UNIQUENESS/NONUNIQUENESS FOR NONNEGATIVE SOLUTIONS OF SECOND ORDER PARABOLIC EQUATIONS OF THE FORM $u_t = Lu + Vu - \gamma u^p$ IN R^n

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Abstract. In this paper we investigate uniqueness and nonuniqueness for nonnegative solutions of the equation

$$\text{(NS)}\qquad \begin{aligned} &u_t = Lu + Vu - \gamma u^p \text{ in } R^n \times (0,\infty);\\ &u(x,0) = f(x),\ x \in R^n;\\ &u \ge 0, \end{aligned}$$

where $\gamma > 0,\ p > 1,\ \gamma, V \in C^\alpha(R^n),\ 0 \le f \in C(R^n)$ and $L = \sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i}$ with $a_{i,j}, b_i \in C^\alpha(R^n)$.

1. Introduction. In this article we study uniqueness for nonnegative solutions

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$u \in C^{2,1}(R^n \times (0,\infty)) \cap C(R^n \times [0,\infty))$ to the semilinear equation

$$
\text{(NS)} \qquad
\begin{aligned}
&u_t = Lu + Vu - \gamma u^p \text{ in } R^n \times (0,\infty);\\
&u(x,0) = f(x),\ x \in R^n;\\
&u \geq 0,
\end{aligned}
$$

where γ, $V \in C^\alpha(R^n)$, $\gamma > 0$, $p > 1$, $0 \leq f \in C(R^n)$ and

$$
L = \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i},
$$

with $a_{i,j}, b_i \in C^\alpha(R^n)$ and $\sum_{i,j=1}^{n} a_{i,j}(x)\nu_i\nu_j > 0$, for all $x \in R^n$ and $\nu \in R^n - \{0\}$.

In the case that V is bounded from above, it will be useful to compare uniqueness in the class of nonnegative solutions for the semilinear equation with uniqueness in the class of bounded solutions $u \in C^{2,1}(R^n \times (0,\infty)) \cap C(R^n \times [0,\infty))$ for the corresponding linear equation:

$$
\text{(BL)} \qquad
\begin{aligned}
&u_t = Lu + Vu \ \text{ in } R^n \times (0,\infty);\\
&u(x,0) = f(x);\\
&\sup_{0 \leq t \leq T} \sup_{x \in R^n} |u(x,t)| < \infty, \text{ for all } T > 0,
\end{aligned}
$$

where $f \in C(R^n)$.

In the sequel we will sometimes use the notation NS_f, $NS(L,V,\gamma)$ or $NS_f(L,V,\gamma)$ to specify the dependence respectively on the initial condition, on the particular operator or on both the intial condition and the particular operator. Similarly, we will sometimes use the notation $BL(L,V)$. (In the linear case, the initial condition is of course irrelevant with regard to the question of uniqueness.)

In Section 2 we prove a basic result asserting the existence of a minimal and a maximal solution to the nonnegative semilinear equation NS_f. For some related results in the case $L = \Delta$, see [10] and [1]. This result, of interest in its own right, is also useful for the study of uniqueness—indeed, uniqueness occurs if and only if the minimal and maximal solutions coincide.

In section three, we begin the study of uniqueness for the semilinear equation. One of the two main results in that section is a sufficiency condition for uniqueness which is given in terms of pointwise bounds on the coefficients of the semilinear operator. The other main result in that section is a sufficiency condition for nonuniqueness which states that if $\inf_{x\in R^n} \frac{V(x)}{\gamma(x)} > 0$ and if nonuniqueness holds for the linear problem $BL(L,0)$, then nonuniqueness also holds for $NS_0(L,V,\gamma)$. In order to implement this result, we also present a result on uniqueness for the linear problem.

In section 4, we develop a connection between uniqueness for the semilinear parabolic problem and uniqueness for the corresponding steady state elliptic equation, which turns out to be very useful in applications. In sections 5 and 6, we apply the results of sections 3 and 4 to two specific classes of problems. We also show how our results can be used to give an alternative proof to a classical result of Ni [11], Kenig and Ni [7] and Lin [9] on uniqueness/nonuniqueness of positive solutions to the semilinear elliptic equation $\Delta w - \gamma w^p = 0$ in R^n, for $n \geq 3$, and how they lead to a new result for this equation when $n = 1, 2$.

Since the proof of Theorem 1 in section 2 is long and technical, one may prefer, at least on the first reading, to read the statement of Theorem 1 and then preceed directly to sections 3-6.

2. Existence of a Maximal and a Minimal Solution. In this section we prove the following theorem on the existence of minimal and maximal solutions.

Theorem 1. *Let $f \in C(R^n)$. There exist solutions $u_{f;min}$ and $u_{f;max}$ of NS_f with* *the property that any solution u to NS_f satisfies*

$$u_{f;min} \le u \le u_{f;max}.$$

Before giving the proof, we present a standard semilinear parabolic maximum principle and then apply it to obtain an a priori estimate on the size of any solution to NS.

Proposition 1. *Let $D \subset R^n$ be a bounded domain and let $0 \le u_1, u_2 \in C^{2,1}(D \times (0,\infty)) \cap C(\bar{D} \times [0,\infty))$ satisfy*

$$Lu_1 + Vu_1 - \gamma u_1^p - \frac{\partial u_1}{\partial t} \le Lu_2 + Vu_2 - \gamma u_2^p - \frac{\partial u_2}{\partial t}, \text{ for } (x,t) \in D \times (0,\infty),$$

$$u_1(x,t) \ge u_2(x,t), \text{ for } (x,t) \in \partial D \times (0,\infty)$$

and

$$u_1(x,0) \ge u_2(x,0), \quad \text{for } x \in D.$$

Then $u_1 \ge u_2$ in $D \times (0,\infty)$.

Proof. . Let $W = u_1 - u_2$ and define $H(x) = \frac{u_1^p(x) - u_2^p(x)}{W(x)}$, if $W(x) \ne 0$, and $H(x) = 0$ otherwise. We have $LW + (V - H)W - \frac{\partial W}{\partial t} \le 0$ in $D \times (0,\infty)$, $W(x,0) \ge 0$ in D, and $W(x,t) \ge 0$ on $\partial D \times (0,\infty)$. Thus, by the standard linear maximum principle, $u_1 \ge u_2$. $\square$

In the sequel we will frequently use the notation

$$B_R = \{x \in R^n : |x| < R\}.$$

Proposition 2. *Let $u \in C^{2,1}(B_R \times (0,\infty)) \cap C(\bar{B}_R \times [0,\infty))$ satisfy*

$$u_t = Lu + Vu - \gamma u^p \text{ in } B_R \times (0,\infty);$$

$$u(x,0) = f(x), \; x \in \bar{B}_R;$$

$$u \ge 0,$$

where $f \in C(\bar{B}_R)$. *Let* $V_R = \sup_{x\in B_R} V(x)$, *if* $\sup_{x\in B_R} V(x) > 0$, *and let* $V_R > 0$ *be arbitrary otherwise. Let* $\gamma_R = \inf_{x\in B_R} \gamma(x)$. *Then there exists a constant* K_R *such that*

$$
\begin{aligned}
&u(x,t) \le (\frac{V_R}{\gamma_R})^{\frac{1}{p-1}}(1-\exp(-(p-1)V_R(t+\epsilon)))^{-\frac{1}{p-1}} \\
(2.1)\qquad &+((R+\epsilon)^2-|x|^2)^{-\frac{2}{p-1}}\exp(K_R(t+1)),\ \textit{for}\ (x,t)\in \bar{B}_R\times[0,\infty), \\
&\textit{for sufficiently small}\ \epsilon>0.
\end{aligned}
$$

Proof. For $\epsilon > 0$, let $w_{1,\epsilon}(t) = (\frac{V_R}{\gamma_R})^{\frac{1}{p-1}}(1-\exp(-(p-1)V_R(t+\epsilon)))^{-\frac{1}{p-1}}$ and $w_{2,\epsilon}(x,t) = ((R+\epsilon)^2-|x|^2)^{-\frac{2}{p-1}}\exp(K_R(t+1))$. We will show below that $Lw_{i,\epsilon} + Vw_{i,\epsilon} - \gamma w_{i,\epsilon}^p - \frac{\partial w_{i,\epsilon}}{\partial t} \le 0,\ i=1,2$. Using the fact that $(w_{1,\epsilon}+w_{2,\epsilon})^p \ge w_{1,\epsilon}^p + w_{2,\epsilon}^p$, it will then follow that the function $W_\epsilon \equiv w_{1,\epsilon}+w_{2+\epsilon}$ satisfies $LW_\epsilon + VW_\epsilon - \gamma W_\epsilon^p - \frac{\partial W_\epsilon}{\partial t} \le 0$. Since $\lim_{\epsilon\to 0} w_{1,\epsilon}(0) = \infty$ and $\lim_{\epsilon\to 0} w_{2,\epsilon}(x,t) = \infty$ for $|x| = R$, we conclude from Proposition 1 that $u(x,t) \le W_\epsilon(x,t)$ for $\epsilon > 0$ sufficiently small.

Returning to the inequalities above, an easy calculation shows that $W(t) \equiv c(1-\exp(-k(t+\epsilon)))^{-\frac{1}{p-1}}$ satisfies $VW - \gamma W^p - \frac{\partial W}{\partial t} \le 0$ if one choses $c = (\frac{V_R}{\gamma_R})^{\frac{1}{p-1}}$ and $k = (p-1)V_R$. This proves that $Lw_{1,\epsilon} + Vw_{1,\epsilon} - \gamma w_{1,\epsilon}^p - \frac{\partial w_{1,\epsilon}}{\partial t} \le 0$.

Letting $W(x,t) = ((R+\epsilon)^2-|x|^2)^{-\frac{2}{p-1}}\exp(K(t+1))$, for $|x|<R$, we have

$$
\begin{aligned}
&((R+\epsilon)^2-|x|^2)^{\frac{2p}{p-1}}\exp(K(t+1))(LW+VW-\gamma W^p-\frac{\partial W}{\partial t}) = \\
&\frac{4(p+1)}{(p-1)^2}\sum_{i,j=1}^n a_{i,j}(x)x_ix_j - \gamma(x)\exp(K(p-1)(t+1)) \\
&+\frac{2}{p-1}((R+\epsilon)^2-|x|^2)\sum_{i=1}^n(a_{i,i}(x)+2b_i(x)x_i) + ((R+\epsilon)^2-|x|^2)^2(V(x)-K).
\end{aligned}
$$

From this it is clear that if $K = K_R$ is chosen sufficiently large, then the right hand side above will be nonpositive. This proves that $Lw_{2,\epsilon} + Vw_{2,\epsilon} - \gamma w_{2,\epsilon}^p - \frac{\partial w_{2,\epsilon}}{\partial t} \le 0$.
□

Proof of Theorem 1. *Construction of the minimal positive solution to* NS_f. Using [8, Theorem 12.16], there exists a nonnegative solution $u_m \in C^{2,1}(B_m \times$

$(0,\infty)) \cap C(\bar{B}_m \times [0,\infty))$ to the equation

$$u_t = Lu + Vu - \gamma u^p,\ (x,t) \in B_m \times (0,\infty);$$

$$u(x,0) = f_m(x),\ x \in B_m; \tag{2.2}$$

$$u(x,t) = 0,\ (x,t) \in \partial B_m \times (0,\infty),$$

where $f_m \in C^2(B_m)$ is nonnegative and compactly supported in B_m. (Actually, to apply the existence result in [8], one must make a truncation as follows. Letting $G(x,z) = V(x)z - \gamma(x)|z|^{1+p}$, and letting $G_k(x,z)$ be an appropriately truncated version of G which agrees with G on $\{|z| \le k\}$, one applies the existence result in [8] to obtain a solution to (2.2) with $V(x)u(x,t) - \gamma(x)u^{1+p}(x,t)$ replaced by $G_k(x,u(x,t))$. Then using the maximum principle in Proposition 1 and the a priori estimate in Proposition 2, it follows that the solution is in fact nonnegative and bounded, in which case the term $G_k(x,u(x,t))$ agrees with $V(x)u(x,t) - \gamma(x)u^{1+p}(x,t)$ if k is sufficiently large.) By the maximum principle in Proposition 1, the solution to (2.2) is unique.

