Points with Large Quadrant-Depth

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Abstract

We characterize the set of all pairs (a, b) such that for every set P of n points in general position in $\mathbb{R}^2$ there always exists a point $p \in P$ and two opposite quadrants determined by the axes-parallel lines through p such that one quadrant contains at least an points of P and the other quadrant contains at least bn points of P. We consider also some generalizations of this problem in the plane and in higher dimensions.

1 Introduction

Let $z = (z_1, \ldots, z_d)$ be any point in $\mathbb{R}^d$. The *orthants* determined by z are the connected components of $\mathbb{R}^d$ minus the union of the d axes-parallel hyper-planes through z, that is, the hyper-planes $H_i = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_i = z_i\}$. Clearly, z determines 2^d orthants. Two orthants determined by z are called *opposite* if the are separated by each of the d axes-parallel hyper-planes through z. Sometimes we will consider the *closed* orthants determined by z. These are just the closures of the orthants that are determined by z.

In [BLP], Brönnimann, Lenchner, and Pach define the notion of an opposite-quadrant depth of a point in a point set in the plane. Specifically, let P be a set of n points in general position in the plane. The opposite-quadrant depth of a point x in P is the maximum number $opp(x, P)$ such that x determines two opposite closed orthants (in fact, quadrants) each containing at least $opp(x, P)$ points of P.

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In [BLP] the following theorem is proved:

Theorem 1 ([BLP]). *1. Any set P of n points in general position in the plane has a point $p \in P$ such that opp$(p,P) \geq \frac{1}{8}n$.*

2. If P is in convex position, then there is a point $p \in P$ such that opp$(p,P) \geq \frac{1}{4}n$.

In Section 2 we bring an extremely simple argument proving the result in [BLP] and even strengthening it. We also provide a construction that shows that the result in Theorem 1 is best possible for every n.

In Section 3 we refine the notion of quadrant-depth in the plane and consider the set of all pairs a and b such that any set P of n points in the plane contains a point $p \in P$ that determines two opposite quadrants containing at least an and bn points of P, respectively. Section 4 contains a discussion of the same problem in the plane and in higher dimensions, except that then we seek for a point $z \in mathbbR^d$, not necessarily in the set P such that two opposite orthants, determined by z, contain at least an and bn points of P, respectively. Further generalizations are considered as well.

Finally, Section 5 contains some concluding remarks and several open problems.

2 A simple proof for Theorem 1

Definition 1. For a point p in the plane we denote by $Q_1(p), Q_2(p), Q_3(p)$, and $Q_4(p)$ the first, second, third, and forth quadrants, respectively, that are determined by p.

We now bring a very short argument proving a strengthened version of Theorem 1.

Theorem 2. *(i) Let P be a set of n points in general position in the plane. Then there exists a point $p \in P$ such that:*

$$\min(|Q_1(p) \cap P|, |Q_3(p) \cap P|) + \min(|Q_2(p) \cap P|, |Q_4(p) \cap P|) \geq \frac{1}{4}(|P|-1).$$

In particular either each of $Q_1(p)$ and $Q_3(p)$ contains $\frac{1}{8}|P|$ points of P, or each of $Q_1(p)$ and $Q_3(p)$ contains $\frac{1}{8}|P|$ points of P. That is, opp$(p,P) \geq \frac{1}{8}n$.

(ii) If the points of P are in convex position, then there exists a point $p \in P$ such that opp$(p,P) \geq \frac{1}{4}n$.

Proof. To see the first part of the theorem, let P_L be the set of $\lfloor \frac{1}{4}(n-1) \rfloor$ points of P with the least x-coordinates. Let P_R be the set of $\lfloor \frac{1}{4}(n-1) \rfloor$ points of P with the largest x-coordinates. Similarly, let P_U be the set of $\lfloor \frac{1}{4}(n-1) \rfloor$ points of P with the largest y-coordinates and let P_D be the set of $\lfloor \frac{1}{4}(n-1) \rfloor$ points of P with the least y-coordinates.

Let p be any point in $P \setminus (P_L \cup P_R \cup P_U \cup P_D)$. We will show that p is the desired point. Let s denote $s = \min(|P \cap Q_1(p)|, |P \cap Q_3(p)|)$ and assume without loss of generality that $s = |P \cap Q_1(p)|$. Since $P_U \subset Q_1(p) \cup Q_2(p)$ it follows that $|P \cap Q_2(p)| \geq |P_U \cap Q_2(p)| > \lfloor \frac{1}{4}(n-1) \rfloor - s$. Similarly, since $P_R \subset Q_1(p) \cup Q_4(p)$ it follows that $|P \cap Q_4(p)| \geq |P_R \cap Q_4|(p) > \lfloor \frac{1}{4}(n-1) \rfloor - s$.

Hence $\min(|P \cap Q_2(p)|, |P \cap Q_4(p)|) \geq \lfloor \frac{1}{4}(n-1) \rfloor - s$. Therefore,

$$\min(|Q_1(p) \cap P|, |Q_3(p) \cap P|) + \min(|Q_2(p) \cap P|, |Q_4(p) \cap P|) \geq s + \lfloor \frac{1}{4}(n-1) \rfloor - s = \lfloor \frac{1}{4}(n-1) \rfloor.$$

The result now follows considering the *closures* of $Q_1(p), Q_2(p), Q_3(p)$, and $Q_4(p)$ all of which contain the point p as well.

The second part of Theorem 1 where the points of P are in convex position, follows from the first part. This is because if the points of P are in convex position, then one of $Q_1(p), Q_2(p), Q_3(p)$, and $Q_4(p)$, does not contain any point of P, or otherwise there are four points in $P \setminus \{p\}$ containing p in their convex hull. ∎

It is easy to see that the second part of Theorem 1 is best possible for every n by taking P to be the set of vertices of a regular n-gon. It is shown in [BLP] that the first part of Theorem 1 is best possible for arbitrarily large values of n of the form $n = 4 \cdot 3^k$.

We will now show by a much simpler construction that the first part of Theorem 1 cannot be improved for any n. The construction is sketched in Figure 1. Let Q be a regular octagon centered at the origin, tilted slightly so that its highest vertex is slightly to the left of the y-axis. Very close to each vertex v of Q position $n/8$ points along a line through the origin and v. It is left to the reader to verify that that resulting set P of n points satisfies $opp(x, P) \leq n/8$ for every $x \in P$.

