e' \neq e$ , where  $e'$  is an edge of  $S$ . Denote by  $M$  the union of the subarc of  $\beta$ ,  $\{\beta(t) : t_1 \leq t \leq c\}$  with the straight line segment between  $\beta(t_1)$  and  $\beta(c)$ . Then  $M$  is an embedded closed curve. Looking at  $M$  and  $e$  as curves in the completion of  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ , we see that the intersection number  $M \circ e$ , mod 2, equals one. This is a contradiction, because  $M$  is contractable, hence  $M \circ e = 0$ .  $\square$

### 3 Defining $\epsilon$

We will now define a number that we denote by  $\epsilon$ . This number satisfies a certain isoperimetric inequality and will play a crucial role in the sequel. Let  $P_0 = (0, 0)$  and  $Q_0 = (1, 0)$ . Consider the family  $\mathcal{H}$  of all simple paths  $\beta$  with  $P_0$  and  $Q_0$  as endpoints, that lie in the planar region  $\{(x, y) | 0 \leq x \leq 1, y \geq 0\}$ .

For every such  $\beta$  let  $L(\beta)$  denote the length of  $\beta$  and  $A(\beta)$  denote the area enclosed by  $\beta$  and the interval  $P_0Q_0$ .

We define  $\epsilon$  to be  $\inf_{\beta \in \mathcal{H}} \frac{L(\beta)}{A(\beta) + 1/2}$ .

**Claim 3.1.**  $\epsilon$  is a minimum which is obtained for a curve in  $\mathcal{H}$ .

**Proof.** First, observe that by considering  $\beta$  to be the half-circle  $\{(x, y) | y = \sqrt{1 - x^2}, 0 \leq x \leq 1\}$  we conclude that  $\epsilon < 1.8$ . Moreover, it is enough to consider only the subfamily of  $\mathcal{H}$  of all the curves  $\beta$  with the property that no  $y$ -coordinate of a point of  $\beta$  exceeds 100. Indeed, if the largest  $y$ -coordinate of a point on  $\beta$  is  $k$ , then  $L(\beta) \geq 2k$  and  $A(\beta) \leq k$ . Therefore,

$$\frac{L(\beta)}{A(\beta) + \frac{1}{2}} \geq \frac{2k}{k + \frac{1}{2}} > 2 - \frac{1}{k}.$$

If  $k > 100$ , then  $2 - \frac{1}{k} > 1.8$ . It now follows easily by principles of compactness that there exists an optimal curve in  $\mathcal{H}$ .  $\square$

**Claim 3.2.**  $\epsilon$  is obtained for  $\beta$  which is a circular arc.

Claim 3.2 will be a consequence of the following theorem.

**Theorem 3.3.** Let  $P$  and  $Q$  be two points on the  $x$ -axis of  $\mathbb{R}^2$ . Denote by  $\mathcal{T}$  be the family of all curves  $\beta \subset \mathbb{R}^2$  with the following two properties:

1.  $\beta$  is a simple curve with endpoints  $P$  and  $Q$ .
2.  $\beta$  lies above the segment  $PQ$  and in the region bounded by the lines through  $P$  and  $Q$  that are perpendicular to  $PQ$ .

Let  $c > 0$  be a positive constant such that  $c|PQ| < 2$ . Then the minimum over all  $\beta \in \mathcal{T}$  of the expression  $L(\beta) - cA(\beta)$ , is obtained for  $\beta \in \mathcal{T}$  which is a circular arc.

**Proof.** Without loss of generality we assume that  $P$  is to the left of  $Q$ . Clearly, an optimal curve  $\beta$  must be a concave curve. We can also assume that the optimal curve  $\beta$  (that may not be unique) is symmetric with respect to the line which is the perpendicular bisector to the segment  $PQ$ .

We need the following claim.

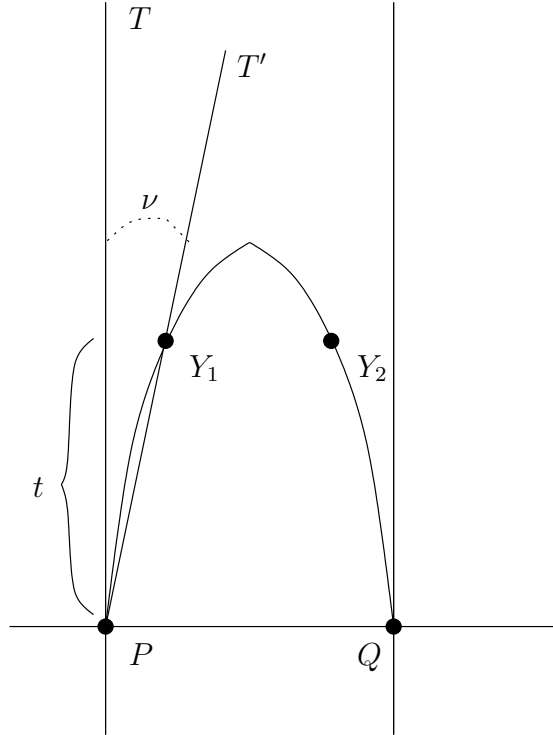


Figure 1:

**Claim 3.4.** *Let  $T$  be a line which passes through  $P$  such that  $\beta$ , is fully contained in a closed half-plane bounded by  $T$ . Assume further that  $T$  has the smallest positive slope among all such lines. Then the slope of  $T$  (with respect to the segment  $PQ$ ) is strictly smaller than  $\pi/2$ .*

**Proof.** Assume to the contrary that  $T$  is vertical. Let  $T'$  be a line through  $P$  whose slope equals  $\pi/2 - \nu$ , where  $\nu$  is a small positive number to be determined later.

Let  $Y_1$  be the point on  $\beta \cap T'$  with the largest  $y$  coordinate. Denote the value of the  $y$ -coordinate of  $Y_1$  by  $t$ . Let  $Y_2$  be the symmetric point to  $Y_1$  with respect to the vertical line through the midpoint of  $PQ$ . See Figure 3.

Let  $\beta'$  be the subarc of  $\beta$  between  $Y_1$  and  $Y_2$ . Now  $L(\beta') \leq L(\beta) - 2t$ .  $A(\beta') \geq A(\beta) - t|PQ|$ .

Let  $s = |PQ|/|Y_1Y_2|$  and let  $\beta''$  be the curve obtained from  $\beta'$  by applying a similarity transformation with ratio  $s$ . The endpoints of  $\beta''$  are at distance  $|PQ|$  from each other, therefore, by identifying them with  $P$  and  $Q$  we may regard  $\beta''$  as a curve in  $\mathcal{T}$ . Moreover,  $L(\beta'') = sL(\beta')$  and  $A(\beta'') = s^2A(\beta')$ . Denote  $x = \frac{2t \tan \nu}{|PQ| - 2t \tan \nu}$ . Observe that  $s = 1 + x$ .

Therefore,

$$\begin{aligned}
L(\beta'') - cA(\beta'') &= sL(\beta') - cs^2A(\beta') \leq \\
&\leq s(L(\beta) - 2t) - cs^2(A(\beta) - t|PQ|) = \\
&= (1+x)(L(\beta) - 2t) - c(1+x)^2(A(\beta) - t|PQ|) = \\
&= L(\beta) - cA(\beta) + (c|PQ|t - 2t + x(L(\beta) - 2t) - c(2x+x^2)(A(\beta) - t|PQ|))
\end{aligned}$$

For  $\nu$  small enough  $x$  is much smaller than  $t$  as it is of the order of  $t \tan \nu$ . We thus obtain a contradiction to the minimality of  $\beta$ , assuming that  $c|PQ| - 2 < 0$ .  $\square$

Let  $X$  be any point on the curve  $\beta$ . We will show that the angle  $\angle PXQ$  is independent of  $X$ . This will clearly show that  $\beta$  is a circular arc.

Let  $\alpha_0 = \angle PXQ$ . For every  $\alpha$  in a small neighborhood of  $\alpha_0$  we define a curve in  $\mathcal{T}$  in the following way:

We think of  $\beta_{PX}$ , the subarc of  $\beta$  between  $P$  and  $X$ , as solid and similarly of  $\beta_{XQ}$ , the subarc of  $\beta$  between  $X$  and  $Q$ . On the other hand we think of the point  $X$  as an axis about which the solid parts  $\beta_{PX}$  and  $\beta_{XQ}$  can rotate. We then rotate the parts  $\beta_{PX}$  and  $\beta_{XQ}$  in such a way that  $\angle PXQ$  becomes equal to  $\alpha$ . We thus obtain a curve the distance between whose endpoints is  $r_\alpha = \sqrt{|PX|^2 + |XQ|^2 - 2|PX||XQ|\cos(\alpha)}$ . We then apply a similarity transformation with ratio equals  $|PQ|/r_\alpha$  to obtain a new curve  $\beta_\alpha$  whose endpoints (that are at distance  $|PQ|$  from each other) we identify with  $P$  and  $Q$ . It follows from Claim 3.4 that if  $\alpha$  is in a small enough neighborhood of  $\alpha_0$ , then  $\beta_\alpha$  belongs to the class  $\mathcal{T}$  - the crucial point is that if  $\alpha$  is very close to  $\alpha_0$ ,  $\beta_\alpha$  is still contained in the region bounded by the vertical lines through  $P$  and  $Q$ .

Let  $g(\alpha) = L(\beta_\alpha) - cA(\beta_\alpha)$ . We know that  $g(\alpha)$  is minimized for  $\alpha = \alpha_0$ .

We will now obtain a more direct formula for  $g(\alpha)$  and interpret the condition  $g'(\alpha_0) = 0$ .