We now use an interior parabolic Schauder estimate, an interior L^p estimate and the Sobelev embedding theorem to show that there exists a unique nonnegative solution to (2.2) under the assumption that $0 \le f_m \in C_b(B_m)$. This same technique will be used numerous times in the sequel and will be referred to as the *standard compactness argument.*

Let $\{f_{m,k}\} \subset C^2(B_m)$ be a uniformly bounded sequence of compactly supported, nonnegative functions which converge pointwise to f_m in B_m and let $u_{m,k}$ denote the corresponding solution to (2.2). For $R > 0$ and $0 < \epsilon < T < \infty$, let $\Omega_{R,T,\epsilon} = \{(x,t) : x \in B_R,\ t \in (\epsilon,T)\}$. Since $Lu_{m,k} + Vu_{m,k} - \frac{\partial u_{m,k}}{\partial t} = \gamma u^p_{m,k}$, it follows from an interior parabolic Schauder estimate [8, Theorem 4.9] and the assumption on L, V and γ that there exists a $C_\epsilon > 0$ such that

$$||u_{m,k}||_{2+\alpha,1+\frac{\alpha}{2};\Omega_{m-\epsilon,T,\epsilon}} \le C_\epsilon ||u_{m,k}||_{\alpha,\frac{\alpha}{2};\Omega_{m-\frac{\epsilon}{2},T+\epsilon,\frac{\epsilon}{2}}}. \tag{2.3}$$

($||\cdot||_{2+\alpha,1+\alpha;A}$ denotes the space of $C^{2,1}$- functions on $A \subset R^n \times (0,\infty)$ whose second order mixed partial derivatives in x are uniformly α-Hölder and whose first order derivative in t is uniformly $\frac{\alpha}{2}$-Hölder, and $||\cdot||_{\alpha,\frac{\alpha}{2};A}$ denotes the space of continuous functions on A which are uniformly α-Hölder in x and uniformly $\frac{\alpha}{2}$-Hölder in t.)

By Proposition 2, the solutions $u_{m,k}$ are uniformly bounded on $B_m \times (0,\infty)$; thus by an interior L^p estimate [8, Theorem 7.13], it follows that $\{\frac{\partial^2 u_{m,k}}{\partial x_i \partial x_j}\}_{k=1}^{\infty}$ and $\{\frac{\partial u_{m,k}}{\partial t}\}_{k=1}^{\infty}$ are uniformly bounded in $L^p(B_{\Omega_{m-\frac{\epsilon}{4},T+2\epsilon,\frac{\epsilon}{4}}})$ for each $p > 1$. It then follows from the Sobolev embedding theorems [5] that $\{||u_{m,k}||_{\alpha,\frac{\alpha}{2};\Omega_{m-\frac{\epsilon}{2},T+\epsilon,\frac{\epsilon}{2}}}\}_{k=1}^{\infty}$ is uniformly bounded. Using this in conjunction with (2.3) shows that the sequence $\{u_{m,k}\}_{k=1}^{\infty}$ is precompact in the $||\cdot||_{2,1;\Omega_{m-\epsilon,T,\epsilon}}$−norm. Thus there exists a subsequence which converges to a function u_m which satisfies the parabolic equation in (2.2).

It remains to show that u_m satisfies the initial condition and the boundary condition. This is done via appropriate barrier functions. For $M > 0$, let $W_M^{\pm} \in C^{2,1}(B_m \times (0,\infty)) \cap C(\bar{B}_m \times (0,\infty)) \cap C(B_m \times [0,\infty))$ denote the solutions to the linear inhomogeneous boundary-initial value problems

$$
\begin{aligned}
&w_t = Lw \pm M,\ (x,t) \in B_m \times (0,\infty);\\
&w(x,0) = f(x),\ x \in B_m;\\
&w(x,t) = \pm M,\ x \in \partial B_m, t > 0.
\end{aligned}
$$

By the a priori bound (2.1), it follows that for sufficiently large M, $|Vu_{m,k} - \gamma u_{m,k}^p| \le M$ and $0 \le u_{m,k} \le M$ on $B_m \times (0,1)$, for $k = 1,2,...$ Thus, for such an M, it follows by the linear maximum principle that $W_M^- \le u_{m,k} \le W_M^+$ on $B_m \times (0,1)$. Thus, $\lim_{t\to 0} u_m(x,t) = \lim_{t\to 0}\lim_{k\to\infty} u_{m,k}(x,t) = f(x)$, for $x \in B_m$.

To show that the zero Dirichlet boundary value is satisfied, one makes a similar argument using the barrier functions Z_M which satisfies

$$z_t = Lz + Mz,\ (x,t) \in B_m \times (0,\infty);$$

$$z(x,0) = M,\ x \in B_m;$$

$$z(x,t) = 0,\ x \in \partial B_m, t > 0.$$

For each $T > 0$, choose M_T such that $0 \le u_{m,k} \le M_T$ and $|Vu_{m,k} - \gamma u_{m,k}^p| \le M_T$ on $B_m \times [0,T]$, for $k = 1, 2, \ldots$ Then $0 \le u_{m,k} \le Z_{M_T}$ on $B_m \times [0,T)$; thus $\lim_{x\to\partial B_m} u_m(x,t) = 0$, for $t > 0$.

We are now ready to construct the minimal solution to NS_f. Assume first that $0 \le f \in C(R^n)$ is compactly supported. Let m_0 be such that f is supported in B_{m_0}. For $m \ge m_0$, let u_m denote the solution constructed above in B_m with initial condition f. Arguing as above, the sequence $\{u_m\}_{m=k}^\infty$ is compact in the $C^{2,1}$-norm on $\Omega_{B_k,T,\epsilon}$, for any integer $k \ge m_0$ and $0 < \epsilon < T < \infty$. By the maximum principle in Proposition 1, the sequence $\{u_m\}_{m=m_0}^\infty$ is nondecreasing. Thus $u_f \equiv \lim_{m\to\infty} u_m$ exists and is a classical solution to the semilinear equation in $R^n \times (0,\infty)$.

We now show that $\lim_{t\to 0} u_f(x,t) = f(x)$. Fix $x_0 \in R^n$ and let $B_1(x_0) \subset R^n$ denote the ball of radius 1 centered at x_0. For $M > 0$, let $W_M^\pm \in C^{2,1}(B_1(x_0) \times (0,\infty)) \cap C(\bar{B}_m \times (0,\infty)) \cap C(B_m \times [0,\infty))$ denote the solutions to the linear inhomogeneous boundary-initial value problems

$$w_t = Lw \pm M \text{ in } B_1(x_0) \times (0,\infty);$$

$$w(x,0) = f(x),\ x \in B_1(x_0);$$

$$w(x,t) = \pm M,\ x \in \partial B_1(x_0),\ t \in (0,\infty).$$

By the a priori bound (2.1), it follows that for sufficiently large M, $|Vu_f - \gamma u_f^p| \le M$ and $0 \le u \le M$ on $B_1(x_0) \times (0,1)$. Thus, for such an M, it follows by the linear maximum principle that $W_M^- \le u_f \le W_M^+$ on $B_1(x_0) \times (0,1)$, which proves that $\lim_{t\to 0} u_f(x_0,t) = f(x_0)$.

To show the minimality of u_f, let U be any solution of NS_f. In light of the zero Dirichet boundary condition on u_m, it follows from the maximum principle in Proposition 1 that $u_m \leq U$. Letting $m \to \infty$ shows that $u_f \leq U$. This completes the proof of the existence of a minimal solution to NS_f when the initial condition f is compactly supported.

Now consider the case that the initial condition satisfies $0 \leq f \in C(R^d)$. Take an increasing sequence of continuous, compactly supported functions $\{f_m\}$ satisfying $f = \lim_{m\to\infty} f_m$ and let u_{f_m} be the minimal solution to NS_{f_m}. By the maximum principle, it follows that $\{u_{f_m}\}_{m=1}^{\infty}$ is monotone. By the a priori estimate in (2.1) and the parabolic estimates and Sobolev embedding theorem used above, it follows that $u_f \equiv \lim_{m\to\infty} u_{f_m}$ solves the semilinear equation. The same argument used in the case that f is compactly supported shows that $\lim_{t\to 0} u_f(x,t) = f(x)$. The proof of minimality follows easily from the minimality in the compactly supported case. This completes the proof of the existence of a minimal solution to NS_f.

Construction of the maximal positive solution to NS_f. For $m > 0$ and a positive integer k, let $\psi_{m,k} \in C^\infty(R^n)$ satisfy

$$\begin{aligned}
&\psi_{m,k}(x) = 0,\ |x| \leq m \text{ and } |x| > 2m+1\\
&\psi_{m,k}(x) = k, m + \frac{1}{k} \leq |x| \leq 2m \qquad (2.4)\\
&0 \leq \psi_{m,k} \leq k.
\end{aligned}$$

Using [8, Theorem 12.16] again, there exists a nonnegative solution $U_{m,k} \in C^{2,1}(B_{2m} \times (0,\infty)) \cap C(\bar{B}_{2m} \times [0,\infty))$ to the equation

$$\begin{aligned}
&u_t = Lu + Vu - \gamma u^p + \psi_{m,k},\ (x,t) \in B_{2m} \times (0,\infty);\\
&u(x,0) = f_m(x),\ x \in B_{2m}; \qquad (2.5)\\
&u(x,t) = 0,\ (x,t) \in \partial B_{2m} \times (0,\infty),
\end{aligned}$$

where $f_m \in C^2(B_{2m})$ is nonnegative and compactly supported in B_{2m}. Using the standard compactness argument and barrier functions as above in the proof of the existence of a minimal solution, this then extends to the case that the initial data f_m is continuous, nonnegative, and compactly supported in B_{2m}.