3 The Set of Quadrants-Admissible Pairs

Definition 2. A pair (α, β) of real nonnegative numbers will be called *quadrants-admissible* if for every set P of n points in the plane such that no two points of P have the same x-coordinate or the same y-coordinate, one can find $p \in P$ such that the horizontal and vertical lines through p determine two opposite quadrants one containing at least $\alpha n + o(n)$ points of P and the other containing at least $\beta n + o(n)$ points of P. We denote the set of all quadrants-admissible pairs by $\mathcal{F}$.

In this section we provide a full characterization of the set $\mathcal{F}$. We begin with some easy observations. Clearly, $\mathcal{F}$ is contained in the unit cube $[0,1]^2$. Moreover, $\mathcal{F}$ is symmetric in the sense that if $(a, b) \in \mathcal{F}$, then also $(b, a) \in \mathcal{F}$. $\mathcal{F}$ is also monotone decreasing, that is, if $(a, b) \in \mathcal{F}$ and $0 \leq a' \leq a$ and $0 \leq b' \leq b$, then $(a', b') \in \mathcal{F}$.

Theorem 1 implies that $(\frac{1}{8}, \frac{1}{8}) \in \mathcal{F}$ and that if both a and b are greater than $\frac{1}{8}$, then $(a, b) \notin \mathcal{F}$.

The next observation implies that $(\frac{1}{2}, 0)$, and hence also $(0, \frac{1}{2})$, belong to the boundary of $\mathcal{F}$.

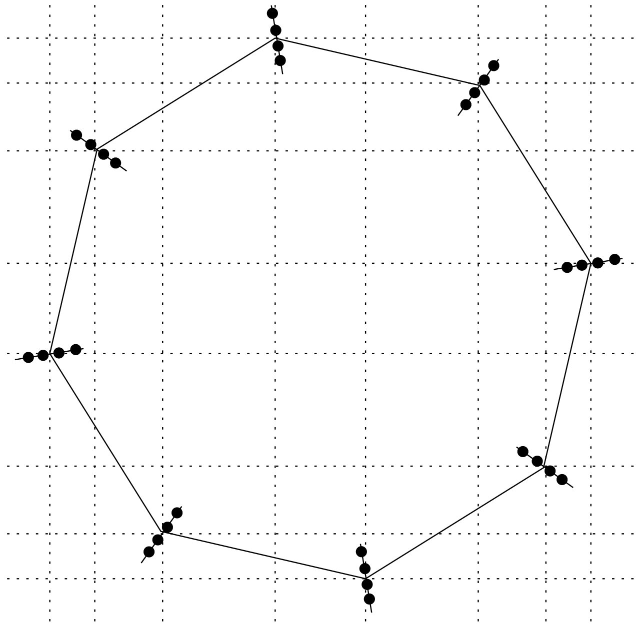

Figure 1: A set P with $opp(x, P) \geq n/8$ for every $x \in P$.

Observation 1. *The pair* $(\frac{1}{2}, 0)$ *(and therefore also the pair* $(0, \frac{1}{2})$*) belongs to* $\mathcal{F}$*. Moreover, for every* $\alpha > \frac{1}{2}$ *the pair* $(\alpha, 0)$ *does not belong to* $\mathcal{F}$*.*

Proof. Let P be a set of n points in the plane such that no two points of P share the same x-coordinate or the same y-coordinate. Let $z \in P$ be the point of P with the largest x-coordinate. Clearly, the points of $P \setminus \{z\}$ are contained in the second and third quadrants determined by z. Hence, at least $\frac{1}{2}n$ points of P are contained either in the third quadrant determined by z or in the forth quadrant determined by z. This shows that $(\frac{1}{2}, 0)$ belongs to $\mathcal{F}$.

To see that $(a, 0)$ does not belong to $\mathcal{F}$ for any $a > \frac{1}{2}$, consider a set P of n points evenly distributed on a circle (see Figure 8). We leave it to the reader to verify that no point $z \in P$ determines a quadrant which contains more than $\frac{1}{2}n$ points of P. ∎

From Observation 1 and Theorem 1, together with the fact that $\mathcal{F}$ is symmetric and monotone decreasing, we conclude that $\mathcal{F}$ is contained in the area shown in Figure 2.

In the next theorems and constructions we provide a finer description of the set $\mathcal{F}$ and in fact a complete characterization of it.

Theorem 3. *For every* $0 \leq a \leq \frac{1}{12}$ *and* b *such that* $3a + 3b = 1$*, the pair* (a, b) *is in* $\mathcal{F}$*.*

Proof. Let P be a set of n points in the plane with no two points sharing the same x-coordinate or the same y-coordinate. Assume to the contrary that there is no point $x \in P$ which determines two opposite quadrants containing at least an and bn points respectively. Let L denote the set of $(a + b)n$ leftmost points of P. Let R denote the set of $(a + b)n$

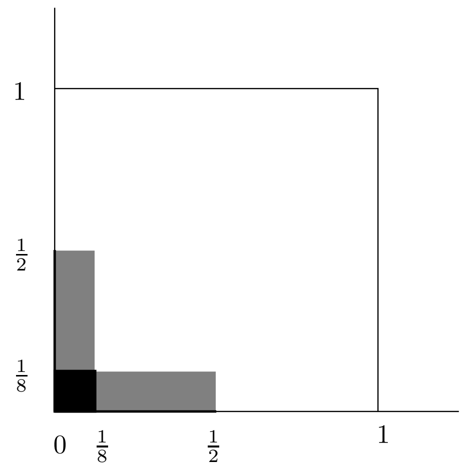

Figure 2: $\mathcal{F}$ contains the black region and is contained in the union of the black and gray regions.

rightmost points of P. Let M denote the set $P \setminus (L \cup R)$. Let T denote the set of $2an$ top most points in M and let B denote the set of $2an$ bottom most points in M. Note that because of the restrictions on a and b, the sets T and B are disjoint, or in other words every point in T has y-coordinate greater than the y-coordinate of every point in B. This is because $|M| = (a+b)n \geq 4an = |T| + |B|$.

Let p be the lowest point in T and let q be the highest point in B. Suppose that $Q_4(p)$ contains bn or more points of P. Then $Q_2(p)$ must contain less than an points of P. Because $L \subset Q_2(p) \cup Q_3(p)$ it follows that $Q_3(p)$ contains more than bn points of P. Therefore, $Q_1(p)$ contains less than an points of P. We have thus reached a contradiction because each of $Q_1(p)$ and $Q_2(p)$ contains less than an points of P while the set $T \setminus \{p\}$, which is a subset of $Q_1(p) \cup Q_2(p)$, consists of $an - 1$ points of P.

