INTEGRATING CURVATURE: FROM UMLAUFSATZ TO $J^+$ INVARIANTS.

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Abstract. Hopf’s Umlaufsatz relates the total curvature of a closed immersed plane curve to its rotation index. While the curvature of a curve changes under local deformations, its integral over a closed curve is invariant under regular homotopies. A natural question is whether one can find some natural densities on a curve, such that the corresponding integrals are (possibly after some corrections) also invariant under regular homotopies of the curve. We construct a family of such densities using indices of points relative to the curve. The corresponding generating function in a formal variable $q$ may be considered as a quantization of the total curvature. The linear term in the Taylor expansion at $q = 1$ coincides, up to a normalization, with Arnold’s $J^+$ invariant.

Let $\Gamma$ be a closed oriented immersed plane curve $\Gamma : S^1 \to \mathbb{R}^2$. One of the fundamental notions related to $\Gamma$ is its curvature $\kappa$. Another important notion is that of a rotation index $\text{rot}(\Gamma)$, i.e. the number of turns made by the tangent vector as we follow $\Gamma$ along its orientation.

Hopf’s Umlaufsatz [2] is one of the simplest versions of the Gauss-Bonnet theorem and one of the fundamental theorems in the theory of plane curves. It relates two different types of data: local geometric characteristic of a plane curve – its curvature $\kappa$ – and a global topological characteristic – its rotation index $\text{rot}(\Gamma)$. Although the curvature of a plane curve changes under local deformations, the theorem states that its average (integral) over a closed curve is invariant under homotopies in the class of immersed curves:

**Theorem 1** (Hopf’s Umlaufsatz).

\[
\frac{1}{2\pi} \int_{S^1} \kappa(t) \, dt = \text{rot}(\Gamma)
\]

A natural question is whether one can find some natural densities $\rho$ on $\Gamma$ such that the average $I_\rho(\Gamma) = \int_{S^1} \kappa(t) \rho(t) \, dt$ is (possibly after

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some corrections) also invariant under local deformations of $\Gamma$. Since the rotation index is (up to normalization) the only invariant of $\Gamma$ in the class of immersed curves, we cannot expect $I_p(\Gamma)$ to remain invariant under arbitrary homotopies. We can hope, however, that the result is invariant under regular homotopies. Here by a regular homotopy we mean homotopy in the class of generic immersions, i.e. immersions with a finite set $X$ of transversal double points as the only singularities. Invariants of such a type were originally introduced by Arnold [1] and include the celebrated $J^\pm$ and $St$ invariants (see [1] for details).

We construct a family of such densities using the index $\text{ind}_p(\Gamma)$ of $\Gamma$ relative to a point $p$. Given $p \in \mathbb{R}^2 \setminus \Gamma$, we define $\text{ind}_p(\Gamma)$ as the number of turns made by the vector pointing from $p$ to $\Gamma(t)$, as we follow $\Gamma$ along its orientation. This defines a locally-constant function on $\mathbb{R}^2 \setminus \Gamma$. See Figure 1a. Suppose that $\Gamma$ is generic. Then we can extend $\text{ind}_p(\Gamma)$ to a $\frac{1}{2}\mathbb{Z}$-valued function on $\mathbb{R}^2$. To define $\text{ind}_p(\Gamma)$ for $p \in \Gamma$, average its values on the regions adjacent to $p$ – two regions if $p$ is a regular point of $\Gamma$, and four regions if $p$ is a double point of $\Gamma$. See Figure 1b. For each double point $d = (t_1) = (t_2) \in X$, define $\theta_d \in (0, \pi)$ as the (non-oriented) angle between two tangent vectors $\Gamma'(t_1)$ and $-\Gamma'(t_2)$. For $q \in \mathbb{R} \setminus \{0\}$, define $I_q(\Gamma) \in \mathbb{R}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ by

$$I_q(\Gamma) = \frac{1}{2\pi} \left( \int_{\mathbb{S}^1} \kappa(t) \cdot q^{\text{ind}_p(\Gamma)} dt - \sum_{d \in X} \theta_d \cdot q^{\text{ind}_d(\Gamma)} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \right)$$

**Theorem 2.** $I_q(\Gamma)$ is invariant under regular homotopies of $\Gamma$.

**Proof.** Note that we can generalize all above notions and formulas to the case of a multi-component curve $\Gamma : \bigcup_n \mathbb{S}^1 \to \mathbb{R}^2$ by a summation of the appropriate indices over the components of $\Gamma$.