Since $U_{m,k}$ satisfies the homogeneous semilinear equation in B_m (because $\psi_{m,k}$ vanishes there), the functions $\{U_{m,k}\}_{k=1}^{\infty}$ all satisfy the a priori estimate (2.1) with $\epsilon = 0$ (and with R replaced by m). By the maximum principle in Proposition 1, $U_{m,k}$ is increasing in k. From this and the standard compactness argument it follows that $U_m \equiv \lim_{k\to\infty} U_{m,k}$ exists, $U_m \in C^{2,1}(B_m \times (0,\infty))$, and U_m satisfies the semilinear equation in B_m. The barrier function argument given above in the case of the minimal solution shows that $\lim_{t\to 0} U_m(x,t) = f_m(x)$, for $x \in B_m$. We will prove below that

$$\lim_{x\to\partial B_m} U_m(x,t) = \infty,\ t \in (0,\infty). \tag{2.6}$$

Using this, the proof of the existence of a maximal solution goes as follows. For $f \in C(R^n)$, let f_m and U_m be as above with f_m chosen so that $f_m = f$ on B_m and so that $\{f_m\}$ is nondecreasing. By the same reasoning as has already been used several times above, $U_f \equiv \lim_{m\to\infty} U_m$ exists and solves the semilinear equation. Again by the proof used in the case of the minimal solution, we have $\lim_{t\to 0} U_f(x,t) = f(x)$; thus, U_f solves NS_f. To see that U_f is maximal, let u be any solution to NS_f. Then by (2.6) and the maximal principle in Proposition 1, we have $u \le U_m$ on B_m; thus, $u \le U_f$.

We now turn to the proof of (2.6). For $\epsilon > 0$, we will constuct a function w_ϵ which satisfies $Lw_\epsilon + Vw_\epsilon - \gamma w_\epsilon^p - \frac{\partial w_\epsilon}{\partial t} \ge 0$ in $B_{m+\epsilon}$ and $w_\epsilon(m+\epsilon,t) = \infty$. From the maximum principle, we then obtain $U_m \ge w_\epsilon$ in $B_{m+\epsilon}$. From the construction, it will follow that $w \equiv \lim_{\epsilon\to 0} w_\epsilon$ satisfies $\lim_{x\to\partial B_m} w(x) = \infty$. To implement this, we need a number of preliminary results.

We first show that

$$\lim_{k\to\infty} U_{m,k}(x,t) = \infty, \text{ for } m < |x| < 2m \text{ and } t > 0. \tag{2.7}$$

Fix $N > 0$ and define $W(x,t) = Nt\left(l^2 - (m + \frac{1}{k} + l - |x|)^2\right)$, where $l = \frac{1}{2}(m - \frac{1}{k})$. Note that $W > 0$ in the annulus $A_{m+\frac{1}{k},2m} \equiv \{m + \frac{1}{k} < |x| < 2m\}$ and vanishes on $\partial A_{m+\frac{1}{k},2m}$. Fix $T > 0$. Clearly $LW + VW - \gamma W^p - W_t$ is bounded in $A_{m+\frac{1}{k},2m} \times [0,T]$. Thus for k sufficiently large, we have $LW + VW - \gamma W^p - W_t + \psi_{m,k} \geq 0$ in $A_{m+\frac{1}{k},2m} \times [0,T]$. Since $W(x,0) = 0$ and W vanishes on $\partial A_{m+\frac{1}{k},2m}$, it follows by the maximum principle in Proposition 1 that $U_{m,k} \geq W$ in $A_{m+\frac{1}{k},2m} \times [0,T]$, for k sufficiently large. Letting $k \to \infty$, we obtain $U_m(x,t) \geq Nt\left((\frac{m}{2})^2 - (\frac{3m}{2} - |x|)^2\right)$, for $m \leq |x| \leq 2m$ and $0 \leq t \leq T$. Since N and T are arbitrary, (2.7) follows.

We will need the function g described below. It is well-known from the theory of travelling waves [4] that for $\rho > 0$ sufficiently small, there exists a strictly increasing function $g \in C^2([0,\infty))$ satisfying

$$\begin{aligned} & g'' - \rho g' + g - g^p = 0 \text{ on } [0,\infty); \\ & g(0) = 0, \ \lim_{s\to\infty} g(s) = 1; \\ & g' \geq 0, \ g'' \leq 0. \end{aligned} \tag{2.8}$$

For $m > 0$ define

$$\phi_m(x) = \lambda(m^{2l} - |x|^{2l})^{-\frac{2}{p-1}}, \ x \in B_m,$$

where $\lambda, l > 0$. We have

(2.9)

$$\frac{1}{\lambda}(m^{2l} - |x|^{2l})^{\frac{2p}{p-1}}(L\phi_m + V\phi_m - \gamma\phi_m^p) = \frac{8l^2(p+1)}{p-1}|x|^{4l-4}\sum_{i,j=1}^n a_{i,j}(x)x_ix_j$$

$$+ \frac{8l(l-1)}{p-1}(|x|^{2l-4}(m^{2l} - |x|^{2l}))\sum_{i,j=1}^n a_{i,j}(x)x_ix_j +$$

$$\frac{4l}{p-1}|x|^{2l-2}(m^{2l} - |x|^{2l})\sum_{i=1}^n(a_{i,i}(x) + x_ib_i(x)) + V(x)(m^{2l} - |x|^{2l})^2 - \lambda^{p-1}\gamma(x).$$

In light of the strict ellipticity, it is easy to see that if $l > 0$ is chosen sufficiently large, then the sum of the second and third terms on the right hand side of (2.9) is nonnegative on B_m, and that if $\lambda > 0$ is chosen sufficiently small, then the sum of the first term and the last two terms on the right hand side of (2.9) is nonnegative. Fixing such an l and a λ, we conclude that

$$L\phi_m + V\phi_m - \gamma\phi_m^p \geq 0 \text{ in } B_m. \tag{2.10}$$

We can now define the function w_ϵ as follows:

$$w_\epsilon(x,t) \equiv \begin{cases} \phi_{m+\epsilon}(x)g(c(t+|x|^2-(m+\epsilon)^2)), \\ \ \text{if } (x,t) \in B_{m+\epsilon} \cap \{t+|x|^2-(m+\epsilon)^2 > 0\}; \\ 0, \ \text{if } (x,t) \in B_{m+\epsilon} \cap \{t+|x|^2-(m+\epsilon)^2 \leq 0\}. \end{cases} \tag{2.11}$$

Using the ellipticity and the fact that g' and $\nabla\phi_{m+\epsilon} \cdot \frac{x}{|x|}$ are nonnegative, it follows that

$$\begin{aligned} &\sum_{i,j=1}^n a_{i,j} \frac{\partial(g(c(t+|x|^2-(m+\epsilon)^2)))}{\partial x_i} \frac{\partial(\phi_{m+\epsilon}(x))}{\partial x_j} \geq 0, \\ &\text{if } (x,t) \in B_{m+\epsilon} \cap \{t+|x|^2-(m+\epsilon)^2 > 0\}. \end{aligned} \tag{2.12}$$

In the sequel, when g appears without an argument, it is to be understood that the argument is $c(t+|x|^2-(m+\epsilon)^2)$. From (2.10)-(2.12) we have

$$\begin{aligned} &Lw_\epsilon + Vw_\epsilon - \gamma w_\epsilon^p - \frac{\partial w_\epsilon}{\partial t} \geq \phi_{m+\epsilon} L(g(c(t+|x|^2-(m+\epsilon)^2))) \\ &+ gL\phi_{m+\epsilon} + Vg\phi_{m+\epsilon} - \gamma g^p \phi_{m+\epsilon}^p - \phi_{m+\epsilon}\frac{\partial(g(c(t+|x|^2-(m+\epsilon)^2)))}{\partial t} \\ &\geq \phi_{m+\epsilon} L(g(c(t+|x|^2-(m+\epsilon)^2))) - \phi_{m+\epsilon}\frac{\partial(g(c(t+|x|^2-(m+\epsilon)^2)))}{\partial t} \\ &+ \gamma\phi_{m+\epsilon}^p(g-g^p), \ \text{if } (x,t) \in B_{m+\epsilon} \cap \{t+|x|-(m+\epsilon)^2 > 0\}. \end{aligned} \tag{2.13}$$

Using the fact that $g' \geq 0$ and $g'' \leq 0$, it's easy to check that for any $\delta > 0$, one can choose $c = c_\delta > 0$ sufficiently small so that

$$\begin{aligned} &L(g(c(t+|x|^2-(m+\epsilon)^2))) - \frac{\partial(g(c(t+|x|^2-(m+\epsilon)^2)))}{\partial t} \geq \delta(g'' - \rho g'), \\ &\text{if } B_{m+\epsilon} \cap \{t+|x|^2-(m+\epsilon)^2 > 0\}. \end{aligned} \tag{2.14}$$

Choosing $\delta = \inf_{x \in B_{m+\epsilon}} \gamma(x)\phi^{p-1}_{m+\epsilon}(x)$, we conclude from (2.8), (2.13), and (2.14) that

$$
\begin{aligned}
(2.15) \qquad & Lw_\epsilon + Vw_\epsilon - \gamma w_\epsilon^p - \frac{\partial w}{\partial t} \geq (\gamma\phi^p_{m+\epsilon} - \delta\phi_{m+\epsilon})(g - g^p) \geq 0, \\
& \text{if } (x,t) \in B_{m+\epsilon} \cap \{t + |x|^2 - (m+\epsilon)^2 > 0\}.
\end{aligned}
$$

Let $D_\epsilon = B_{m+\frac{\epsilon}{2}} \cap \{t + |x|^2 - (m+\epsilon)^2 > 0\}$. Note that w_ϵ vanishes on the part of ∂D_ϵ where $t + x^2 - (m+\epsilon)^2 = 0$. Also, since w_ϵ is bounded on D_ϵ, it follows from (2.7) that $w_\epsilon \leq U_{m,k}$ on $\partial B_{m+\frac{\epsilon}{2}}$, for k sufficiently large. Thus, since $U_{m,k} \geq 0$ and satisfies the semilinear equation in D_ϵ, it follows from the maximum principle of Proposition 1 that for k sufficiently large, $w_\epsilon \leq U_{k,m}$ in D_ϵ. Letting $k \to \infty$ and then letting $\epsilon \to 0$ gives

$$
\begin{aligned}
(2.16) \qquad & U_m(x,t) \geq \lambda(m^{2l} - |x|^{2l})^{-\frac{2}{p-1}} g(c(t + |x|^2 - m^2)), \\
& \text{if } (x,t) \in B_m \cap \{t + |x|^2 - m^2 \geq 0\}.
\end{aligned}
$$

Now (2.6) follows from (2.16). □

3. Uniqueness/Nonuniqueness for the Semilinear Parabolic Equation. Note that by Theorem 1, uniqueness holds for NS_f if and only if $u_{f;min} \equiv u_{f;max}$.

We begin with a couple of useful comparison results.

Proposition 3. *Let $0 \leq f_1 \leq f_2$. If uniqueness holds for NS_{f_1}, then it also holds for NS_{f_2}.*

Remark. In particular, it follows from the proposition that if uniqueness holds for $f \equiv 0$, then it holds for all $0 \leq f \in C(R^n)$. In fact, we suspect that uniqueness either holds for all f or no f.

Proposition 4. *Assume that*

$$V_1 \leq V_2$$

and

$$0 < \gamma_2 \leq \gamma_1.$$

If uniqueness holds for $NS_0(L, V_2, \gamma_2)$, then uniqueness also holds for $NS_f(L, V_1, \gamma_1)$, for all f.