We start with  $g(\alpha) = L(\beta_\alpha) - cA(\beta_\alpha)$ . Clearly,  $L(\beta_\alpha) = \frac{|PQ|}{r_\alpha}L(\beta)$ .  $A(\beta_\alpha)$  is given by  $(\frac{|PQ|}{r_\alpha})^2(A(\beta_{PX}) + A(\beta_{XQ}) + \frac{1}{2}|PX||XQ|\sin \alpha)$ , where  $A(\beta_{PX})$  ( $A(\beta_{XQ})$ ) is the area enclosed by  $\beta_{PX}$  ( $\beta_{XQ}$ ) and the line segment  $PX$  ( $XQ$ ).

We can now compute  $g'(\alpha)$  and obtain

$$\begin{aligned}
g'(\alpha) &= -L(\beta)\frac{|PQ|}{r_\alpha^3}|PX||XQ|\sin \alpha - \\
&c\frac{|PQ|^2}{r_\alpha^4}(2|PX||XQ|\sin \alpha)(A(\beta_{PX}) + A(\beta_{XQ}) + \frac{1}{2}|PX||XQ|\sin \alpha) - \\
&c\frac{|PQ|^2}{r_\alpha^2}(\frac{1}{2}|PX||XQ|\cos \alpha). \tag{1}
\end{aligned}$$

We know that  $g'(\alpha_0) = 0$ . Moreover,  $r_{\alpha_0} = |PQ|$  and  $A(\beta_{PX}) + A(\beta_{XQ}) + \frac{1}{2}|PX||XQ|\sin \alpha_0 = A(\beta)$ .

Therefore, by plugging  $\alpha = \alpha_0$  in (1), we get

$$0 = -L(\beta) \frac{|PX||XQ| \sin \alpha_0}{|PQ|^2} - \frac{c}{|PQ|^2} (2|PX||XQ| \sin \alpha_0 A(\beta)) - c \frac{1}{2} |PX||XQ| \cos \alpha_0. \quad (2)$$

After dividing (2) by  $|PX||XQ|$ , and some easy manipulations we get:

$$\tan \alpha_0 = \frac{\sin \alpha_0}{\cos \alpha_0} = \frac{-c|PQ|^2}{2L(\beta) + 4cA(\beta)}.$$

It is evident that  $\tan \alpha_0$  does not depend on  $X$  which is what we wanted to prove.

□

**Proof of Claim 3.2:** Let  $\beta_0$  be an optimal curve. We know that for every  $\beta \in \mathcal{H}$ ,  $\frac{L(\beta)}{A(\beta) + \frac{1}{2}} \geq \epsilon$ , or in other words  $L(\beta) - \epsilon A(\beta) \geq \frac{\epsilon}{2}$ . Since we have equality for  $\beta_0$ , it follows from Theorem 3.3 (with  $|PQ| = 1$  and  $c = \epsilon$ ) that  $\beta_0$  is a circular arc. □

The following easy claim can be verified via direct calculations:

**Claim 3.5.** *Let  $\beta$  be a subarc of a circle. Let  $P$  and  $Q$  denote the endpoints of  $\beta$  and assume that  $|PQ| = 1$ . Let  $\alpha$  denote the constant angle  $\angle PXQ$  for any point  $X$  on  $\beta$ . Then  $\frac{L(\beta)}{A(\beta) + \frac{1}{2}}$  is given by*

$$F(\alpha) = \frac{\frac{\pi - \alpha}{\sin \alpha}}{\frac{\pi - \alpha}{4 \sin^2 \alpha} + \frac{\cos \alpha}{4 \sin \alpha} + \frac{1}{2}}.$$

By definition,  $F(\alpha) \geq \epsilon$ , hence

$$\frac{\pi - \alpha}{\sin \alpha} - \epsilon \left( \frac{\pi - \alpha}{4 \sin^2 \alpha} + \frac{\cos \alpha}{4 \sin \alpha} \right) \geq \frac{\epsilon}{2}. \quad (3)$$

We will need the following theorem regarding isoperimetric inequality.

**Theorem 3.6.** *Let  $0 \leq t \leq 1$  and let  $\beta$  be a simple curve whose endpoints are  $(0, 0)$  and  $(t, 0)$  and which is fully contained in the region  $\{(x, y) | 0 \leq x \leq t, y \geq 0\}$ . Then  $L(\beta) - \epsilon A(\beta) \geq t \frac{\epsilon}{2}$ .*

**Remark:** observe that when  $t = 1$  Theorem 3.6 follows immediately from the definition of  $\epsilon$ .

**Proof.** By Theorem 3.3, the curve  $\beta$  for which the expression  $L(\beta) - \epsilon A(\beta)$  is minimum, is a subarc of a circle