Let us smooth the original curve $\Gamma$ in each double point respecting the orientation to get a multi-component curve $\tilde{\Gamma} = \bigcup_n \tilde{\Gamma}_n$ without double points. Then values of $I_q$ on $\Gamma$ and $\tilde{\Gamma}$ differ by an easily computable
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factor, which depends only on the regular homotopy class of \( \Gamma \). Indeed, consider a small neighborhood \( U_d \) of a double point \( d \) of index \( i \), see Figure 1c. Under smoothing of \( d \), the total curvature of \( \Gamma \cap U_d \) differs from that of \( \Gamma \cap U_d \) by \( \pm (\pi - \theta_d) \) for the fragment with index \( i \pm \frac{1}{2} \), see Figure 1c. Thus the integral part of \( I_q \) changes by \( \frac{1}{2\pi}(\pi - \theta_d)(q^{i+\frac{1}{2}} - q^{-\frac{1}{2}}) \). Also, the double point \( d \) contributes \( \frac{1}{2}\theta_d q^i(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \) to \( I_q(\Gamma) \). Smoothing removes \( d \), so this summand disappears from \( I_q(\Gamma) \). Thus, the total change of \( I_q \) under smoothing of \( d \) equals \( \frac{1}{2} q^i(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \). Hence

\[
I_q(\Gamma) = I_q(\widetilde{\Gamma}) - \frac{1}{2} \sum_d q^{\text{ind}_d(\Gamma)}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}).
\]

Since \( \sum_d q^{\text{ind}_d(\Gamma)}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \) is invariant under regular homotopies of \( \Gamma \), it remains to prove the invariance of \( I_q(\widetilde{\Gamma}) = \sum_q I_q(\widetilde{\Gamma}) \).

Note that \( \text{ind}_{\eta(t)}(\widetilde{\Gamma}) \) is constant on each component \( \widetilde{\Gamma}_n \) of \( \widetilde{\Gamma} \), so

\[
I_q(\widetilde{\Gamma}) = \frac{1}{2\pi} \int_{\beta_t} \kappa_n(t) \cdot q^{\text{ind}_{\eta(t)}(\Gamma)} \, dt = q^{\text{ind}_{\eta(t)}(\Gamma)} \frac{1}{2\pi} \int_{\beta_t} \kappa_n(t) \, dt
\]

and by Umlaufsatz (1) we get \( I_q(\widetilde{\Gamma}_n) = \pm q^{\text{ind}_{\eta(t)}(\Gamma)} \), depending on \( \text{rot}(\widetilde{\Gamma}_n) = \pm 1 \). Thus, \( I_q(\widetilde{\Gamma}_n) \) is invariant under regular homotopies of \( \widetilde{\Gamma} \). But a regular homotopy of \( \Gamma \) induces a regular homotopy of \( \widetilde{\Gamma} \) and the theorem follows.

\[\square\]

Any two immersions with the same rotation number can be connected by regular homotopy and a finite sequence of self-tangency and triple-point modifications, shown in Figure 2. Depending on orientations and

\[
\begin{array}{c}
\raisebox{-0.5cm}{\includegraphics[width=0.3\textwidth]{figure2.png}}
\end{array}
\]

\textbf{Figure 2.} Self-tangency and triple-point modifications.

indices of adjacent regions, one can distinguish several types of these modifications. Self-tangencies can be separated into direct and opposite, shown in Figure 3a and 3b respectively. An index of a self-tangency modification is the index of two new-born double points (e.g., modifications in Figure 3 are of index \( i \)). Triple-point modifications can be separated into weak (or acyclic) and strong (or cyclic), shown in Figure 4a and 4b.
respectively. An index of a triple-point modification\(^1\) is the minimum of indices of double points involved in this modification (e.g., modifications in Figure 4 are of index \(i\)).

![Figure 3: Direct and opposite self-tangency modifications of index \(i\).](image)

Invariants of regular homotopy are uniquely determined by their behavior under these modifications, together with normalizations on standard curves \(K_i\) of \(\text{rot}(K_i) = i, i = 0, \pm 1, \pm 2, \ldots\) shown in Figure 5. Basic invariants \(J^\pm\) and \(St\) of (regular homotopy classes of) generic plane curves were introduced axiomatically by Arnold [1]. In particular, \(J^+\) is uniquely determined by the following axioms:

- \(J^+\) does not change under an opposite self-tangency or triple-point modifications.
- Under a direct self-tangency modification which increases the number of double points, \(J^+\) jumps by 2.