Proof of Proposition 3. To prove the proposition, it suffices to show that

$$u_{f_1;max} - u_{f_1;min} \geq u_{f_2;max} - u_{f_2;min}, \text{ if } 0 \leq f_1 \leq f_2. \tag{3.1}$$

The construction of the minimal and maximal solutions revealed that for $f \in C(R^n)$, $u_{f;max} = \lim_{m\to\infty} u_{f_m;max}$ and $u_{f;min} = \lim_{m\to\infty} u_{f_m;min}$, where $\{f_m\}$ is an increasing sequence of compactly supported functions which converges pointwise to f. Thus, it suffices to prove (3.1) in the case that f_1, f_2 are compactly supported. That construction also revealed that $u_{f_i;max} = \lim_{m\to\infty} \lim_{k\to\infty} U^{(i)}_{m,k}$, where for m sufficiently large so that $supp(f_i) \subset B_m$, $U^{(i)}_{m,k}$ solves (2.5) with f_m replaced by f_i. Since f_i is compactly supported, the constuction also showed that $u_{f_i;min} = \lim_{m\to\infty} u^{(i)}_m$, where for m sufficiently large so that $supp(f_i) \subset B_m$, $u^{(i)}_m \in C^{2,1}(B_m \times (0,\infty)) \cap C(\bar{B}_m \times [0,\infty))$ and satisfies

$$\begin{aligned} &u_t = Lu + Vu - \gamma u^p \text{ in } B_m \times (0,\infty); \\ &u(x,0) = f_i(x), \text{ for } x \in \bar{B}_m; \\ &u(x,t) = 0, \text{ for } x \in \partial B_m \text{ and } t > 0. \end{aligned} \tag{3.2}$$

Thus (3.1) will follow if we show that

$$U^{(1)}_{m,k} - u^{(1)}_{2m} \geq U^{(2)}_{m,k} - u^{(2)}_{2m}, \text{ for } (x,t) \in B_{2m} \times (0,\infty) \text{ and } m,k = 1,2,... \tag{3.3}$$

Fix m and k and let $W_i = U^{(i)}_{m,k} - u^{(i)}_{2m}$. By the strong maximum principle, $W_i > 0$ in $B_{2m} \times (0,\infty)$. We have

$$LW_i + (V - \gamma G_i)W_i - \frac{\partial W_i}{\partial t} = -\psi_{m,k}, \; (x,t) \in B_{2m} \times (0,\infty), \tag{3.4}$$

where $G_i(x,t) = \frac{(U^{(i)}_{m,k}(x,t))^p - (u^{(i)}_{2m}(x,t))^p}{U^{(i)}_{m,k}(x,t) - u^{(i)}_{2m}(x,t)}$ Since $f_1 \leq f_2$, it follows from the maximum principle in Proposition 1 that $U^{(2)}_{m,k} \geq U^{(1)}_{m,k}$ and $u^{(2)}_{2m} \geq u^{(1)}_{2m}$. One can easily check that the function $H(x,y) \equiv \frac{x^p - y^p}{x-y}$, for $0 \leq y < x < \infty$ is increasing in each of its variables. Thus, we have

$$G_2 \geq G_1 \geq 0. \tag{3.5.}$$

Letting $Z = W_1 - W_2$, and using the fact that $W_2 \geq 0$, we obtain from (3.4) and (3.5) that

$$LZ + (V - G_1)Z - \frac{\partial Z}{\partial t} \leq 0. \tag{3.6}$$

Noting that $Z(x,0) = 0$ for $x \in \bar{B}_{2m}$ and that $Z(x,t) = 0$ for $x \in \partial B_{2m}$ and $t > 0$, it follows from (3.6) and the standard linear parabolic maximum principle that $Z \geq 0$. $\square$.

Proof of Proposition 4. Let $u^{(i)}_{0;max}$ denote the maximal solution for $NS_0(L, V_i, \gamma_i)$. In light of Proposition 3, to prove the theorem, it suffices to show that

$$u^{(1)}_{0;max} \leq u^{(2)}_{0;max}. \tag{3.7}$$

Similar to the proof of Proposition 3, we have $u^{(i)}_{0;max} = \lim_{m\to\infty} \lim_{k\to\infty} U^{(i)}_{m,k}$, where $U^{(i)}_{m,k}$ solves (2.5) with V, γ and f_m replaced respectively by V_i, γ_i and 0. Thus, to show (2.7) it suffices to prove that

$$U^{(2)}_{m,k} \geq U^{(1)}_{m,k}, \text{ for } (x,t) \in B_{2m} \times (0,\infty) \text{ and } m,k = 1,2,... \tag{3.8}$$

Since $LU^{(1)}_{m,k} + V_1 U^{(1)}_{m,k} - \gamma_1 (U^{(1)}_{m,k})^p - \frac{\partial U^{(1)}_{m,k}}{\partial t} = -\psi_{m,k}$ while $LU^{(2)}_{m,k} + V_1 U^{(2)}_{m,k} - \gamma_1 (U^{(2)}_{m,k})^p - \frac{\partial U^{(2)}_{m,k}}{\partial t} = -\psi_{m,k} + (V_1 - V_2) U^{(2)}_{m,k} + (\gamma_2 - \gamma_1)(U^{(2)}_{m,k})^p \leq -\psi_{m,k}$, (3.8) follows from the maximum principle in Proposition 1. $\square$

We now come to our first main result, which guarantees uniqueness for NS if the coefficients satisfy appropriate pointwise estimates.

Theorem 2. *Assume that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\nu_i\nu_j \le C|\nu|^2(1+|x|)^2; \tag{3.9a}$$

$$|b(x)| \le C(1+|x|); \tag{3.9b}$$

$$V(x) \le C, \tag{3.9c}$$

for some $C > 0$. Assume in addition that

$$\inf_{x\in R^n} \gamma(x) > 0.$$

Then uniqueness holds for NS_f, for all f.

Proof. By Proposition 3, it suffices to consider the case $f = 0$. We need to show that $u_{0;max} = 0$. We will build an appropriate family of test functions which will be compared to $u_{0;max}$. Fix $\epsilon \in (0,1)$. For $R > 1$, choose $\phi_R(x) \in C^2(B_R)$ such that

$$\phi_R(x) = (1+|x|)^{\frac{2}{p-1}}(R-|x|)^{-\frac{2}{p-1}}, \text{ for } |x| > \epsilon, \tag{3.10}$$

and such that

$$\sum_{i=1}^{n} |\frac{\partial \phi_R}{\partial x_i}| + \sum_{i,j=1}^{n} |\frac{\partial^2 \phi_R}{\partial x_i \partial x_j}| \le C_\epsilon \phi_R(x), \text{ for } |x| \le \epsilon, \tag{3.11}$$

where $C_\epsilon > 0$ is independent of R. This is possible because from the definition of ϕ_R in (3.10), it follows that the inequality in (3.11) holds for $|x| = \epsilon$. Define

$$u_R(x,t) = \phi_R(x)\exp(K(t+1)), \text{ for } x \in B_R \text{ and } t \ge 0.$$

We have

$$\exp(-K(t+1))(Lu_R + Vu_R - \gamma u_R^p - \frac{\partial u_R}{\partial t})(x,t)$$

(3.12) $$= L\phi_R(x) + V\phi_R(x) - \gamma(x)\phi_R^p(x)\exp(K(p-1)(t+1)) - K\phi_R(x),$$

for $x \in R^n$ and $t > 0$.

We will show below that

(3.13) $$\frac{L\phi_R}{\phi_R^p} \text{ is bounded above uniformly in } R.$$

From (3.12) and (3.13), we conclude that there exists a K independent of R such that

(3.14) $$Lu_R + Vu_R - \gamma u_R^p - \frac{\partial u_R}{\partial t} \leq 0 \text{ for } (x,t) \in B_R \times (0,\infty).$$

Since $u_{0;max}(x,0) = 0$, $u_R \geq 0$, and $\lim_{x \to \partial B_R} u_R(x,t) = \infty$, it follows from (3.14) and the maximum principle in Proposition 1 that

(3.15) $$u_{0;max}(x,t) \leq u_R(x,t),\ (x,t) \in B_R \times [0,\infty).$$

Letting $R \to \infty$, it follows from (3.10) and (3.15) that

$$u_{0;max}(x,t) = 0,\ (x,t) \in (R^n - B_\epsilon) \times [0,\infty).$$

Since $\epsilon > 0$ is arbitrary we conclude that $u \equiv 0$.

We now return to prove (3.13). Letting $r = |x|$ and resolving L into spherical coordinates, we have

$$L = A(x)\frac{\partial^2}{\partial r^2} + B(x)\frac{\partial}{\partial r} + \text{ terms involving differentiation not only in } r.$$

By assumption, there exists a $C > 0$ such that $0 < A(x) \leq C(1+|x|)^2$ and $|B(x)| \leq C(1+|x|)$, for $x \in R^n$. A simple, direct calculation now reveals that (3.13) holds. □

The second main result in this section relates nonuniqueness of the semilinear equation to nonuniqueness of the corresponding linear problem obtained by setting both γ and V equal to 0.

Theorem 3. *Assume that uniqueness does not hold for $BL(L,0)$ and that*

$$\inf_{x\in R^n} \frac{V(x)}{\gamma(x)} > 0.$$

Then uniqueness does not hold for $NS_0(L,V,\gamma)$.

Remark. For an example where the condition $\inf_{x\in R^n} \frac{V(x)}{\gamma(x)} > 0$ holds and there is uniqueness for BL but not for NS, one can turn to the applications in section five and take the class of equations in (5.2) with $V = C > 0$ and γ as in Theorem 7-(ii).

In order for Theorem 3 to be useful, we need to know when uniqueness holds for the bounded linear problem $BL(L,0)$. Before proving Theorem 3, we make a small digression to study the linear problem. We have the following result which actually considers more generally $BL(L,V)$.

Proposition 5. *i-a. If V is bounded from above and uniqueness holds for $BL(L,0)$, then uniqueness holds for $BL(L,V)$.*
i-b.If V is bounded from below and uniqueness holds for $BL(L,V)$, then uniqueness holds for $BL(L,0)$.
ii-a. If there exist $m_0,\lambda > 0$ and a positive function ϕ satisfying $L\phi \leq \lambda\phi$ in $R^n - B_{m_0}$ and $\lim_{|x|\to\infty} \phi(x) = \infty$, then uniqueness holds for $BL(L,0)$.
ii-b. If there exist $m_0,\lambda > 0$, an $x_0 \in R^n$ satisfying $|x_0| > m_0$, and a bounded, positive function ϕ satisfying $L\phi \geq \lambda\phi$ in $R^n - B_{m_0}$ and $\phi(x_0) \geq \sup_{|x|=m_0} \phi(x)$, then uniqueness does not hold for $BL(L,0)$.

Remark 1. Recall from Theorem 2 that if the pointwise bound (3.9-a,b,c) on the coefficients of the linear part of the semilinear equation is in effect along with the condition $\inf_{x\in R^n}\gamma(x) > 0$ on the nonlinear part, then uniqueness holds for the semilinear equation. It is interesting to note how (3.9-a,b,c) relates to uniqueness for the linear equation. Using the function $\phi(x) = |x|^2$ in part (ii-a) of Proposition 5 and then using part (i-a) shows that if (3.9-a,b,c) is in force, then uniqueness holds for $BL(L, V)$. As far as pointwise polynomial-type bounds are concerned, condition (3.9-a,b) is sharp for the uniqueness of $BL(L, 0)$. Indeed, applying part (ii-b) with the function $\phi(x) = 1-|x|^{-l}$, where $l > 0$ is sufficiently small, shows that uniqueness does not hold for $BL(L, 0)$ in the following two cases: (1) $L = (1 + |x|)^{2+\delta}\Delta$ with $\delta > 0$ and $n \geq 3$; (2) $L = \Delta + b\nabla$ and $n \geq 1$, where $b(x) \cdot \frac{x}{|x|} \geq c|x|^{1+\delta}$ for large $|x|$ and some $\delta, c > 0$.