Let  $\frac{\pi}{2} \leq \alpha \leq \pi$  be the constant angle defined by the cord between  $(0, 0)$  and  $(t, 0)$ . It can then be verified by direct calculations that the expression  $L(\beta) - \epsilon A(\beta)$  is given by

$$g(\alpha) = t \frac{\pi - \alpha}{\sin \alpha} - t^2 \epsilon \left( \frac{\pi - \alpha}{4 \sin^2 \alpha} + \frac{\cos \alpha}{4 \sin \alpha} \right).$$

A direct calculation gives:

$$g'(\alpha) = \frac{\sin \alpha + (\pi - \alpha) \cos \alpha}{\sin^2 \alpha} \left( \frac{t^2 \epsilon}{2 \sin \alpha} - t \right).$$

It is easy to see that  $\sin \alpha + (\pi - \alpha) \cos \alpha > 0$  for every  $\frac{\pi}{2} < \alpha < \pi$  and that  $\frac{t^2 \epsilon}{2 \sin \alpha} - t$  is an increasing function of  $\alpha$  in the range  $\frac{\pi}{2} < \alpha < \pi$ . Therefore  $g(\alpha)$  obtains a minimum when  $\frac{t^2 \epsilon}{2 \sin \alpha} - t = 0$ , that is,  $\sin \alpha = \frac{t \epsilon}{2}$ .

It follows from (3) that for every  $\frac{\pi}{2} \leq \alpha \leq \pi$ .

$$\epsilon \left( \frac{\pi - \alpha}{4 \sin^2 \alpha} + \frac{\cos \alpha}{4 \sin \alpha} \right) \leq \frac{\pi - \alpha}{\sin \alpha} - \frac{\epsilon}{2}.$$

Therefore, for every  $\frac{\pi}{2} \leq \alpha \leq \pi$ ,

$$g(\alpha) \geq (t - t^2) \frac{\pi - \alpha}{\sin \alpha} + t^2 \frac{\epsilon}{2} \geq t \frac{\epsilon}{2}.$$

□

**Remark:** We can now obtain an implicit equation for  $\epsilon$ . That is,

$$\epsilon = \frac{\frac{\pi - \alpha}{\sin \alpha}}{\frac{\pi - \alpha}{4 \sin^2 \alpha} + \frac{\cos \alpha}{4 \sin \alpha} + \frac{1}{2}},$$

where  $\sin \alpha = \frac{\epsilon}{2}$  and  $\frac{\pi}{2} \leq \alpha \leq \pi$ . An approximated solution to this equation is 1.71579.... Nevertheless, we will not make use of this observation through the rest of the paper.

## 4 Proof of the Main Theorem

We will now get to the proof of Theorem 1.2. We may assume that the curves in question are only curves that are transverse to every edge. Indeed, this can be achieved by a small perturbation of the given curve  $\alpha$ .

The following claim shows that in order to prove theorem 1.2 it is enough to consider curves that intersect every edge in  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  at exactly two points or none.

**Claim 4.1.** *Let  $\alpha : S^1 \rightarrow \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  be a simple closed curve, then there exists a simple closed curve  $\beta : S^1 \rightarrow \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  such that  $L(\beta) \leq L(\alpha)$  and  $A(\beta) \geq A(\alpha)$  and such that  $\beta$  intersects every edge at exactly two points or none.*

**Proof.**  $\alpha$  is a compact boundaryless one-dimensional manifold, which is the boundary of the 2-dimensional Riemannian manifold. We denote by  $A$  the domain bounded by  $\alpha$ . We orient  $\alpha$  so as the domain  $A$  is to the right of  $\alpha$ . Let  $e$  be an edge in  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  which intersects

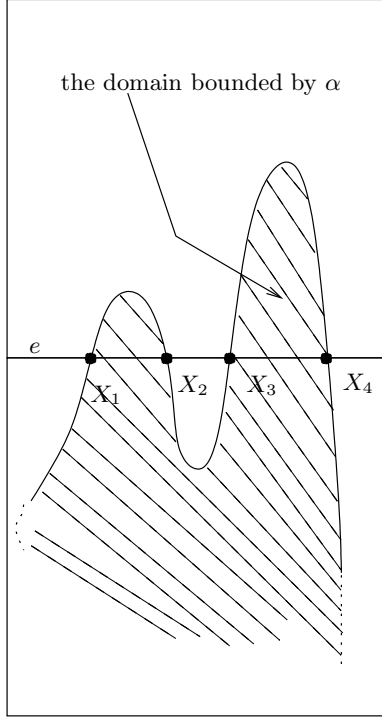


Figure 2:

$\alpha$  at  $l > 2$  points. Assume that  $e$  is horizontal. Let  $X_1, \dots, X_l$  be the intersection points of  $\alpha$  with  $e$ . We assume that the points are indexed consecutively from the left to the right. See Figure 4.