We conclude that $Q_4(p)$ contains less than bn points of P. By symmetry we may assume that also $Q_3(p)$ contains less than bn points of P. Arguing similarly about q we conclude that each of $Q_1(q)$ and $Q_2(q)$ contains less than b points of P.

We have thus reached a contradiction because

$$4bn > |Q_3(p) \cap P| + |Q_4(p) \cap P| + |Q_1(q) \cap P| + |Q_2(q) \cap P| \geq n + |M| - |T| - |B| = n(1 + b - 3a),$$

which implies $3a + 3b > 1$. ∎

Note that the proof of Theorem 3 in the special case where $\alpha = \frac{1}{8}$ provides an alternative proof for Theorem 1. Therefore, Theorem 3 is best possible for $\alpha = \frac{1}{8}$, in the sense that the point $(\frac{1}{8}, \frac{1}{8})$ lies on the boundary of $\mathcal{F}$.

We will now show by a construction that Theorem 3 is best possible. That is, we will show by construction that for every a and b which satisfy the conditions of Theorem 3, the pair (a, b) lies on the boundary of $\mathcal{F}$.

The construction is shown in Figure 3.

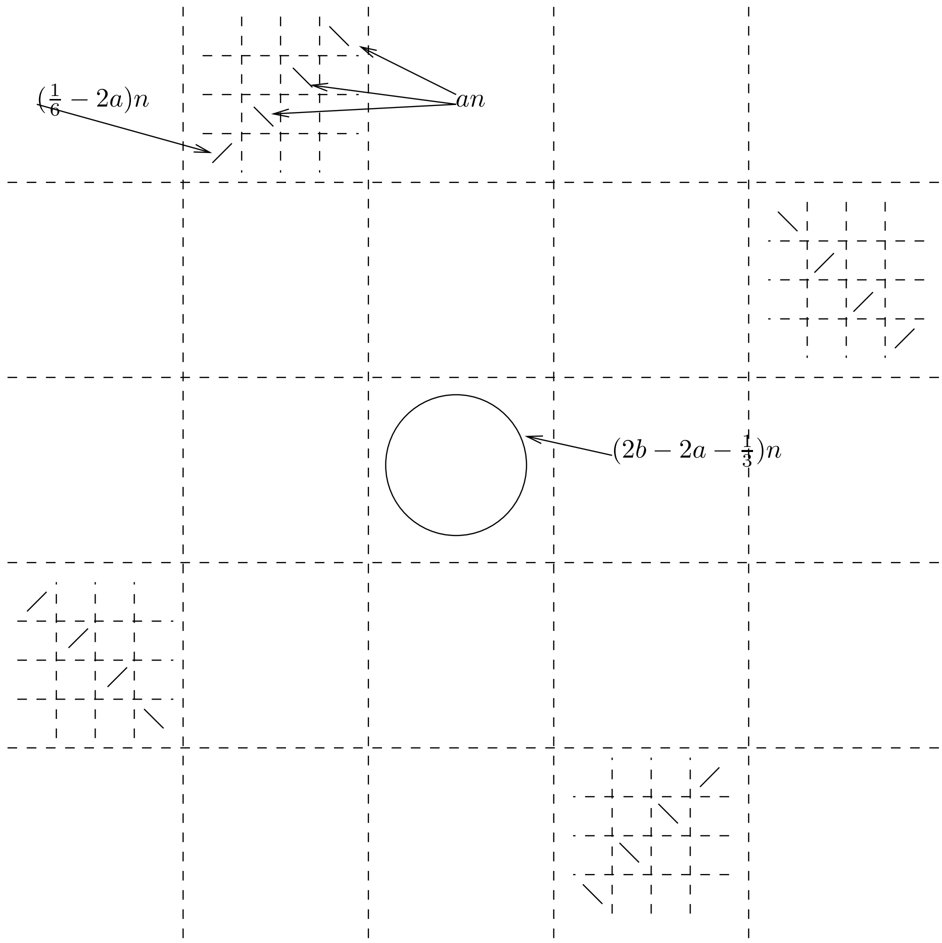

Figure 3: A construction by which Theorem 3 is best possible.

This construction describes a set P of n points arranged on 16 segments and a circle as shown in Figure 3. We note that the $(2b - 2a - \frac{1}{3})n$ points on the circle are evenly distributed on the circle. One can easily check that no point of the set determines two opposite quadrants, one containing more than an points of P and the other containing more than bn points of P. Hence (a, b) is on the boundary of $\mathcal{F}$.

Theorem 4. *For every* $\frac{1}{12} \leq a \leq \frac{1}{10}$ *and* $b \leq \frac{1}{4}$ *such that* $6a + 2b = 1$*, the pair* (a, b) *is in* $\mathcal{F}$*.*

Proof. Let P be a set of n points in the plane with no two points sharing the same x-coordinate or the same y-coordinate. Assume to the contrary that there is no point $x \in P$ which determines two opposite quadrants containing at least an and bn points respectively. Let L denote the set of $2an$ leftmost points of P. Let R denote the set of $2an$ rightmost points of P. Let C denote the set $P \setminus (L \cup R)$. Consider the points p which is the median point of C with respect to the value of its y-coordinate.

Without loss of generality assume that $|Q_2(p) \cap P| \geq \frac{1}{4}n \geq bn$. It follows that $|Q_4(p) \cap P| < an$. Therefore, $|Q_1(p) \cap P| \geq an$ and consequently $|Q_3(p)| < bn$. Observe that

$|C| \leq 2(|Q_3(p) \cap P| + |Q_4(p) \cap P|$. Hence, $|C| < (2b+2a)n$ and we reached a contradiction as $n = |R| + |L| + |C| < 2an + 2an + (2b+2a)n = (6a+2b)n$, while we assume that $6a+2b = 1$. ∎

The estimates in Theorem 4 are best possible. To see this consider the following construction illustrated in Figure 4.