In passing, we note that the question of uniqueness of *positive* solutions to the linear equation has a long history in the partial differential equations literature, going back to Widder. It is known that uniqueness of positive solutions holds if (3.9-b,c) is in force and if (3.9a) is replaced by a two-sided bound of the form $C_1|\nu|^2(1 + |x|)^q \leq \sum_{i,j=1}^n a_{ij}(x)\nu_i\nu_j \leq C_2|\nu|^2(1 + |x|)^q$, for some $q \in [0, 2]$. See, for example, [6] and references therein.

Remark 2. It's well-known in the probability literature that uniqueness holds for $BL(L, 0)$ if and only if the Markov diffusion process corresponding to the operator L is *nonexplosive*; that is, the process does not run out to infinity in finite time. In the case that $p \in (1, 2]$, the equation NS is also connected with a Markov process; namely, with a measure-valued diffusion process. The so-called *compact support property* for measure valued diffusions can be thought of as the parallel to nonexplosiveness for ordinary diffusions. We have shown elsewhere that uniqueness for NS_0 is equivalent to the compact support property holding [3]. (Actually, the case $p = 2$ is treated in [3] but it extends immediately to $p \in (1, 2]$.) Certain results in this paper appeared in the case $p = 2$ with probabilistic proofs in [3] or [2].

We now give the proof of Proposition 5 followed by the proof of Theorem 3.

Proof of Proposition 5. *i-a.* Let $u_{m,V}$ denote the solution to $u_t = (L+V)u$ in $B_m \times (0,\infty)$ with $u(x,0) = 0$ in B_m and $u(x,t) = 1$ on $\partial B_m \times (0,\infty)$. By the maximum principle, uniqueness holds for $BL(L,V)$ if and only if $\lim_{m\to\infty} u_{m,V} = 0$. We will show that if V is bounded from above and $\lim_{m\to\infty} u_{m,0} = 0$, then $\lim_{m\to\infty} u_{m,V} = 0$. Let $\lambda = \sup_{x\in R^n} V(x)$ and define $Z(x,t) = u_{m,0}(x,t)\exp(\lambda t)$. Then $LZ + VZ - \frac{\partial Z}{\partial t} \le 0$ in $B_m \times (0,\infty)$, $Z(x,0) = 0$ in B_m and $Z(x,t) \ge 1$ on $\partial B_m \times (0,\infty)$. Thus, by the maximum principle, $0 \le u_{m,V} \le u_{m,0}\exp(\lambda t)$ and consequently, $\lim_{m\to\infty} u_{m,V} = 0$.

i-b. The proof is very similar to the proof of (i-a).

ii-a. Denote by u_m the function that was called $u_{m,0}$ in the proof of part(i). We need to show that $\lim_{m\to\infty} u_m = 0$. Continue the function ϕ appearing in the statement of the proposition so that it is defined on all of R^n as a smooth, positive function. By increasing λ if necessary, we have $L\phi \le \lambda\phi$ in R^n. Let $U_m(x,t) = \frac{\phi(x)}{\inf_{|y|=m}\phi(y)}\exp(\lambda t)$. Then $LU_m - \frac{\partial U_m}{\partial t} \le 0$ in $B_m \times (0,\infty)$, $U_m(x,0) \ge 0$ in B_m, and $U_m(x,t) \ge 1$ on $\partial B_m \times (0,\infty)$. Thus, it follows from the maximum principle that $U_m \ge u_m \ge 0$ in $B_m \times (0,\infty)$. Using the assumption that $\lim_{|x|\to\infty}\phi(x) = \infty$, we obtain $\lim_{m\to\infty} U_m = 0$, and thus, $\lim_{m\to\infty} u_m = 0$.

ii-b. Assume to the contrary that uniqueness does hold for $BL(L,0)$. Let $Z(x,t) = \exp(-\lambda t)\phi(x)$ in $(R^n - B_{m_0}) \times [0,1]$. By assumption, we have $LZ + \frac{\partial Z}{\partial t} \ge 0.$ in $(R^n - \bar{B}_{m_0}) \times (0,1)$. For $m > m_0$, let U_m denote the solution to the equation

$$\begin{aligned} &u_t + Lu = 0 \text{ in } (B_m - \bar{B}_{m_0}) \times (-\infty,1);\\ &u(x,1) = \exp(-\lambda)\phi(x) \text{ on } B_m - \bar{B}_{m_0};\\ &u(x,t) = \phi(x) \text{ on } (\partial B_m \cup \partial B_{m_0}) \times (-\infty,1). \end{aligned} \tag{3.16}$$

By the maximum principle,

$$Z \le U_m \text{ in } (B_m - B_{m_0}) \times [0,1]. \tag{3.17}$$

Now Let V_m denote the solution to

$$
(3.18)\qquad
\begin{aligned}
&u_t + Lu = 0 \text{ in } (B_m - \bar{B}_{m_0}) \times (-\infty, 1);\\
&u(x,1) = \exp(-\lambda)\phi(x) \ \text{ on } B_m - \bar{B}_{m_0};\\
&u(x,t) = 0 \text{ on } \partial B_m \times (-\infty,1),\ u(x,t) = \phi(x) \text{ on } \partial B_{m_0} \times (-\infty, 1).
\end{aligned}
$$

We will now show that the uniqueness assumption for $BL(L,0)$ guarantees that

$$
(3.19)\qquad \lim_{m\to\infty} U_m = \lim_{m\to\infty} V_m.
$$

Let $W_m = U_m - V_m$. From (3.16) and (3.18) we have

$$
(3.20)\qquad
\begin{aligned}
&\frac{\partial W_m}{\partial t} + LW_m = 0 \text{ in } (B_m - \bar{B}_{m_0}) \times (-\infty, 1);\\
&W_m(x,1) = 0 \ \text{ on } B_m - \bar{B}_{m_0};\\
&W_m(x,t) = \phi(x) \text{ on } \partial B_m \times (-\infty,1),\ W_m(x,t) = 0 \text{ on } \partial B_{m_0} \times (-\infty, 1).
\end{aligned}
$$

Let $v_m(x,t) = cu_m(x, 1-t)$, where u_m is as in part(ii-a) and $c = \sup_{|y|=m_0}\phi(y)$. Then v_m satisfies $\frac{\partial v_m}{\partial t} + Lv_m = 0$ in $(B_m - \bar{B}_{m_0}) \times (-\infty, 1)$. Taking into account the boundary conditions, we conclude from (3.20) and the maximum principle that $0 \le W_m(x,t) \le v_m(x,t) = cu_m(x, 1-t)$. We have assumed that uniqueness holds for $BL(L,0)$ which is equivalent to the assumption that $\lim_{m\to\infty} u_m = 0$. Thus we conclude that $\lim_{m\to\infty} W_m = 0$, which proves (3.19).

From (3.17) and (3.19) we conclude that

$$
(3.21)\qquad Z \le \lim_{m\to\infty} V_m \ \text{ in } (R^n - B_{m_0}) \times [0,1].
$$

By the maximum principle,

$$
(3.22)\qquad \lim_{m\to\infty} V_m \le \max(\sup_{|y|=m_0}\phi(y), \exp(-\lambda)\sup_{|x|\ge m_0}\phi(x)) \text{ in } (R^n - B_{m_0}) \times [0,1].
$$

By the assumption on ϕ in the proposition, there exists an $x_0 \in R^n - \bar{B}_{m_0}$ such that $\phi(x_0)$ is strictly larger than the righthand side of (3.22). Recall that $Z(x_0, 0) =$

$\phi(x_0)$. Using these two facts along with (3.21) and (3.22) gives a contradiction. Thus, in fact uniqueness for $BL(L,0)$ does not hold. $\square$

Proof of Theorem 3. We must show that $u_{0;max} \not\equiv 0$. Recall from its construction that $u_{0;max} = \lim_{m\to\infty} U_m$, where $U_m \geq 0$ satisfies the semilinear equation in B_m and $\lim_{x\to\partial B_m} U_m(x,t) = \infty$, for $t > 0$.

By assumption, uniqueness does not hold for $BL(L,0)$. Thus there exists a function $w_0 \not\equiv 0$ satisfying $(w_0)_t = Lw_0$, $w_0(x,0) = 0$ and $\sup_{0\leq t\leq T}\sup_{x\in R^n} |w_0(x,t)| < \infty$, for all $T > 0$. In fact then, there exists a nonnegative function $w^+ \not\equiv 0$ satisfying the same conditions. To see this, note that if w_0 does not change sign, then we can choose $w^+ = \pm w_0$. Thus, assume that w_0 changes sign. Fix $T > 0$ such that $\sup_{0\leq t\leq T}\sup_{x\in R^n} w_0(x,t) > 0$. Let w_m^+ denote the solution to $u_t = Lu$ in B_m with $u(x,0) = 0$, for $x \in B_m$, and $u(x,t) = N$, for $x \in \partial B_m$ and $t > 0$, where $N = \sup_{0\leq t\leq T}\sup_{x\in R^n} w_0(x,t) > 0$. By the maximum principle,

$$\max(0, w_0) \leq w_m^+,\ x \in R^n,\ 0 \leq t \leq T, \tag{3.23}$$

and w_m^+ is monotone nonincreasing in m. By the standard compactness argument, it follows that $w^+ \equiv \lim_{m\to\infty} w_m^+$ is a solution to $BL(L,0)$, and by (3.23), $w^+ \gneq 0$.

Now let $Z = kw^+$, where $k > 0$. Then

$$LZ + VZ - \gamma Z^p - \frac{\partial Z}{\partial t} = VZ - \gamma Z^p = \gamma k w^+ (\frac{V}{\gamma} - (kw^+)^{p-1}). \tag{3.24}$$

Since w^+ is bounded on $R^n \times [0,T]$ and since by assumption, $\inf_{x\in R^n} \frac{V}{\gamma}(x) > 0$, it follows that the right hand side of (3.24) is nonnegative on $R^n \times [0,T]$ if $k > 0$ is chosen sufficiently small. Since $Z(x,0) = 0$, it then follows from (3.24) and the maximum principle in Proposition 1 that

$$U_m \geq Z, \text{ on } B_m \times [0,T].$$

Letting $m \to \infty$, we conclude that $u_{0;max} \geq kw^+$ in $R^n \times [0,T]$. □

4. The Interplay Between Uniqueness/Nonuniqueness of the Parabolic Equation and of the Corresponding Steady-State Elliptic Equation. Consider the elliptic semilinear equation corresponding to steady state solutions of NS:

$$Lw + Vw - \gamma w^p = 0 \text{ and } w \geq 0 \text{ in } R^n. \tag{4.1}$$

The next theorem gives conditions for uniqueness/nonuniqueness in terms of solutions to the elliptic equation. As we shall see in the next section, this result can be very useful.