We orient  $e$  in such a way that it enters  $A$  at the point  $X_1$ .  $\alpha$  is contractable and hence, if we look at  $\alpha$  and  $e$  as in the completion of  $\mathbb{R}^2 \setminus \mathbb{Z}^2$ , we have  $e \circ \alpha = 0$ . Therefore, the number of points in the set  $\alpha \cap e$  is even. It is easy to see that the line segments  $X_{2j}X_{2j+1}$  for  $j = 1, \dots, \frac{l}{2} - 1$ , do not intersect the interior of  $A$ .

Denote by  $\alpha_{X_2, X_3}$  the subarc of alpha that starts at  $X_2$  and ends at  $X_3$  (according to the orientation of  $\alpha$ ). We replace  $\alpha_{X_2, X_3}$  by the line segment  $X_2X_3$  to get another embedded curve that we denote by  $\alpha_1$ . Observe that  $L(\alpha) \geq L(\alpha_1)$ . Moreover,  $A(\alpha) \leq A(\alpha_1)$ , because  $A$  lies inside the domain bounded by  $\alpha_1$ .

We now perform a small perturbation to  $\alpha_1$  at a small neighborhood of  $e$  so that the following statements hold:

- $\alpha_1$  is transverse to  $e$ .
- There are no other intersection points of  $\alpha_1 \cap e$  beside  $X_1$  and  $X_4, \dots, X_n$ .
- $L(\alpha) \geq L(\alpha_1)$  and  $A(\alpha) \leq A(\alpha_1)$ .

We apply successively the above procedure for the pairs  $(X_4, X_5), \dots, (X_{2n-2}, X_{2n-1})$ . We

obtain an embedded curve  $\alpha_{\frac{l}{2}}$  which satisfies:  $L(\alpha_{\frac{l}{2}}) \leq L(\alpha)$ ,  $A(\alpha_{\frac{l}{2}}) \geq A(\alpha)$ , and  $\alpha_{\frac{l}{2}}$  intersects  $e$  at exactly two points:  $X_1$  and  $X_n$ .

We apply the above procedure to every edge  $e'$  which intersects  $\alpha$ . We get an embedded curve  $\beta$  which satisfies:  $L(\beta) \leq L(\alpha)$ ,  $A(\beta) \geq A(\alpha)$ . Moreover,  $\beta$  intersects any edge  $e'$  at exactly two points or none. This completes the proof.  $\square$

From now on we consider only curves that intersect any edge of  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  at exactly two points or none.

**Definition 4.2.** A  $\theta$ -curve is a simple curve in  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  whose both endpoints lie on the same edge of a fundamental square. For a  $\theta$ -curve  $\beta$ ,  $L(\beta)$  will denote its length,  $A(\beta)$  will denote the area enclosed by the closed curve obtained from  $\beta$  by joining its two endpoints by a straight line segment.  $d(\beta)$  will denote the distance between the two endpoints of  $\beta$ .

Theorem 1.2 is an easy consequence of the following lemma:

**Lemma 4.3.** Let  $\beta$  be a  $\theta$ -curve, then

$$L(\beta) \geq \epsilon A(\beta) + \frac{\epsilon}{2} d(\beta). \quad (4)$$

We first show how Theorem 1.2 follows from Lemma 4.3. Without loss of generality  $\alpha$  intersects (at exactly two points) an edge of a fundamental square. Let  $Q$  and  $P$  be these two intersection points.  $P$  and  $Q$  divide  $\alpha$  into two curves each of which is a  $\theta$ -curve. Let  $\beta_1$  and  $\beta_2$  be those two curves. By Lemma 4.3,  $L(\beta_i) \geq \epsilon A(\beta_i) + \frac{\epsilon}{2} |PQ|$ , for  $i = 1, 2$ . Therefore,

$$L(\alpha) = L(\beta_1) + L(\beta_2) \geq \epsilon(A(\beta_1) + A(\beta_2)) + 2 \frac{\epsilon}{2} (|PQ|) \geq \epsilon A(\alpha).$$

**Proof of Lemma 4.3.** We prove the Lemma by induction on the number  $n$  of fundamental squares that intersect the relative interior of  $\beta$ . The *relative interior* of  $\beta$  relates to the interior of the simple closed curve obtained from  $\beta$  and the straight line segment joining its two endpoints.