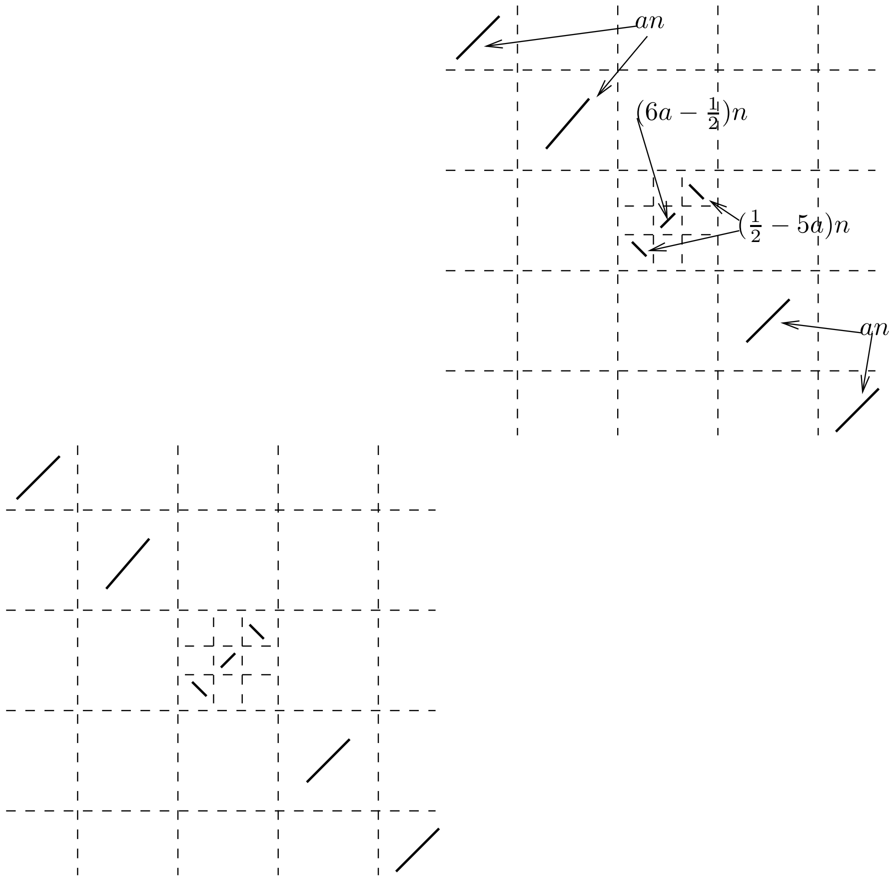

Figure 4: A construction that shows that Theorem 4 is best possible

The construction consists of n points laying on a union of 14 segments as shown in Figure 4. It is easy to see by inspection that in this construction for every point $p \in P$, there are at most an points of P either in $Q_1(p)$ or in $Q_3(p)$. Moreover, there are at most $bn = (2a + (\frac{1}{2} - 5a))n$ points of P either in $Q_2(p)$ or in $Q_4(p)$. The resulting set of points demonstrates that (a, b) is a point on the boundary of $\mathcal{F}$.

Theorem 5. *For every $\frac{1}{8} \leq a \leq \frac{1}{10}$ and b such that $4b + 2a = 1$, the pair (a, b) is in $\mathcal{F}$.*

Proof. Let P be a set of n points in the plane with no two points sharing the same x-coordinate or the same y-coordinate. Assume to the contrary that there is no point $p \in P$ which determines two opposite quadrants containing at least an and bn points, respectively. Let L denote the set of $2an$ leftmost points of P. Let R denote the set of $2an$ rightmost points of P. Let C denote the set $P \setminus (L \cup R)$. Consider the point p which is the median point of C with respect to the value of its y-coordinate.

Without loss of generality $|Q_4(p) \cap P| \geq \frac{1}{4}n \geq bn$. Therefore, $|Q_2(p) \cap P| < an$. If follows that $|Q_3(p) \cap | \geq an$. Note that $|Q_1(p) \cap P| + |Q_2(p) \cap P| \geq |C|/2 = (\frac{1}{2} - 2a)n \geq 2an$. Hence, $|Q_2(p) \cap P| < an$ implies $|Q_1(p) \cap P| \geq an$. Observe that $|Q_4| \leq \frac{1}{2}n$. Therefore, $|Q_1(p) \cap P| + |Q_3(p) \cap P| \geq (\frac{1}{2} - a)n$. Therefore one of $Q_1(p)$ and $Q_3(p)$ contains at least $(\frac{1}{4} - \frac{a}{2})n = bn$ points of P, while each of $Q_1(p)$ and $Q_3(p)$ contains at least an points of P. We have thus reached a contradiction, as $Q_1(p)$ and $Q_3(p)$ are two opposite quadrants, determined by p. ▮

The result of Theorem 5 is best possible in the sense that if $\frac{1}{10} \leq a \leq \frac{1}{8}$ and b is such that $4b + 2a = 1$, then the pair (a, b) lies on the boundary of $\mathcal{F}$.

Indeed, this can be seen from the following construction illustrated in Figure 5. The constructed set consists of n points distributed on 10 segments as shown in Figure 5.

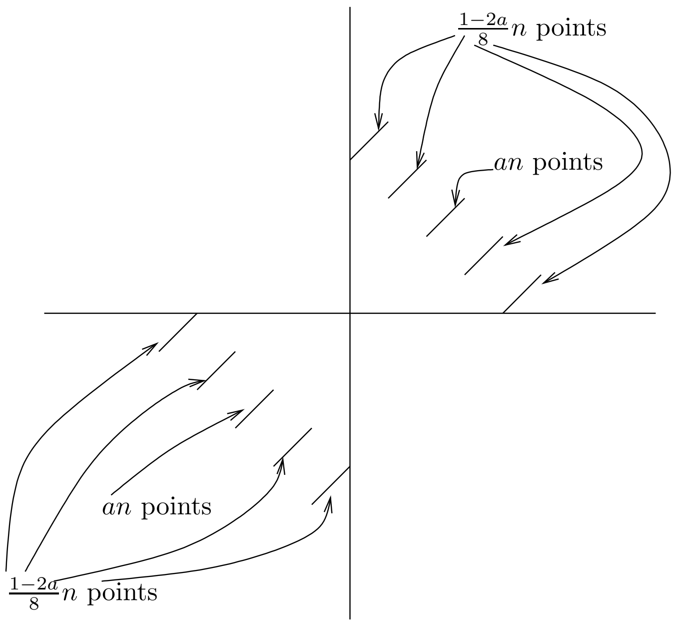

Figure 5: A construction that shows that Theorem 5 is best possible

Observe that because $a \geq \frac{1}{10}$, then $\frac{1-2a}{8}n \leq an$. It is easy so see by inspection that for every *epsilon* > 0 there is no point p in the resulting set P such that two opposite quadrants determined by p contain $(a + \epsilon)n$ and $(b + \epsilon)n$ points of P, respectively. This shows that (a, b) lies on the boundary of $\mathcal{F}$.

