From 3-manifolds to planar graphs and cycle-rooted trees

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"Confirming the belief that music and math are related, I will now sing some lovely French equations."
Encode 3-manifolds by planar weighted graphs
Pass from various presentations of 3-manifolds to graphs and back
Similar encodings for related objects: links in 3-manifolds, manifolds with $Spin$- or $Spin^c$-structures, elements of the mapping class group, etc.
Encoding is not unique: finite set of simple moves on graphs (related to electrical networks)
Various invariants of 3-manifolds transform into combinatorial invariants
Configuration space integrals $\rightarrow$ counting of subgraphs
Low-degree invariants $\rightarrow$ counting of rooted forests
A chainmail graph is a planar graph $G$, decorated with $\mathbb{Z}$-weights:

- Each vertex $v$ is decorated with a weight $d(v)$; A vertex is balanced, if $d(v) = 0$ (can think about $d(v)$ as a “defect” of $v$); a graph is balanced, if all of its vertices are.

- Each edge $e$ is decorated with a weight $w(e)$. A 0-weighted edge may be erased. Multiple edges are allowed. Two edges $e_1, e_2$ connecting the same pair of vertices may be redrawn as one edge of weight $w(e_1) + w(e_2)$. Looped edges are also allowed; a looped edge may be erased.
Given a chainmail graph $G$ with vertices $v_i$ and edges $e_{ij}$, $i,j = 1, 2, \ldots, n$ we construct a surgery link $L$ as follows:

- **vertex** $v_i$  $\rightarrow$ standard planar unknot $L_i$
- **$\pm 1$-weighted edge** $e_{ij}$  $\rightarrow$ $\pm 1$-clasped ribbon linking $L_i$ and $L_j$
Linking numbers and framings of components are given by a graph Laplacian matrix $\Lambda$ with entries

$$l_{ij} = \begin{cases} w_{ij}, & i \neq j \\ d_{ii} - \sum_{k=1}^{n} w_{ik}, & i = j \end{cases}$$

**Example (Constructing a surgery link)**

Different graphs and surgery links for the Poincare homology sphere
It turns out, that

**Theorem**

*Any (closed, oriented) $3$-manifold can be encoded by a chainmail graph.*

- Moreover, there are simple direct constructions starting from many different presentations of a manifold: surgery, Heegaard decompositions, plumbing, double covers of $S^3$ branched along a link, etc.
- Similar constructions work also for a variety of similar objects: links in $3$-manifolds, $3$-manifolds with $Spin$- or $Spin^c$-structures, elements of the mapping class group, etc.

Some info about $M$ can be immediately extracted from $G$. In particular, $M$ is a $\mathbb{Q}$-homology sphere iff $\det \Lambda \neq 0$ and then $|H_1(M)| = |\det \Lambda|$; also, signature of $M$ is the signature $\text{sign}(\Lambda)$ of $\Lambda$. 
Proofs and explicit constructions ...

... No time to present here.
Calculus of chainmail graphs

An encoding of a manifold by a chainmail graph is non-unique. However, there is a finite set of simple moves which allow one to pass from one chainmail graph encoding a manifold to any other graph encoding the same manifold. The most interesting moves are

They are related to a number of topics: Kirby moves, relations in the mapping class group, electrical networks and cluster algebras, and Reidemeister moves for link diagrams (via balanced median graphs) -
Combinatorial invariants of 3-manifolds

Chern-Simons theory leads to a lot of knot and 3-manifold invariants. Attempts to understand the Jones polynomial in these terms led to quantum knot invariants, the Kontsevich integral, configuration space integrals and other constructions. In particular,

\[
\text{Perturbative CS-theory} \xrightarrow{\text{Feynman diagrams}} \text{Configuration space integrals}
\]

- Rather powerful: contain universal finite type invariants of knots and 3-manifolds
- Very complicated technically
- Extremely hard to compute

We expect a similar combinatorial setup in our case: An appropriate CS-theory on graphs \[\xrightarrow{\text{discrete}}\text{Discrete sums over subgraphs}\]

Types of subgraphs are suggested by the theory: uni-trivalent graphs for links; trivalent graphs for 3-manifolds.
This actually works! Here is the setup: we pass from the manifold $M$ to its combinatorial counter-part $\rightarrow$ a chainmail graph $G$. In both cases we use summations over similar Feynman graphs.

- Vertices of a Feynman graph:
  configurations of $n$ points in $M \rightarrow$ sets of $n$ vertices in $G$

- Edges of a Feynman graph:
  propagators in $M \rightarrow$ paths of edges in $G$

- Integration over the configuration space $\rightarrow$ sum over subgraphs

- Compactifications and anomalies due to collisions of points in $M \rightarrow$
  appearance of degenerate graphs when several vertices merge together
Let’s see this on an example of the simplest non-trivial perturbative invariant, corresponding to the Feynman graph with 2 vertices, i.e., the \( \Theta \)-graph:

\[
\begin{array}{c}
1 & 2 \\
\end{array}
\]

We count maps \( \phi : \Theta \to G \) with weights and multiplicities. One can think about such a map as a choice of two vertices \( v_i \) and \( v_j \) of \( G \), connected by 3 paths of edges which do not have any common internal vertices:

The weight \( W(\phi) \) of \( \phi \) is the product \( L(\phi) \prod_{e \in \phi(G)} l_e \), where \( L(\phi) \) is the minor of \( \Lambda \), corresponding to all vertices of \( G \) not in \( \phi(\Theta) \).
Degenerate maps should be counted as well. Such degeneracies appear when two vertices of the $\Theta$-graph collide together to produce a figure-eight graph:

$$\begin{align*}
1 & \rightarrow \\
2 &
\end{align*}$$

Diagonal entries of $\Lambda$ also enter in the formula, when one lobe (or possibly both) of the figure-eight graph becomes a looped edge in the 4-valent vertex. The weight of such a loop in $v_i$ is $l_{ii}$. E.g., for the map

$$\begin{align*}
i & \rightarrow \\
\rightarrow & \\
j & \rightarrow \\
\rightarrow & \\
k &
\end{align*}$$

we have $W(\phi) = L(\phi) \cdot l_{ij} \cdot l_{jk} \cdot l_{ki} \cdot l_{ii}$. In the most degenerate cases — a triple edge or double looped edge — weights need to be slightly adjusted.
“I think you should be more explicit here in step two.”
Theorem

\[ \Theta(G) = \sum_\phi W(\phi) \] is an invariant of \( M \). If \( M \) is a \( \mathbb{Q} \)-homology sphere (i.e., \( \det \Lambda \neq 0 \)), we have

\[ \Theta(G) = \pm 12 |H_1(M)| (\lambda_{CW}(M) - \frac{\text{sign}(M)}{4}) \]

where \( \lambda_{CW}(M) \) is the Casson-Walker invariant.

Conjecture

The next perturbative invariant can be obtained in a similar way by counting maps of \( \triangle \) and \( \square \) to \( G \).

Note that \( \Theta(G) \) is a polynomial of degree \( n + 1 \) in the entries of \( \Lambda \). This leads to

Conjecture

Any finite type invariant of degree \( d \) of 3-manifolds (with an appropriate normalization) is a polynomial of degree at most \( n + d \) in the entries of \( \Lambda \).
Remark

Instead of counting maps $\phi : \Theta \to G$, we may count $\Theta$-subgraphs of $G$, taking symmetries into account:

Example

For the (negatively oriented) Poincare homology sphere one has $G = \frac{3}{2} \frac{2}{5}$. Thus $\Lambda = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $\det \Lambda = -1$ (so $M$ is a $\mathbb{Z}$-homology sphere), $\text{sign}(\Lambda) = 0$, and to compute $\Theta(G)$ we count

$$2 \cdot (\begin{array}{c} \blackcirc \\ \blackcirc \end{array} + \begin{array}{c} \blackcirc \\ \blackcirc \end{array}) + (\begin{array}{c} \blackcirc \\ \blackcirc \end{array} + \begin{array}{c} \blackcirc \\ \blackcirc \end{array}) + 2 \cdot \begin{array}{c} \blackcirc \\ \blackcirc \end{array}$$


to get

$$\Theta(G) = 2 \cdot (1 \cdot 2^2 + 3 \cdot 2^2) + (1^2 + 2)(-3) + (3^2 + 2)(-1) + 2 \cdot (2^3 - 2) = 24$$

and obtain $\lambda_{CW}(M) = -2$. 
Counting cycle-rooted trees

Recall that the matrix $\Lambda$ was defined as the graph Laplacian for the weight matrix $W$:

$$l_{ij} = \begin{cases} w_{ij}, & i \neq j \\ d_{ii} - \sum_{k=1}^{n} w_{ik}, & i = j \end{cases}$$

An expression for $\Theta(M)$ in terms of the original weight matrix $W$ (with $d_{ii}$ on the diagonal) is even simpler and can be achieved by a certain generalized version of the celebrated Matrix Tree Theorem. For this purpose, we add to $G$ a new balancing "super-vertex" $v_0$, connecting every vertex $v_i$ of $G$ to $v_0$ by an edge of the weight $w_{0i} = -d_{ii}$. We also change weights of all old vertices to 0 to get a balanced graph $\hat{G}$:
The classical Matrix Tree Theorem states that \( \det \Lambda \) equals to the weighted number of the spanning trees of \( \widehat{G} \), where a tree \( T \) is counted with the weight \( \prod_{e \in T} w(e) \).

It turns out, that one can pass from \( \Theta(G) \) to a similar count of spanning cycle-rooted trees in \( \widehat{G} \):

This approach has a number of interesting applications and ramifications:

- Simpler computational formulas: no more degenerated cases, simpler graphs.
- Counting spanning cycle-rooted trees in \( \widehat{G} \) to get \( \Theta(G) \) leads to a new generalized version of the classical theorem Matrix Tree Theorem.
Finally, cycle-rooted trees can be interpreted as closed orbits of vector fields on a graph:

A **discrete vector field on a graph** is a choice of at most one outgoing edge at each vertex.

Critical vertices are those with no outgoing edges. An orbit may end in a critical point - these are trees with roots in critical points (and all edges oriented toward the root).

There are also **closed orbits**; these are cycle-rooted trees (with all edges oriented towards the cycle).

In these terms, the determinant $\det \Lambda$ counts vector fields on $G$ with no closed orbits. The $\Theta$-invariant counts vector fields with one closed orbit.
Time for speculations:

There is a highly suggestive continuous analogue for such a closed orbits counting: Gopakumar-Vafa’s Gauge Theory/Geometry duality between the CS theory and closed strings on a resolved conifold. The closed strings theory suggested by Gopakumar-Vafa leads to a certain Floer-type symplectic homology setup.

It seems that in our discrete setting Gopakumar-Vafa duality boils down to the Laplace transform on graphs and corresponds to a generalized Matrix Tree Theorem.

We thus expect that there is a suitable chain complex and a homology theory in the cycle-rooted trees setup. Its construction is challenging.
"On the other hand, my responsibility to society makes me want to stop right here."