The case  $n = 1$  follows directly from Theorem 3.6, after some suitable reductions: Denote by  $S$  the single fundamental square which contains  $\beta$ . Let  $P$  and  $Q$  be the endpoints of  $\beta$  and assume without loss of generality that  $P$  is to the left of  $Q$  and that the interval  $PQ$  lies on the bottom edge of  $S$  (see Figure...). We may assume that  $\beta$  lies entirely to the right of the perpendicular line to the segment  $PQ$  which touches  $P$ . Indeed, otherwise consider the line  $y$  that is perpendicular to  $PQ$  and is the tangent to  $\beta$  so that  $\beta$  lies entirely to the right of  $y$ . Let  $T$  be a point at which  $y$  touches  $\beta$ . Let  $P'$  be the intersection point of  $y$  with the bottom edge of  $S$ . Now modify  $\beta$  by replacing the subarc of  $\beta$  between  $P$  and  $T$  by the straight line segment on  $y$  between  $P'$  and  $T$ . We thus obtain a new  $\theta$ -curve  $\beta'$  such that  $L(\beta') \leq L(\beta)$ ,  $A(\beta') \geq A(\beta)$ , and  $d(\beta') \geq d(\beta)$  (and therefore  $\frac{\epsilon}{2} d(\beta') \geq \frac{\epsilon}{2} d(\beta)$ ). Observe that now  $\beta'$  lies entirely to the right of the perpendicular line through  $P'$ .

In a similar manner we can assume that  $\beta$  lies entirely to the left of the perpendicular line to the segment  $PQ$  which touches  $Q$ . We can now use Theorem 3.6 and conclude the case  $n = 1$ .

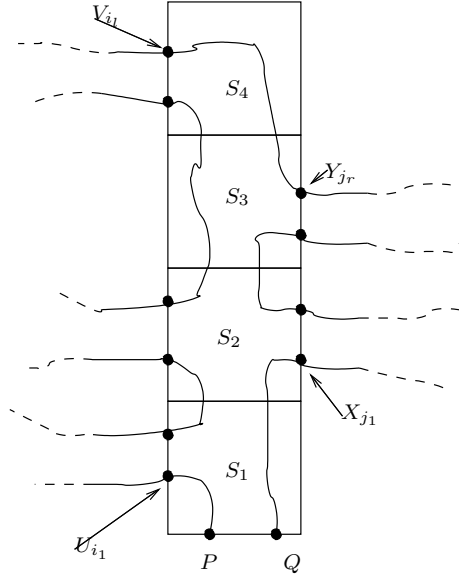


Figure 3: Case 2 with  $l = 3$  and  $r = 2$ .

In order to complete the proof of Lemma 4.3 we have to consider the case  $n > 1$ , and complete the induction step. To this end, we consider the square  $S$  which contains the endpoints  $P$  and  $Q$  of  $\beta$ . Without loss of generality we assume that both  $P$  and  $Q$  lie on the bottom edge of  $S$  so that  $P$  is to the left of  $Q$ . Let us denote the four vertices of  $S$  in the clockwise order starting from the lower left vertex, by  $A, B, C$ , and  $D$ .

Denote  $S_1 = S$  and for each integer  $m > 1$  let  $S_m$  be the fundamental square adjacent to  $S_{m-1}$  in its top edge. Let  $i_1 < i_2 < \dots < i_l$  be all the indices such that  $\beta$  intersects the left edge of  $S_{i_k}$  (at exactly two points). For every  $0 \leq k \leq l$ , let us denote the two points of intersection of  $\beta$  and the left edge of  $S_{i_k}$  by  $U_{i_k}$  and  $V_{i_k}$  so that  $V_{i_k}$  is above  $U_{i_k}$ .

Similarly, let  $j_1 < j_2 < \dots < j_r$  be all the indices such that  $\beta$  intersects the right edge of  $S_{j_k}$  (at exactly two points). For every  $0 \leq k \leq r$ , let us denote the two points of intersection of  $\beta$  and the right edge of  $S_{j_k}$  by  $X_{j_k}$  and  $Y_{j_k}$ , so that  $Y_{j_k}$  is above  $X_{j_k}$ . See Figure 4.

We distinguish between three cases:

**Case 1.**  $r = l = 0$ . This case follows directly from Theorem 3.6 after similar adjustments to those made in the case  $n = 1$ .

**Case 2.**  $r > 0$  and  $l > 0$ . In this case for every  $1 \leq k \leq l - 1$  we may replace the subarc of  $\beta$  between  $V_{i_k}$  and  $U_{i_{k+1}}$  by a straight line segment (with a small perturbation in order that  $\beta \subset \mathbb{R}^2 \setminus \mathbb{Z}^2$ ) since this will decrease  $L(\beta)$  and will increase  $A(\beta)$ . We may also replace the subarc of  $\beta$  between  $P$  and  $U_{i_1}$  by a straight line segment and thus assume that  $P$  almost coincides with  $A$ . Indeed, this will result in decreasing  $L(\beta)$  increasing  $A(\beta)$  and increasing  $d(\beta)$ , and it will be enough to prove that even after this adjustment  $L(\beta) - \epsilon A(\beta) \geq \frac{\epsilon}{2} d(\beta)$  still holds.