Theorems 3, 4, and Theorem 5 together with Theorem 1 and the constructions above thus fully characterize the set $\mathcal{F}$. The set $\mathcal{F}$ is shown in Figure 6.

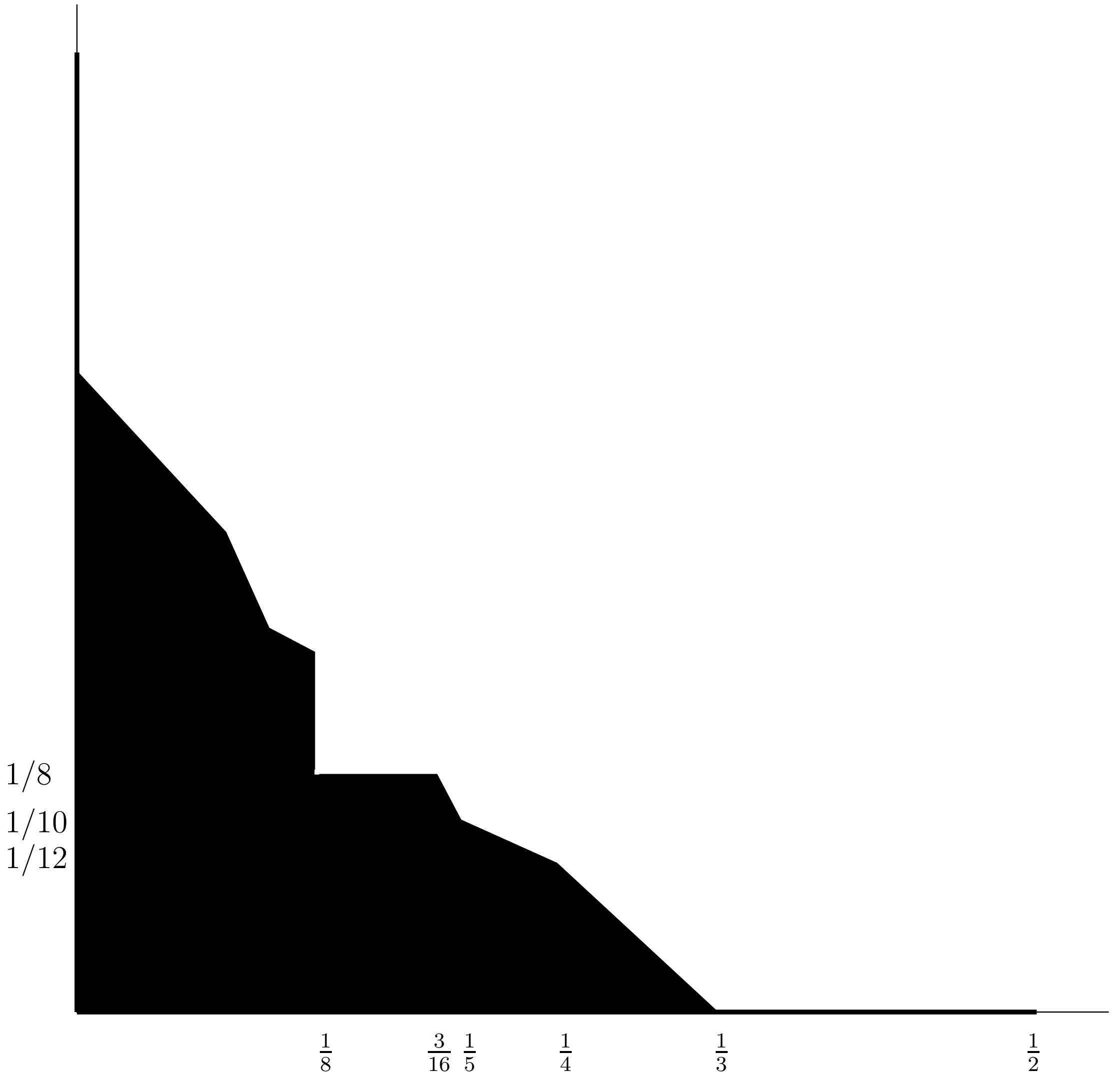

Figure 6: The set $\mathcal{F}$.

4 Generalizations to higher dimensions

The set $\mathcal{F}$ of all quadrants-admissible pairs was defined as the set of all pairs (α, β) such that every set P of n points in the plane contains a point $p \in P$ that determines two opposite quadrants, one containing at least αn points of P and the other containing at least βn points of P. What if we do not require the point p to belong to the set P, and only seek for a point z in the plane with the same property of two opposite quadrants determined by it. In other words, we would like to find all pairs (α, β) such that for any set P of n points in the plane there exist a vertical line and a horizontal line that determine two opposite quadrants one containing at least αn points of P and the other containing at least βn points of P.

One way to generalize this is to consider two opposite quadrants determined by a vertical and a horizontal lines as a diagonal in the 2-by-2 array of cells determined by these two lines.

Then one may want to consider $n-1$ vertical lines and $n-1$ horizontal lines and to find a *generalized diagonal* of cells in the arrangement of n-by-n cells, determined by these lines, such that each of the cells contains "many" points of P. By a generalized diagonal we mean a set of n cells such that no two are in the same row or in the same column (see Figure 7).