Theorem 4.

i. Let $\{f_m\}_{m=1}^{\infty} \subset C(R^n)$ be an increasing sequence of nonnegative compactly supported functions satisfying $\lim_{m\to\infty} f_m = \infty$. Let $u_{f_m;min}$ denote the minimal solution to NS_{f_m}. Then

$$w^*(x) \equiv \lim_{t\to\infty} \lim_{m\to\infty} u_{f_m;min}(x,t) \tag{4.2}$$

exists and is a nonnegative solution to (4.1). There exists a maximal solution w_{max} to (4.1), and if $w_{max} \gneq w^*$, then uniqueness does not hold for NS_f, for any f. Furthermore, if $\inf_{x\in R^n} \gamma(x) > 0$, then w^* satisfies the bound

$$\sup_{x\in R^n} w^*(x) \leq \left(\frac{\sup_{x\in R^n} V^+(x)}{\inf_{x\in R^n} \gamma(x)}\right)^{\frac{1}{p-1}}, \tag{4.3}$$

where $V^+ = \max(V, 0)$.

ii. If $w = 0$ is the only solution to (4.1), then uniqueness holds for NS_f, for all f.

We prepare for the proof of Theorem 4 with the following result which is of independent interest.

Proposition 6. *Let* $\{f_m\}_{m=1}^{\infty}$ *be an increasing sequence of nonnegative compactly supported functions satisfying* $\lim_{m\to\infty} f_m = \infty$. *Then*

$$u_{\infty;min} \equiv \lim_{m\to\infty} u_{f_m;min}$$

and

$$u_{\infty;max} \equiv \lim_{m\to\infty} u_{f_m;max}$$

exist and are independent of the particular sequence $\{f_m\}$. *They solve* NS *with initial condition* $f = \infty$ *and they are monotone nonincreasing in* t. *Furthermore*

$$w^*(x) \equiv \lim_{t\to\infty} u_{\infty;min}(x,t) \tag{4.4}$$

is a solution to (4.1) and

$$w_{max}(x) \equiv \lim_{t\to\infty} u_{\infty;max}(x,t) \tag{4.5}$$

is the maximal, nonnegative solution to (4.1).

Proof. By the maximum principle and the construction of minimal and maximal solutions, $u_{f_m;min}$ and $u_{f_m;max}$ are monotone in m. Thus, the existence of the limits and the fact that $u_{\infty;min}$ and $u_{\infty;max}$ satisfy NS with initial condition $f = \infty$ follow from the standard compactness argument and the a priori bounds in (2.1). The fact that the above procedure is independent of the particular sequence follows from the existence plus the fact that given two such sequences, one can construct a new increasing sequence of compactly supported functions using infinitely many of the functions from each of the two original sequences.

We now turn to the monotonicity in t. Fix $t_0 > 0$. Let $v_m(x,t) = u_{f_m;min}(x, t + t_0)$ and $v(x,t) = u_{\infty;min}(x, t + t_0)$. By the already-proved part of the theorem, we have $\lim_{m\to\infty} v_m = v$ and v solves NS with initial condition $f(x) = u_{\infty;min}(x, t_0)$. Let Z be any solution to NS with initial condition $f(x) = u_{\infty;min}(x, t_0)$. Since

v_n is the minimal solution to NS with initial condition $f(x) = u_{f_n;min}(x,t_0)$ and since $u_{f_n;min}(x,t_0) \le u_{\infty;min}(x,t_0)$, it follows from the maximum princple and the construction of minimal solutions that v_n is less than or equal to the minimal solution of NS with initial condition $f(x) = u_{\infty;min}(x,t_0)$. Consequently, $v_n \le Z$, and letting $n \to \infty$ gives $v \le Z$. Thus v is in fact the minimal solution of NS with initial condition $f(x) = u_\infty(x,t_0)$. But then again by the maximum principle and the construction of minimal solutions, and by the definition of $u_{\infty;min}$, it follows that $v \le u_{\infty;min}$, which proves the monotonicity of $u_{\infty;min}$ in t.

Now let $V(x,t) = u_{\infty;max}(x,t+t_0)$. By the already-proved part of the theorem, V is a solution of NS with initial condition $f(x) = u_{\infty;max}(x,t_0)$. Thus, V is less than or equal to the maximal solution of NS with initial condition $f(x) = u_{\infty;max}(x,t_0)$, and by the maximum principle, the construction of maximal solutions, and the definition of $u_{\infty;max}$, the maximal solution of NS with initial condition $f(x) = u_{\infty;max}(x,t_0)$ is less than or equal to $u_{\infty;max}$. Thus $V \le u_{\infty;max}$, which proves the monotonicity of $u_{\infty;max}$ in t.

We now show that w^* and w_{max} are solutions to (4.1). Let $v_s(x,t) = u_{\infty;min}(x,t+s)$. Then from the monotonicity in t, the standard compactness argument and the a priori bounds in (2.1), it follows that $\lim_{s\to\infty} v_s$ exists and solves NS. Since $w^*(x) = \lim_{s\to\infty} v_s$, we conclude that w^* is a solution to (4.1). A similar proof works for w_{max}.

Finally, we show that w_{max} is the maximal nonnegative solution to (4.1). To show this, we will prove that if w is a nonnegative solution to (4.1), then $u_{\infty;max}(x,t) \ge w(x)$ for $(x,t) \in R^n \times [0,\infty)$. From the definition of $u_{\infty;max}$, it suffices to prove the above inequality with $u_{\infty;max}$ replaced by $u_{f_m;max}$ and R^n replaced by B_{l_m} for m sufficiently large, where $\lim_{m\to\infty} l_m = \infty$. From the construction of the maximal solution, it follows that $u_{f_m;max} = \lim_{k\to\infty} U_k^{(m)}$, where $U_k^{(m)}$ solves the semilinear equation in B_k, $U_k^{(m)}(x,0) = f_m(x)$ in B_k and $\lim_{x\to\partial B_k} U_k^{(m)}(x,t) = \infty$ for $t > 0$. Thus, it suffices to show that $U_k^{(m)}(x,t) \ge w(x)$ in $B_{l_m} \times [0,\infty)$. Since w satisfies the semilinear parabolic equation, it follows from the maximal principle in Proposition 1 that $U_k^{(m)}(x,t) \ge w(x)$ in $B_l \times [0,\infty)$ if l satisfies $f_m \ge w$ in B_l. Since f_m is increasing and converges pointwise to ∞, we can construct a sequence l_m satisfying $\lim_{m\to\infty} l_m = \infty$ and such that $f_m \ge w$ on B_{l_m}. This completes the proof of the maximality of w_{max}. □

Proof of Theorem 4. *i.* The ground work for the proof has been prepared in Proposition 6 above. Note that the claims that w^* solves (4.1) and that there exists a maximal solution w_{max} to (4.1) follow from Proposition 6. The key additional step is the following inequality:

$$u_{\infty;max} - u_{\infty;min} \le u_{0;max}. \tag{4.6}$$

Letting $t \to \infty$ in (4.6) and using (4.4) and (4.5) shows that if $w_{max} \gneq w^*$, then $u_{0;max} \not\equiv 0$. This, in conjunction with Proposition 3, proves that uniqueness does not hold for NS_f, for any f, and completes the proof except for (4.3).

We now prove (4.6). From the definition of $u_{\infty;max}$ and $u_{\infty;min}$ in Proposition 6, (4.6) will follow if we show that

$$u_{f;max} - u_{f;min} \le u_{0,max}, \tag{4.7}$$

for compactly supported, nonnegative f. From the construction of the maximal solution, $u_{f;max} = \lim_{m\to\infty} \lim_{k\to\infty} U_{m,k}^{(f)}$ and $u_{f;min} = \lim_{m\to\infty} u_m^{(f)}$, where for m sufficiently large so that $supp(f) \subset B_{2m}$, $U_{m,k}^{(f)}$ satisfies (2.5) with f_m replaced by f and $u_m^{(f)}$ satisfies (3.2) with f_i replaced by f. Thus, (4.7) will follow if we show that

$$U_{m,k}^{(f)} - u_{2m}^{(f)} \le U_{m,k}^{(0)} \text{ in } B_{2m} \times [0,\infty). \tag{4.8}$$

Let $W = U^{(f)}_{m,k} - u^{(f)}_{2m}$. It follows from the maximum principle in Proposition 1 that $W \geq 0$. From that maximum principle, (4.8) will hold if we show that

$$LW + VW - \gamma W^p - \frac{\partial W}{\partial t} \geq -\psi_{m,k} \text{ in } B_{2m} \times [0,\infty), \tag{4.9}$$

where $\psi_{m,k}$ is as in (2.5). We have $LW+VW-\frac{\partial W}{\partial t} = -\psi_{m,k}+\gamma[(U^{(f)}_{m,k})^p-(u^{(f)}_{2m})^p]$. Thus,

$$LW + VW - \gamma W^p - \frac{\partial W}{\partial t} = -\psi_{m,k} + \gamma[(U^{(f)}_{m,k})^p - (u^{(f)}_{2m})^p - (U^{(f)}_{m,k} - u^{(f)}_{2m})^p]. \tag{4.10}$$

Now (4.9) follows from (4.10) and the inequality $b^p - a^p - (b-a)^p \geq 0$, for $0 \leq a \leq b$.

We now turn to the proof of (4.3). Let $\beta = \sup_{x\in R^n} V^+(x)$ and let $\alpha = \inf_{x\in R^n} \gamma(x)$. By assumption, $\alpha > 0$ and we may assume that $\beta < \infty$ since otherwise there is nothing to prove. Define

$$H(t) = \begin{cases} (\frac{\beta}{\alpha})^{\frac{1}{p-1}}(1-\exp(-(p-1)\beta t))^{-\frac{1}{p-1}}, \text{ if } \beta > 0 \\ (\frac{1}{(p-1)\alpha t})^{\frac{1}{p-1}}, \text{ if } \beta = 0. \end{cases}$$

Then an easy calculation shows that

$$LH + VH - \gamma H^p - \frac{\partial H}{\partial t} \leq 0. \tag{4.11}$$

By the construction of the minimal solution, $u_{f_m;min} = \lim_{l\to\infty} u_{m,l}$, where for l sufficiently large so that $supp(f_m) \subset B_l$, $u_{m,l}$ solves (3.2) with f_i replaced by f_m and B_m replaced by B_l. By (4.11) and the maximum principle of Proposition 1, it follows that $u_{m,l}(x,t) \leq H(t)$ for $(x,t) \in B_l \times [0,\infty)$. Thus, $u_{f_m;min}(x,t) \leq H(t)$ for $(x,t) \in B_l \times [0,\infty)$. Letting $m \to \infty$ and then letting $t \to \infty$ now shows that (4.3) holds.

ii. By the construction of the maximal solution, $u_{0;max} = \lim_{m\to\infty} \lim_{k\to\infty} U^{(0)}_{m,k}$ where $U^{(0)}_m$ solves (2.5) with f_m replaced by 0. Let $t_0 > 0$ and define $W(x,t) = U^{(0)}_m(x,t+t_0)$. It follows by the maximum principle in Proposition 1 that $W \geq U^{(0)}_{m,k}$

on $B_m \times [0,\infty)$; thus $U^{(0)}_{m,k}$ is monontone nondecreasing in t and the same is true of $u_{0;max}$. By the same type of argument used to show that w^* solves (4.1), if follows that $\lim_{t\to\infty} u_{0;max}$ solves (4.1). By assumption, $w \equiv 0$ is the only nonnegative solution to (4.1); thus, $\lim_{t\to\infty} u_{0;max} = 0$. In light of the monotonicity in t, we conclude that $u_{0;max} = 0$. This proves uniqueness for NS_0, and in conjunction with Proposition 3, uniqueness for all f. □

5. Applications. In this section we use the array of results in sections three and four to prove theorems on uniqueness/nonuniqueness for two classes of semilinear parabolic equations. We will also show how some of the results in this paper can be used to give an alternative proof and an extension of a classical result in semilinear elliptic theory.