Similarly, for every  $1 \leq k \leq r - 1$  we may replace the subarc of  $\beta$  between  $Y_{j_k}$  and  $X_{j_{k+1}}$

by a straight line segment (with a small perturbation in order that  $\beta \subset \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ ) since this will decrease  $L(\beta)$  and will increase  $A(\beta)$ . We may also replace the subarc of  $\beta$  between  $Q$  and  $X_{j_1}$  by a straight line segment and thus assume that  $Q$  almost coincides with  $D$ .

Note that we may assume that the subarc of  $\beta$  between  $V_{i_l}$  and  $Y_{j_r}$  is concave. Consider the line  $m$  in  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  that is the perpendicular bisector of  $AD$ , and denote its (unique) intersection point with  $\beta$  by  $M$ . We now reflect the subarc of  $\beta$  between  $P$  and  $M$  with respect to the reflection line  $m$ . We thus obtain a  $\theta$ -curve that we denote by  $\beta_l$ . For every  $1 \leq k \leq l$ , denote by  $\beta_{i_k}$  the  $\theta$ -curve which is the subarc of  $\beta$  between  $U_{i_k}$  and  $V_{i_k}$ . Denote by  $V'_{i_l}$  the reflection of  $V_{i_l}$  with respect to  $m$ . Let  $\beta_M$  denote the subarc of  $\beta_l$  between  $V_{i_l}$  and  $V'_{i_l}$  (see Figure 4). We have,

$$L(\beta_l) = 2 \sum_{k=1}^l L(\beta_{i_k}) + L(\beta_M) + 2|PU_{i_1}| + 2 \sum_{k=1}^{l-1} |V_{i_k} U_{i_{k+1}}| \quad (5)$$

$$A(\beta_l) = 2 \sum_{k=1}^l A(\beta_{i_k}) + A(\beta_M) + |PU_{i_1}| + \sum_{k=1}^{l-1} |V_{i_k} U_{i_{k+1}}| + \sum_{k=1}^l |U_{i_k} V_{i_k}| \quad (6)$$

By the induction hypothesis, for every  $1 \leq k \leq l$ ,  $L(\beta_{i_k}) - \epsilon A(\beta_{i_k}) \geq \frac{\epsilon}{2} |U_{i_k} V_{i_k}|$ . By Theorem 3.6,  $L(\beta_M) - \epsilon A(\beta_M) \geq \frac{\epsilon}{2}$ .

Therefore,

$$\begin{aligned} L(\beta_l) - \epsilon A(\beta_l) &\geq \frac{\epsilon}{2} + 2 \sum_{k=1}^l \frac{\epsilon}{2} |U_{i_k} V_{i_k}| + \\ &+ 2|PU_{i_1}| + 2 \sum_{k=1}^{l-1} |V_{i_k} U_{i_{k+1}}| - \epsilon(|PU_{i_1}| + \sum_{k=1}^{l-1} |V_{i_k} U_{i_{k+1}}| + \sum_{k=1}^l |U_{i_k} V_{i_k}|) = \\ &= \frac{\epsilon}{2} + (2 - \epsilon)(|PU_{i_1}| + \sum_{k=1}^{l-1} |V_{i_k} U_{i_{k+1}}|) + \sum_{k=1}^l (\epsilon |U_{i_k} V_{i_k}|) - \epsilon |U_{i_k} V_{i_k}| = \\ &= \frac{\epsilon}{2} + (2 - \epsilon)(|PU_{i_1}| + \sum_{k=1}^{l-1} |V_{i_k} U_{i_{k+1}}|) \geq \\ &\geq \frac{\epsilon}{2} \end{aligned}$$

In a similar manner we reflect the subarc of  $\beta$  between  $Q$  and  $M$  with respect to the reflection line  $m$  and obtain a  $\theta$ -curve that we denote by  $\beta_r$ . The same arguments yield that

$$L(\beta_r) - \epsilon A(\beta_r) \geq \frac{\epsilon}{2}.$$

Observe that  $2L(\beta) = L(\beta_l) + L(\beta_r)$  and that  $2A(\beta) = A(\beta_l) + A(\beta_r)$ . Combining this with the isoperimetric inequalities for  $\beta_l$  and  $\beta_r$ , we obtain the desired result, namely,