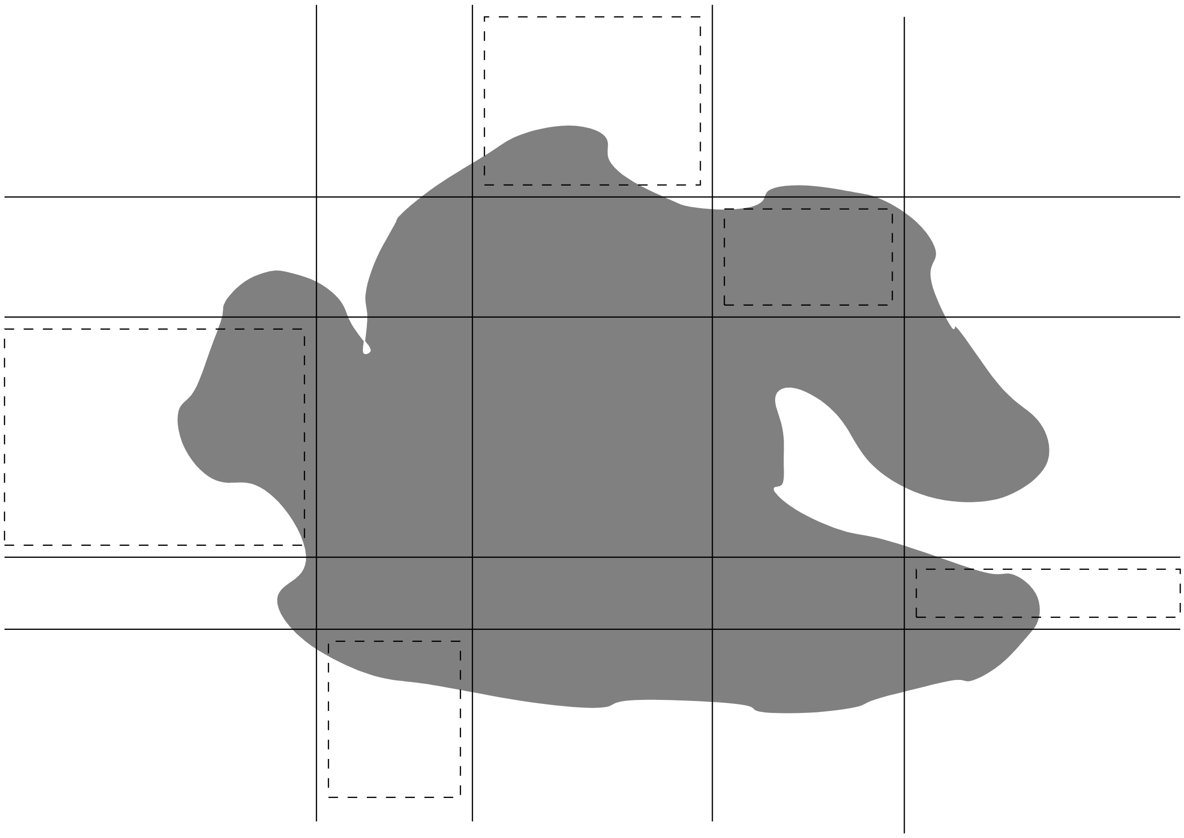

Figure 7: A continuous measure splitted to cells by 4 vertical and 4 horizontal lines. A generalized diagonal is illustrated in the dotted squares.

Since we do not require the vertical and horizontal lines to intersect in points of the set in question, it is equivalent to consider a continuous probability measure in the plane and to seek for $n-1$ vertical lines and $n-1$ horizontal lines and a generalized diagonal of cells in the arrangement of the n-by-n cells, determined by these lines, such that each of the cells has "large" measure (see Figure 7).

The following theorem gives a partial answer to a generalization of this problem to $\mathbb{R}^d$.

Theorem 6. *Let μ be a continuous probability measure in $[0,1]^d$. Let n be a positive integer and let $\alpha_1,\ldots,\alpha_n$ be a sequence of positive real numbers such that $\sum_{i=1}^n \alpha_i = \frac{1}{n^{d-1}}$. Then there exist numbers $x_{i,j}$, where $1 \le i \le d$ and $0 \le j \le n$, and d permutations $\pi_1,\ldots,\pi_d$ on $\{1,\ldots,n\}$ with the following properties:*

1. *For every $1 \le i \le d$ $x_{i,0} = 0 < x_{i,1} < \ldots < x_{i,n-1} < x_{i,n} = 1$.*

2. *For every* $1 \leq j \leq n$ *we have* $\mu([x_{1,\pi_1(j)-1}, x_{1,\pi_1(j)}] \times \ldots \times [x_{d,\pi_d(j)-1}, x_{d,\pi_d(j)}]) \geq \alpha_j$.

Remark. Intuitively speaking, the requirements on the numbers $x_{i,j}$ in Theorem 6 are that the d dimensional array of n^d cells generated by the hyper-planes $\{\{x_i = x_{i,j}\} | 1 \leq i \leq d \text{ and } 0 \leq j \leq n\}$, contains a generalized diagonal of n boxes with measures of at least $\alpha_1, \ldots, \alpha_n$, respectively.

Proof. In fact the proof is not much more complicated than the statement of the theorem. The proof goes by induction on d. For $d = 1$ simply define $x_{1,0} = 0$ and for every $1 \leq j \leq n$ let $x_{1,j} = x_{1,j-1} + \alpha_j$. In this case let $\pi_1 : [n] \to [n]$ be the identity permutation.

For $d > 1$, let $\alpha'_j = n\alpha_j$ for every $1 \leq j \leq n$. Observe that $\sum_{j=1}^{n} \alpha'_j = \frac{1}{n^{d-2}}$. Let μ' denote the measure μ projected on the last dimension. That is, μ' is a continuous probability measure on $[0,1]^{d-1}$ defined by $\mu'([a_1, b_1] \times \ldots \times [a_{d-1}, b_{d-1}]) = \mu([a_1, b_1] \times \ldots \times [a_{d-1}, b_{d-1}] \times [0, 1])$.

By the induction hypothesis there exist numbers $x_{i,j}$, where $1 \leq i \leq d-1$ and $0 \leq j \leq n$, and $d-1$ permutations $\pi_1, \ldots, \pi_{d-1}$ on the elements $\{1, \ldots, n\}$, with the following properties:

1. For every $1 \leq i \leq d-1$ $x_{i,0} = 0 < x_{i,1} < \ldots < x_{i,n-1} < x_{i,n} = 1$.
2. For every $1 \leq j \leq n$ we have $\mu'([x_{1,\pi_1(j)-1}, x_{1,\pi_1(j)}] \times \ldots \times [x_{d-1,\pi_{d-1}(j)-1}, x_{d-1,\pi_{d-1}(j)}]) \geq \alpha'_j$.

We now define the sequence $x_{d,0}, x_{d,1} \ldots, x_{d,n}$ and the permutation π_d as follows. We put $x_{d,0} = 0$. $x_{d,1}$ is defined to be the minimum number greater than $x_{d,0}$ such that there exists some j between 1 and n with $\mu([x_{1,\pi_1(j)-1}, x_{1,\pi_1(j)}] \times \ldots \times [x_{d-1,\pi_{d-1}(j)-1}, x_{d-1,\pi_{d-1}(j)}] \times [x_{d,0}, x_{d,1}]) = \alpha_j$. We set $\pi_d(j) = 1$. Observe that because of the minimality of $x_{d,1}$, for every $1 \leq k \leq n$ we have $\mu([x_{1,\pi_1(k)-1}, x_{1,\pi_1(k)}] \times \ldots \times [x_{d-1,\pi_{d-1}(k)-1}, x_{d-1,\pi_{d-1}(k)}] \times [x_{d,0}, x_{d,1}]) \leq \alpha_k$.