We will determine how uniqueness depends on α for the following class of equations:

$$\begin{aligned} &u_t = \alpha\Delta u - u^p \;\text{ in } R^n \times (0,\infty);\\ &u(x,0) = f(x),\; x \in R^n;\\ &u \geq 0. \end{aligned} \tag{5.1}$$

And with a relatively generic V we will determine how uniqueness depends on γ for the following class of equations:

$$\begin{aligned} &u_t = \Delta u + Vu - \gamma u^p \;\text{ in } R^n \times (0,\infty);\\ &u(x,0) = f(x),\; x \in R^n;\\ &u \geq 0. \end{aligned} \tag{5.2}$$

Concerning the class of equations appearing in (5.1), we have the following result.

Theorem 5.

i-a. Let $n \geq 2$. *If*

$$\alpha(x) \leq C(1+|x|)^2,$$

for some $C > 0$, *then uniqueness holds in (5.1) for all* f.

i-b. Let $n \geq 2$. *If*

$$\alpha(x) \geq C(1+|x|)^{2+\epsilon},$$

for some $\epsilon, C > 0$, *then uniqueness does not hold in (5.1) for any* f.

ii-a. Let $n = 1$. *If*

$$\alpha(x) \leq C(1+|x|)^{1+p},$$

for some $C > 0$, *then uniqueness holds in (5.1), for all* f.

ii-b. Let $n = 1$. *If*

$$\alpha(x) \geq C(1+|x|)^{1+p+\epsilon},$$

for $x > 0$ *or for* $x < 0$ *and some* $\epsilon, C > 0$, *then uniqueness does not hold in (5.1) for any* f.

The proof of Theorem 5, one of whose parts (ii-a) is quite long and involved, is given in the next section.

Before turning to (5.2), we will show how Theorems 2 and 4 can be used to obtain an alternate proof of a classical result concerning nonexistence of nontrivial solutions of a certain semilinear elliptic equation in dimension $n \geq 3$, and how these theorems along with Theorem 5 can be used to extend that result to appropriate corresponding results in the cases $d = 1, 2$. It was shown by Ni [11] and Kenig and Ni [7] that the equation $\Delta w - \gamma w^p = 0$ in $R^n, n \geq 3$, has no nontrivial, nonnegative solution if $\gamma(x) \geq C(1+|x|)^{-2+\epsilon}$, for some $C, \epsilon > 0$, and that nontrivial, nonnegative solutions do exist if $\gamma(x) \leq C(1+|x|)^{-2-\epsilon}$. Lin [9] extended the nonexistence result to the borderline case: there is no nontrivial solution if $\gamma(x) \geq C(1+|x|)^{-2}$. Here is a quick proof of this last result: Let $C > 0$. By Theorem 2, uniqueness holds for

$NS((1+|x|)^2\Delta, 0, C)$. From (4.3) in Theorem 4, it follows that $w^* \equiv 0$. But then since uniqueness holds and $w^* = 0$, it follows again from Theorem 4 that there is no nontrivial nonnegative solution to $(1+|x|)^2\Delta w - Cw^p = 0$.

Note that the above proof is independent of dimension and works just as well for $n = 1, 2$. Using Theorem 5(i), we can also give an alternative proof of the existence part of the above result, and more importantly, we can extend the existence/nonexistence dichotomy to dimensions $n = 1, 2$.

Theorem 6. *Let $p > 1$.*

i. Consider the equation

$$u'' - \gamma u^p = 0 \text{ in } R. \tag{5.3}$$

There exists a positive solution to (5.3) if $\gamma(x) \le C(1+|x|)^{-1-p-\epsilon}$, for some $C, \epsilon > 0$, and there is no positive solution to (5.3) if $\gamma(x) \ge C(1+|x|)^{-1-p}$, for some $C > 0$.

ii. Consider the equation

$$\Delta u - \gamma u^p = 0 \text{ in } R^n, \ n \ge 2. \tag{5.4}$$

There exists a positive solution to (5.4) if $\gamma(x) \le C(1+|x|)^{-2-\epsilon}$, for some $C, \epsilon > 0$, and there is no positive solution to (5.4) if $\gamma(x) \ge C(1+|x|)^{-2}$, for some $C > 0$.

Proof. Consider the semilinear equation

$$u_t = \alpha u'' - u^p \text{ in } R \times (0, \infty). \tag{5.5}$$

If $\alpha(x) \le C(1+|x|)^{1+p}$, then it follows from Theorem 5(ii-a) that uniqueness holds for (5.5). Also, by (4.3) we have $w^* = 0$ for equation (5.5). Thus, we conclude from Theorem 4(i) that there is no positive solution to $\alpha u'' - u^p = 0$ in R. This is equivalent to the nonexistence statement in (i). On the other hand, if

$\alpha(x) \geq C(1+|x|)^{1+p+\epsilon}$, then by Theorem 5(ii-b) uniqueness does not hold for (5.5). Thus, it follows from Theorem 4(ii) that a positive solution exists for $\alpha u'' - u^p = 0$ in R, which is equivalent to the existence statement in (i). Part (ii) is proven in exactly the same manner. □

We now turn to the class of equations in (5.2).

Theorem 7. *i. Let V be bounded from above. If*

$$\gamma(x) \geq C_1 \exp(-C_2|x|^2),$$

for some $C_1, C_2 > 0$, then uniqueness holds in (5.2) for all f.

ii. Let $V \geq 0$. If

$$\gamma(x) \leq C \exp(-|x|^{2+\epsilon}),$$

for some $C, \epsilon > 0$, then uniqueness does not hold in (5.2) for $f \equiv 0$.

Remark. Equation (5.2) with $0 \leq V \leq C$ and $\gamma(x) \leq C\exp(-|x|^{2+\epsilon})$, with $C, \epsilon > 0$ is an example where uniqueness holds for BL but not for NS. For another example, consider $L = (1+|x|)^l \Delta$ with $n = 2$ and $l > 2$ or with $n = 1$ and $l > 1+p$. Let $V = 0$ and $\gamma = 1$. Applying Proposition 5-(ii-a) with $\phi(x) = \log|x|$ if $n = 2$ and with $\phi(x) = |x|$ if $n = 1$ shows that uniqueness holds for BL. On the other hand, by (4.3), we have $w^* = 0$ while by Theorem 6, $w_{max} \neq 0$. Thus, by Theorem 4(i), uniqueness does not hold for NS.

For an example where uniqueness holds for NS but not for BL, consider the operator $L = (1+|x|)^l \Delta$ in R^n, $n \geq 3$, for $l > 2$, and let $V = 0$. Then uniqueness does not hold for BL—see Remark 1 after Proposition 5. On the other hand, if $\gamma \geq (1+|x|)^{l-2}$, then uniqueness does hold for NS. Indeed, by Theorem 4, it suffices to show that there is no nontrivial, nonnegative solution w to $Lw - \gamma w^p = 0$ in R^n, or equivalently, to $\Delta w - \frac{\gamma(x)}{(1+|x|)^l} w^p = 0$ in R^n. But this follows from Theorem

6. Note that in this example, $\inf_{x\in R^n} \frac{V}{\gamma}(x) = 0$, as must be the case in light of Theorem 3.

For the proof of Theorem 7 as well as of Theorem 5, we will need the following semilinear elliptic maximum principle.

Proposition 7. *Let $D \subset R^n$ be a bounded domain and let $0 \le u_1, u_2 \in C^2(D) \cap C(\bar{D})$ satisfy*

$$Lu_1 + Vu_1 - \gamma u_1^p \le Lu_2 + Vu_2 - \gamma u_2^p \quad \text{in } D,$$

and

$$u_1 \ge u_2 \text{ on } \partial D.$$

Assume that $V \le 0$. Then $u_1 \ge u_2$ in D.

Proof. Let $W = u_1 - u_2$ and define $H(x) = \frac{u_1^p(x) - u_2^p(x)}{W(x)}$, if $W(x) \ne 0$, and $H(x) = 0$ otherwise. Then $H \ge 0$ and we have $LW + (V - H)W \le 0$ in D and $W \ge 0$ on ∂D. Since $V - H \le 0$, it follows from the standard linear elliptic maximum principle that $W \ge 0$ in D. □

Proof of Theorem 7. *i.* Let $U(x,t) = u_{0;max}(x,t)\exp(-C|x|^2(t+\delta))$, for some $C, \delta > 0$. Then U satisfies

$$\begin{aligned} &\Delta U + 4C(t+\delta)x \cdot \nabla U + (4|x|^2(t+\delta)^2C^2 + 2nC(t+\delta) + V - C|x|^2)U \\ &- C_1\exp(-C_2|x|^2)\exp(C(p-1)|x|^2(t+\delta))U^p - U_t \ge 0 \text{ in } R^n \times (0,\infty). \end{aligned} \tag{5.6}$$

Fixing $\delta = \frac{C_2}{C(p-1)}$ and $C \ge \frac{16C_2^2}{p-1}$, we obtain from (5.6)

$$\Delta U + 4C(t+\delta)x \cdot \nabla U + (2nC(t+\delta) + V)U - U^p - U_t \ge 0 \text{ in } R^n \times (0,\delta). \tag{5.7}$$

Note that the coefficients of the operator on the left hand side of (5.7) satisfy the requirements in Theorem 2. (They depend on t unlike in Theorem 2, but this is not important.) Thus, it follows from the maximum principle that for any $R > 1$, the super solution in $B_R \times (0, \infty)$ constructed in the proof of Theorem 2 is larger or equal to U in $B_R \times (0, \delta)$. That is,

$$U(x,t) \leq (1+|x|)^{\frac{2}{p-1}}(R-|x|)^{-\frac{2}{p-1}}\exp(K(t+1)) \text{ in } B_R \times (0,\delta).$$

Letting $R \to \infty$ shows that $U \equiv 0$ in $R^n \times (0, \delta)$, and thus the same is true for $u_{0;max}$. As the original equation was time homogeneous, it is clear that in fact $u_{0;max} \equiv 0$ in $R^n \times (0, \infty)$.

ii. Writing $u(x) = \exp((1+|x|^2)^{1+\frac{\epsilon}{4}})\hat{u}$ and dividing through by $\exp((1+|x|^2)^{1+\frac{\epsilon}{4}})$ one sees that nonuniqueness for the initial condition $f = 0$ in (5.2) is equivalent to nonuniqueness for the initial condition $f = 0$ in an equation of the form

$$u_t = \Delta u + B\nabla u + \hat{V}u - \hat{\gamma}u^p, \tag{5.8}$$

where $B(x) \cdot \frac{x}{|x|} \geq C_1|x|^{1+\frac{\epsilon}{2}}$, $\hat{V} \geq C_1$ and $\hat{\gamma} \leq C$, for constants $C_1, C > 0$. Uniqueness does not hold for $BL(\Delta + B\nabla, 0)$ as was shown in the remark following Proposition 5. Thus, by Theorem 3, uniqueness does not hold for the initial condition $f = 0$ in (5.8). □

6. Proof of Theorem 5. *i-a.* The result follows directly from Theorem 2.

i-b. Under the assumption on the coefficients, the right hand side of (4.3) equals 0 and thus $w^* = 0$. Therefore, by Theorem 4, it suffices to show that there exists a nontrivial, nonnegative solution to the elliptic equation

$$\alpha\Delta w - w^p = 0 \text{ in } R^n. \tag{6.1}$$

We note that if a nontrivial, nonnegative solution of (6.1) exists for $\alpha = \alpha_1$, then one also exists for $\alpha = \alpha_2$, if $\alpha_2 \geq \alpha_1$. The reason for this is as follows. The maximal nonnegative solution w_{max} to (6.1) is obtained as $w_{max} = \lim_{k\to\infty} \lim_{m\to\infty} w_{m,k}$ where $w_{m,k}$ satisfies $\alpha\Delta w - w^p$ in B_k and $w(x) = m$ on ∂B_k. (The existence of

$w_{m,k}$ follows from the method of upper and lower solutions—see the paragraph following (6.12) for more detail.) To distinguish between α_i, $i = 1, 2$, we will use the notation $w^{(i)}_{m,k}$ and $w^{(i)}_{max}$. We have $\alpha_1 \Delta w^{(1)}_{m,k} - (w^{(1)}_{m,k})^p = 0$ while

$$\alpha_1 \Delta w^{(2)}_{m,k} - (w^{(2)}_{m,k})^p = (\alpha_1 - \alpha_2)\Delta w^{(2)}_{m,k} = (\frac{\alpha_1}{\alpha_2} - 1)(w^{(2)}_{m,k})^p \le 0.$$

Thus, by the elliptic maximum principle in Proposition 7 $w^{(2)}_{m,k} \ge w^{(1)}_{m,k}$ in B_k, and we conclude that if $w^{(1)}_{max} \ne 0$, then $w^{(2)}_{max} \ne 0$.