\(^1\)Our indices of modifications differ from the ones of [3] by an \(-1\) shift.
On the standard curves $K_i$ we have $J^+(K_0) = 0$ and $J^+(K_i) = -2(|i| - 1)$ for $i = \pm 1, \pm 2, \ldots$.

In a similar way, $I_q(\Gamma)$ is uniquely determined by the following

**Theorem 3.** The invariant $I_q(\Gamma)$ satisfies the following properties:

- $I_q(\Gamma)$ does not change under opposite self-tangencies.
- Under direct self-tangencies of index $i$, the invariant $I_q(\Gamma)$ jumps by $-q^i(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$.
- Under (both weak and strong) triple-point modifications of index $i$, $I_q(\Gamma)$ jumps by $-\frac{1}{2}q^{i+\frac{1}{2}}(q^{\frac{3}{2}} - q^{-\frac{1}{2}})^2$.
- We have $I_q(-\Gamma) = -I_q(\Gamma)$, where $-\Gamma$ denotes $\Gamma$ with the opposite orientation.
- On the standard curves $K_i$ we have $I_q(K_0) = \frac{1}{2}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ and $I_q(K_i) = \frac{1}{2}(i-1)q^{\frac{3}{2}} + \frac{1}{2}(i+1)q^{\frac{1}{2}}$ for $i = 1, 2, \ldots$

**Proof.** A straightforward computation verifies both the behavior of $I_q(\Gamma)$ under self-tangencies and triple-point modifications and its values on the curves $K_i$. To verify the behavior of $I_q(\Gamma)$ under an orientation reversal, note that $\text{ind}_p(-\Gamma) = -\text{ind}_q(\Gamma)$, which corresponds to the involution $q \mapsto q^{-1}$ in terms of $q^{\text{ind}_1(\Gamma)}$ and $q^{\text{ind}_a(\Gamma)}$ of (2). Also, both terms in (2) change signs: the integral due to the change of parametrization, and the sum over double points due to the equality $q^{\frac{1}{2}} - q^{-\frac{1}{2}} = (q^{-1})^{\frac{1}{2}} - (q^{-1})^{-\frac{1}{2}}$.

Substituting $q = 1$ into (2), we readily obtain $I_q(\Gamma) = \frac{1}{2\pi}\int_{S^1} \kappa(t)\, dt = \text{rot}(\Gamma)$ and recover the classical Hopf Umlaufsatz, see Theorem 1. In this sense, invariant $I_q$ may be considered as a quantization of the total curvature (1). Let us study the next term $I'_1(\Gamma)$ of the Taylor expansion of $I_q(\Gamma)$ at $q = 1$. From (2) we immediately get

$$I'_1(\Gamma) = \frac{1}{2\pi} \left( \int_{S^1} \kappa(t) \cdot \text{ind}_1(\Gamma) \, dt - \sum_{d \in X} \theta_d \right).$$

**Proposition 4.** $I'_1(\Gamma)$ can be identified with Arnold’s $J^+$ invariant via $I'_1(\Gamma) = \frac{1}{2}(1 - J^+(\Gamma))$.

**Proof.** Indeed, note that by Theorem 2, $I'_1(\Gamma)$ is invariant under homotopies of $\Gamma$ in the class of generic immersions. Differentiating at $q = 1$ expressions for jumps of $I_q(\Gamma)$ in Theorem 3 we immediately conclude that $I'_1(\Gamma)$ is invariant under opposite tangencies and triple-point modifications. Moreover, under direct tangencies, $I'_1(\Gamma)$ jumps by $-1$. Thus its behavior under all modifications is the same as that of $-\frac{1}{2}J^+(\Gamma)$ (up...
to an additive constant depending on \( \text{rot}(\Gamma) \)). A straightforward computation shows that \( I_1'(K_0) = \frac{1}{2} \) and \( I_1'(K_i) = |i| - \frac{1}{2} \) for \( i = \pm 1, \pm 2, \ldots \) on the standard curves \( K_i \). Thus the proposition follows.

**Remark 5.** An infinite family of invariants, called “momenta of index” \( M_r \), together with their generating function \( P_\Gamma(q) \in \mathbb{Z}[q, q^{-1}] \) were introduced by Viro in [3, Section 5]. A careful check of their behavior under self-tangencies and triple-point modifications, together with their values on the standard curves \( K_i \), allow one to relate \( P_\Gamma(q) \) to \( I_q(\Gamma) \) as follows:

\[
P_\Gamma(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) I_q(\Gamma) + 1 + \frac{1}{2} \sum_{d \in X} q^{\text{ind}_d(\Gamma)} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2
\]

**References**


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