$$L(\beta) - \epsilon A(\beta) \geq \frac{\epsilon}{2}.$$

**Case 3.**  $l > 0$  and  $r = 0$ . As in case 2, for every  $1 \leq k \leq l-1$  we may replace the subarc of  $\beta$  between  $V_{i_k}$  and  $U_{i_{k+1}}$  by a straight line segment (with a small perturbation in order that  $\beta \subset \widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$ ) since this will decrease  $L(\beta)$  and will increase  $A(\beta)$ . We may also replace the subarc of  $\beta$  between  $P$  and  $U_{i_1}$  by a straight line segment and thus assume that  $P$  almost coincides with  $A$ . Indeed, this will result in decreasing  $L(\beta)$  increasing  $A(\beta)$  and increasing  $d(\beta)$ , and it will be enough to prove that even after this adjustment  $L(\beta) - \epsilon A(\beta) \geq \frac{\epsilon}{2}d(\beta)$  still holds (observe that  $\frac{\epsilon}{2}t$  is monotone increasing in  $t$ ). Moreover, we may assume that the subarc of  $\beta$  between  $V_{i_l}$  and  $Q$  is concave, and that  $Q$  is the rightmost point of  $\beta$  in  $S_1 \cup \dots \cup S_l$ . Indeed, otherwise we consider the line  $y$  perpendicular to  $AD$  that is a tangent of  $\beta$  and touches  $\beta$  at a point  $T$ . Then replace the subarc of  $\beta$  between  $Q$  and  $T$  by the straight line segment on  $y$  between  $T$  and  $Q'$ , where  $Q'$  is the intersection point of  $y$  with  $AD$ . In this way we increase  $A(\beta)$ , decrease  $L(\beta)$  and, increase  $\frac{\epsilon}{2}d(\beta)$ .

We consider the line  $m$  in  $\widetilde{\mathbb{R}^2 \setminus \mathbb{Z}^2}$  that is the perpendicular bisector of  $AD$ . We distinguish between two cases.

**Case 3(a):**  $m$  intersects  $\beta$ .

In this case, as in Case 2, we denote the (unique) intersection point of  $m$  with  $\beta$  by  $M$ . We reflect the subarc of  $\beta$  between  $P$  and  $M$  with respect to the reflection line  $m$ . We thus obtain a  $\theta$ -curve that we denote by  $\beta_l$ . The same arguments as in Case 2 yield the following isoperimetric inequality:

$$L(\beta_l) - \epsilon A(\beta_l) \geq \frac{\epsilon}{2}.$$

In a similar manner we reflect the subarc of  $\beta$  between  $Q$  and  $M$  with respect to the reflection line  $m$  and obtain a  $\theta$ -curve that we denote by  $\beta_r$ .  $\beta_r$  satisfies the conditions in Theorem 3.6. Hence,

$$L(\beta_r) - \epsilon A(\beta_r) \geq 2(|PQ| - \frac{1}{2})\frac{\epsilon}{2} = (2|PQ| - 1)\frac{\epsilon}{2}.$$

Observe that  $2L(\beta) = L(\beta_l) + L(\beta_r)$  and that  $2A(\beta) = A(\beta_l) + A(\beta_r)$ . Combining this with the isoperimetric inequalities for  $\beta_l$  and  $\beta_r$  we obtain the desired result, namely,

$$L(\beta) - \epsilon A(\beta) \geq |PQ|\frac{\epsilon}{2}.$$

**Case 3(b):**  $m$  does not intersect  $\beta$ .

In this case we consider the line  $l_S$  that is perpendicular to  $AD$  at  $A$ . We think of the subarc of  $\beta$  between  $V_{i_l}$  and  $Q$  as a curve in  $\mathbb{R}^2$  and reflect it with respect to the line  $l_S$ . We thus obtain a curve that we denote by  $\beta_r$ . Observe that  $\beta_r$  satisfies the conditions of Theorem 3.6. For every  $1 \leq k \leq l$  denote by  $\beta_{i_k}$  the  $\theta$ -curve that is the subarc of  $\beta$  between  $U_{i_k}$  and  $V_{i_k}$ .

We have:

$$L(\beta) \geq \sum_{k=1}^l L(\beta_{i_k}) + \frac{1}{2}L(\beta_r) \quad (7)$$

$$A(\beta) \geq \sum_{k=1}^l A(\beta_{i_k}) + \frac{1}{2}A(\beta_r) \quad (8)$$

By the induction hypothesis, for every  $1 \leq k \leq l$  we have

$$L(\beta_{i_k}) \geq \epsilon A(\beta_{i_k}) + \frac{\epsilon}{2}d(\beta_{i_k}) \geq \epsilon A(\beta_{i_k})$$

By Theorem 3.6,

$$L(\beta_r) \geq \epsilon A(\beta_r) + \frac{\epsilon}{2}d(\beta_r) = \epsilon A(\beta_r) + \frac{\epsilon}{2}2d(\beta).$$

Therefore,

$$\begin{aligned} L(\beta) &\geq \sum_{k=1}^l L(\beta_{i_k}) + \frac{1}{2}L(\beta_r) \geq \\ &\geq \sum_{k=1}^l \epsilon A(\beta_{i_k}) + \frac{1}{2}\epsilon A(\beta_r) + \frac{1}{2}\frac{\epsilon}{2}2d(\beta) = \\ &= \epsilon \left( \sum_{k=1}^l A(\beta_{i_k}) + \frac{1}{2}A(\beta_r) \right) + \frac{\epsilon}{2}d(\beta) = \\ &= \epsilon A(\beta) + \frac{\epsilon}{2}d(\beta). \end{aligned}$$

This completes the proof of Lemma 4.3.  $\square$

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