For $1 < k \leq n$ we define $x_{d,k}$ to be the minimum number greater than $x_{d,k-1}$ such that there exists some j between 1 and n such that j is different from each of $\pi_d^{-1}(1), \ldots, \pi_d^{-1}(k-1)$ and $\mu([x_{1,\pi_1(j)-1}, x_{1,\pi_1(j)}] \times \ldots \times [x_{d-1,\pi_{d-1}(j)-1}, x_{d-1,\pi_{d-1}(j)}] \times [x_{d,k-1}, x_{d,k}]) = \alpha_j$. We set $\pi_d(j) = k$.

Finally, we let $x_{d,n} = 1$ and set $\pi_d(j) = n$, where j is the only index between 1 and n that is not one of $\pi_d^{-1}(1), \ldots, \pi_d^{-1}(n-1)$.

It is straight forward to check that the numbers $x_{i,j}$ where $1 \leq i \leq d$ and $0 \leq j \leq n$ and $\pi_1, \ldots, \pi_d$ satisfy the requirements of the theorem. ∎

The following discrete version of Theorem 6 is an immediate corollary.

Theorem 7. *Let* P *be a finite set of points in* $[0,1]^d$. *Let* n *be a positive integer and let* $\alpha_1, \ldots, \alpha_n$ *be a sequence of positive real numbers such that* $\sum_{i=1}^{n} \alpha_i = \frac{1}{n^{d-1}}$. *Then there exist numbers* $x_{i,j}$, *where* $1 \leq i \leq d$ *and* $0 \leq j \leq n$, *and* d *permutations* $\pi_1, \ldots, \pi_d$ *on* $\{1, \ldots, n\}$ *with the following properties:*

1. *For every* $1 \leq i \leq d$ $x_{i,0} = 0 < x_{i,1} < \ldots < x_{i,n-1} < x_{i,n} = 1$.

2. *For every* $1 \le j \le n$ *we have* $|P \cap [x_{1,\pi_1(j)-1}, x_{1,\pi_1(j)}] \times \ldots \times [x_{d,\pi_d(j)-1}, x_{d,\pi_d(j)}]| \ge \alpha_j |P| + O(1)$.

The case $n = 2$ in Theorem 7 simply says that if P is any finite set of points in $\mathbb{R}^d$ (one can easily replace $[0,1]^d$ in the theorem with $\mathbb{R}^d$ by applying a linear transformation of the from $x \to cx$ for a suitable constant $c > 0$) and α and β are two positive numbers with $\alpha + \beta \le \frac{1}{2^{d-1}}$, then the following statement is true: There exists a point $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$ such that among the 2^d orthants determined by the d axis-parallel hyper-planes through p (namely, $\{x_i = z_i\}$ for $i = 1, \ldots, d$) there are two opposite orthants containing at least $\alpha|P|$ and $\beta|P|$ points of P each, respectively.

When $d = 2$, the example in Figure 8 shows that this result is best possible. Indeed, let P be a finite set of points evenly distributed on a circle. It is not hard to check that for any point z in the plane the horizontal and vertical lines through Z determine four quadrants such that in any two opposite ones there are at most $\frac{1}{2}|P|$ points of P altogether, unless one of the two opposite quadrants is empty and then the other one may contain even all points of P.

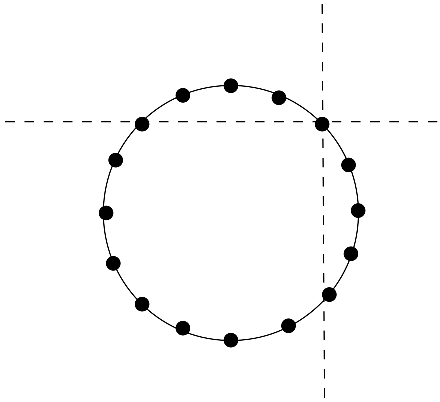

Figure 8: n points evenly distributed on a circle.

As we shall see later, Theorem 7 is not best possible in dimension $d > 2$. However, when $\alpha_1, \ldots, \alpha_n$ in Theorem 7 are equal, then the result in Theorem 7 is best possible in any dimension. This is a consequence of the following simple proposition.

Proposition 1. *Let* λ *be the standard Lebesgue measure on* $[0,1]^d$ *and let* n *be a positive integer. Then one cannot find numbers* $x_{i,j}$*, where* $1 \le i \le d$ *and* $0 \le j \le n$*, and* d *permutations* $\pi_1, \ldots, \pi_d$ *on* $\{1, \ldots, n\}$ *with the following properties:*

1. *For every* $1 \le i \le d$ $x_{i,0} = 0 < x_{i,1} < \ldots < x_{i,n-1} < x_{i,n} = 1$.
2. *For every* $1 \le j \le n$ *we have* $\mu([x_{1,\pi_1(j)-1}, x_{1,\pi_1(j)}] \times \ldots \times [x_{d,\pi_d(j)-1}, x_{d,\pi_d(j)}]) > \frac{1}{n^d}$.

Proof. Assume to the contrary that there number $x_{i,j}$ and permutations π_i as in the statement of the proposition. We will use the means inequality by which for every d positive numbers $a_1, \ldots, a_d$ we have

$$(a_1 \ldots a_d)^{\frac{1}{d}} \le \frac{a_1 + \ldots a_d}{d}.$$

For every $1 \le i \le d$ and $1 \le j \le n$ put $a_{i,j} = x_{i,\pi_i(j)} - x_{i,\pi_i(j)-1}$. Clearly, for every $1 \le i \le d$ we have $\sum_{j=1}^{n} a_{i,j} = 1$. On the other hand because of our assumption, for every $1 \le j \le n$ we have $\prod_{i=1}^{d} a_{i,j} > \frac{1}{n^d}$. Therefore, by the means inequality, for every $1 \le j \le n$ we have:

$$\sum_{j=1}^{d} a_{i,j} \ge d(\prod_{i=1}^{d} a_{i,j})^{\frac{1}{d}} > \frac{d}{n}.$$

Therefore, we reach the desired contradiction as,

$$\begin{aligned} d &= \sum_{i=1}^{d}\sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n}\sum_{i=1}^{d} a_{i,j} > \\ &> \sum_{j=1}^{n} \frac{d}{n} = d. \end{aligned}$$

∎

The following theorem shows that when $d = 3$ and $n = 2$ one can obtain a better result in Theorem 7 in the case where α_1 and α_2 are different.

Theorem 8. *Let P be a finite set of points in $\mathbb{R}^3$. We assume that no two points in P have the same x-coordinates, or the same y-coordinates, or the same z-coordinates. Let α_1 and α_2 be positive numbers such that $3\alpha_1 + \alpha_2 = \frac{1}{3}$. Then there exists a point $z = (z_1, z_2, z_3) \in \mathbb{R}^3$ such that the hyper-planes $\{x_i = z_i\}$ for $i = 1, 2, 3$ determine two opposite orthants, one containing at least $\alpha_1|P|$ points of P and the other at least $\alpha_2|P|$ points of P.*

Proof. Let $0 < \epsilon < \frac{1}{3}$ be given. Define A_1 to be the set of $(\frac{1}{6} + \epsilon)|P|$ points with the least x_1-coordinates and let A_2 be the set of $(\frac{1}{6} + \epsilon)|P|$ points with the largest x_1-coordinates. Similarly, define B_1 and B_2 with respect to the x_2-coordinate and C_1 and C_2 with respect to the x_3-coordinate. Let $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, and $C = C_1 \cup C_2$.

For every $i = 0, 1, 2, 3$, let P_i denote the set of all points in P that belong to precisely i of the sets A, B, and C. Clearly, $|P_0| + |P_1| + |P_2| + |P_3| = |P|$. Moreover, $|P_1| + 2|P_2| + 3|P_3| = |A| + |B| + |C| = (1 + 6\epsilon)|P|$.Hence, $P_2 + 2|P_3| \ge 6\epsilon|P|$. Each point in P_2 belongs to precisely two of the sets A_1, A_2, B_1, B_2, C_1, and C_2. Each point in $P3$ belongs to precisely three of the sets A_1, A_2, B_1, B_2, C_1, and C_2. Keeping in mind that $A_1 \cap A_2 = B_1 \cap B_2 = C_1 \cap C_2 = \emptyset$, we have, by the pigeonhole principle there is some intersection of two sets from A_1, A_2, B_1, B_2, C_1, and C_2 that contain at least $\frac{1}{12}6\epsilon|P|$ elements of $P_2 \cup P_3$, that is, of P.

Without loss of generality assume that $A_1 \cap B_1$ includes at least $\frac{1}{2}\epsilon|P|$ points of P. Observe that

$$|A_1 \cup B_1| = |A_1| + |B_1| - |A_1 \cap B_1| \le (\frac{1}{6} + \epsilon)|P| + (\frac{1}{6} + \epsilon)|P| - \frac{1}{2}\epsilon|P| = (\frac{1}{3} + \frac{3}{2}\epsilon)|P|.$$

Therefore, $P \setminus (A_1 \cup B_1)$ consists of at least $(\frac{2}{3} - \frac{3}{2}\epsilon)|P|$ points of P.

Now let z_1 be a number greater than precisely $(\frac{1}{6} + \epsilon)|P|$ of the x_1-coordinates of the points in P. Then the x_1-coordinate of any of the points in $A_1 \cup B_1$ is smaller than z_1 and any of the x_1-coordinates of points in $P \setminus (A_1 \cup A_2)$ is greater than z_1. Similarly, let z_2 be a number greater than precisely $(\frac{1}{6} + \epsilon)|P|$ of the x_2-coordinates of the points in P. Then the x_2-coordinate of any of the points in $A_1 \cup B_1$ is smaller than z_2 and any of the x_2-coordinates of points in $P \setminus (A_1 \cup A_2)$ is greater than z_2.

Let t_1 be the median of all x_3-coordinates of points in $A_1 \cup B_1$. Let t_2 be the median of all x_3-coordinates of points in $P \setminus (A_1 \cup B_1)$. Let z_3 be any number between t_1 and t_2. Then z_3 separates between at least half of the x_3-coordinates of points in $A_1 \cup B_1$ and at least half of the x_3-coordinates of the points in $P \setminus (A_1 \cup B_1)$. This means that $z = (z_1, z_2, z_3)$ determines two opposite orthants in $\mathbb{R}^3$ one containing at least $\frac{1}{4}\epsilon|P|$ points of P and the other containing at least $(\frac{1}{3} - \frac{3}{4}\epsilon)|P|$ points of P. Put $\epsilon = 4\alpha_1$ and then $\frac{1}{3} - \frac{3}{4}\epsilon = \frac{1}{3} - 3\alpha_1 = \alpha_2$. We thus get the desired result that z determines two opposite orthants, one containing at least $\alpha_1|P|$ points of P and the other containing at least $\alpha_2|P|$ points of P. ∎

We will now show that Theorem 8 is nearly tight when α_1 is close to 0 (and then α_2 is close to $\frac{1}{3}$). The following construction of a set $P \subset \mathbb{R}^3$ of $n = 3m$ points has the property that for any point z and any two opposite orthants, determined by z, such that each contains at least one point of P, we have that there are at most $m = \frac{1}{3}n$ points of P in each of these orthants.

Let S be the following set of m vectors in $\mathbb{R}^3$, very close to the origin: $S = \{\frac{i}{100m}(1,1,1) \mid i = 0, 1, \ldots, m-1\}$. We define the following three Minkowsky sums: $A = S + (-1, 0, 1)$, $B = S + (0, 1, -1)$, and $C = S + (1, -1, 0)$. Finally we let $P = A \cup B \cup C$. Note that P is a set of $n = 3m$ points in $\mathbb{R}^3$ no two of which share the same x_1, x_2, or x_3 coordinates.

Let $z = (z_1, z_2, z_3)$ be any point in $\mathbb{R}^3$. Assume that R and T are two opposite orthants determined by z each containing at least one point of P. We claim that R (and therefore also T) cannot contain more than m points of P. To see this assume that T contains a point $p_A = (a_1, a_2, a_3)$ from A (the other two cases where T contains a point from B or from C are very similar). If $z_1 < a_1$, then R may contain only points of A and in particular it contains at most m points of P. Similarly, if $z_3 > a_3$, then R may contain only points of A and we are done. Therefore, $z_1 > a_1$ and $z_3 < a_3$. This implies that no point of A belongs to R. If $z_2 > a_2$, then no point of C may belong to R and hence R may contain only points of B and in particular at most m points of P. If $z_2 < a_2$, then no point of B may belong to R and hence R may contain only points of C and in particular at most m points of P.

5 Open problems and concluding remarks

References

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