In light of the above, we may assume without loss of generality that $\alpha(x) = C|x|^{2+\epsilon}$ for $|x| \ge 1$, where $\epsilon, C > 0$. Let $0 < h \in C^1(R^n)$ satisfy $h(x) = |x|^\delta$ for $|x| \ge 1$, where $\delta = \frac{\epsilon}{p-1}$. Writing $w = h\hat{w}$ and dividing through by h^p, one sees that the existence of a positive solution to (6.1) is equivalent to the existence of a positive solution to

$$A\Delta w + B\nabla w + \hat{V}w - w^p = 0 \text{ in } R^n, \tag{6.3}$$

where $A(x) = C|x|^2$, $B(x) = 2C\delta x$, and $\hat{V} = C\delta(\delta + n - 2)$ for $|x| \ge 1$. To show that there exists a positive solution to (6.3) we will show that that $w^* \ne 0$ for the parabolic equation

$$u_t = A\Delta u + B\nabla u + \hat{V}u - u^p = 0 \text{ in } R^n \times (0, \infty). \tag{6.4}$$

Let $c_\delta = (C\delta(\delta + n - 2))^{\frac{1}{p-1}}$. (Note that if we had $\hat{V} = C\delta(\delta + n - 2)$ on all of R^n, then the constant c_δ would be a positive solution to (6.3).) For $m > 1$, let u_m denote the solution to

$$\begin{aligned} &u_t = A\Delta u + B\nabla u,\ (x,t) \in (B_m - \bar{B}_1) \times (0, \infty); \\ &u(x,0) = c_\delta,\ x \in B_m - \bar{B}_1; \\ &u(x,t) = 0,\ x \in \partial B_m \cup \partial B_1, t > 0. \end{aligned} \tag{6.5}$$

By the linear maximum principle, $0 \le u_m \le c_\delta$ and u_m is nondecreasing in m and nonincreasing in t. We have $A\Delta u_m + B\nabla u_m + \hat{V}u_m - u_m^p - \frac{\partial u_m}{\partial t} = c_\delta^{p-1}u_m - u_m^p =$

$u_m(c_\delta^{p-1} - u_m^{p-1}) \geq 0$ in $(B_m - \bar{B}_1) \times (0, \infty)$. Recalling the definition of w^* in (4.2), we conclude from the maximum principle in Proposition 1 that

$$w^*(x) \geq \lim_{t\to\infty} \lim_{m\to\infty} u_m(x,t). \tag{6.6}$$

Let $\hat{u}_m$ denote the solution to (6.5) when the boundary condition $u(x,t) = 0$ on ∂B_m is changed to $u(x,t) = c_\delta$. Note that by the maximum principle, $\hat{u}_m$ is nonincreasing in m. By the standard compactness argument, $U \equiv \lim_{m\to\infty} u_m$ and $\hat{U} \equiv \lim_{m\to\infty} \hat{u}_m$ both solve

$$\begin{aligned} &u_t = A\Delta u + B\nabla u,\ (x,t) \in (R_n - \bar{B}_1) \times (0,\infty);\\ &u(x,0) = c_\delta,\ x \in R^n - \bar{B}_1;\\ &u(x,t) = 0,\ (x,t) \in \partial B_1 \times (0,\infty). \end{aligned} \tag{6.7}$$

Because of the bounds given above on A and B, uniqueness holds in the class of bounded solutions for (6.7) as we shall now show. Thus we conclude that $U = \hat{U}$. To see that uniqueness holds, note that the difference v of any two bounded solutions to (6.7) will satisfy the following equation for some $C > 0$ and every $m > 0$:

$$\begin{aligned} &v_t = A\Delta v + B\nabla v,\ (x,t) \in (B_m - \bar{B}_1) \times (0,\infty);\\ &v(x,0) = 0,\ x \in B_m - \bar{B}_1;\\ &v(x,t) = 0,\ (x,t) \in \partial B_1 \times (0,\infty);\\ &|v(x,t)| \leq C,\ (x,t) \in \partial B_m \times (0,\infty). \end{aligned}$$

One can check that $\psi(x,t) = (1+|x|^2)\exp(\lambda t)$ satisfies $A\Delta\psi + B\nabla\psi - \frac{\partial\psi}{\partial t} \leq 0$, if $\lambda > 0$ is sufficiently large. Thus, taking into account the boundary conditions, it follows from the maximum principle that $|v(x,t)| \leq C(1+|m|^2)^{-1}\psi(x,t)$ for $(x,t) \in B_m \times [0,\infty)$. Letting $m \to \infty$ gives $v \equiv 0$.

Letting $r = |x|$, the radial form of the elliptic operator on the right hand side of (6.7) is $Cr^2\frac{\partial^2}{\partial r^2} + C(n-1+2\delta)r\frac{\partial}{\partial r}$, for $r > 1$. Letting $l = n-2+2\delta > 0$, it is easy to show that $\phi_m(x) \equiv c_\delta \frac{1-|x|^{-l}}{1-m^{-l}}$ solves $A\Delta\phi + B\nabla\phi = 0$ in $B_m - \bar{B}_1$ with $\phi(x) = 0$ on ∂B_1 and $\phi(x) = c_\delta$ on ∂B_m. By the maximum principle, $\hat{u}_m(x,t) \geq \phi_m(x)$. Letting $m \to \infty$ and using the fact that $U = \hat{U}$, we conclude from (6.6) that $w^*(x) \geq c_\delta(1 - |x|^{-l})$ in $R^n - \hat{B}_1$.

ii-a. By Theorem 4, it suffices to show that there is no positive solution to the elliptic equation

$$\alpha w'' - w^p = 0 \text{ in } R. \tag{6.8}$$

Let $0 < h(x) \in C^2(R)$ satisfy $h(x) = |x|$ for $|x| \geq 1$. Writing $w = h\hat{w}$ and dividing through by h^p, one sees that the nonexistence of a positive solution for (6.8) is equivalent to the nonexistence of a positive solution to

$$aw'' + bw' + \hat{V}w - w^p = 0 \text{ in } R, \tag{6.9}$$

where $a = \frac{\alpha}{h^{p-1}}, b = 2\alpha\frac{h'}{h^p}$ and $\hat{V} = \frac{\alpha h''}{h^p}$. By the assumption on α, it follows that $a(x) \leq C(1+|x|)^2$, $|b(x)| \leq C(1+|x|)$ and $\hat{V}(x) \leq C$, for some $C > 0$. Thus, it follows from Theorem 2 that uniqueness holds for the parabolic equation

$$u_t = au'' + bu' + \hat{V}u - u^p = 0 \text{ in } R \times (0, \infty) \tag{6.10}$$

associated with (6.9). But then by Theorem 4, the w^* corresponding to the equation (6.10) must coincide with the maximal nonnegative solution of (6.9). Thus to complete the proof, it suffices to show that $w^* = 0$ for (6.10). Since h'' is compactly supported, it follows that $\hat{V}(x) = 0$ except on a bounded set. (This is where the one-dimensionality enters since $\Delta|x| = 0$ only in dimension 1. Also, note that if $\hat{V}$ were everywhere nonpositive then we could conclude from (4.3) that $w^* = 0$.)

Choose $m_0 > 0$ such that $\hat{V} = 0$ on $R - (-m_0, m_0)$. Let ϕ denote the minimal positive solution to

$$\begin{aligned} &aw'' + bw' - w^p = 0 \quad \text{in } \{|x| > m_0\}; \\ &w(\pm m_0) = \infty. \end{aligned} \tag{6.11}$$

The existence of ϕ is proven below. Let $U(x,t) = \frac{1}{p-1}^{\frac{1}{p-1}} t^{-\frac{1}{p-1}} + \phi(x)$ for $|x| > m_0$ and $t > 0$. Using the inequality $(x+y)^p \geq x^p + y^p$, for $x, y \geq 0$, it is easy to check that $aU'' + bU' - U^p - U_t \leq 0$, for $|x| > m_0$ and $t > 0$. Since $U(\pm m_0, t) = U(x, 0) = \infty$, it follows from the maximum principle in Proposition 1 that any solution u of (6.10) with initial condition $f \in C(R)$ satisfies $u(x,t) \leq U(x,t)$ for $|x| > m_0$ and $t > 0$. Letting $t \to \infty$ and recalling the definition of w^* in Theorem 4 then shows that $w^* \leq \phi$. We also know from Theorem 4 that w^* is a solution to (6.9). To show that in fact $w^* = 0$, we will show that the zero solution is the only nonnegative solution to (6.9) which is dominated by ϕ. The proof will require a number of steps. We begin by constucting the function ϕ.

Let $\{\psi_n\}_{n=1}^{\infty}$ be an increasing sequence of smooth functions satisfying $\psi_n(x) = n$ for $|x| \leq m_0 - \frac{1}{n}$, $\psi_n(x) = 0$ for $|x| \geq m_0$ and $0 \leq \psi_n \leq n$. For $m > m_0$, let $\phi_{n,m}$ denote the solution to

$$\begin{aligned} & aw'' + bw' + \hat{V}w - w^p + \psi_n = 0, \ |x| < m; \\ & w(\pm m) = 0. \end{aligned} \tag{6.12}$$

The existence of $\phi_{n,m}$ follows by the standard method of upper and lower solutions. Recall that a lower (upper) solution satisfies (6.12) with the equal sign in the first line changed to $\geq$ ($\leq$) and the equal sign in the second line changed to $\leq$ ($\geq$). If there exists a lower solution $\phi^-_{m,n}$ and an upper solution $\phi^+_{m,n}$ such that $\phi^-_{m,n} \leq \phi^+_{m,n}$, then there exists a solution $\phi_{m,n}$ satisfying $\phi^-_{m,n} \leq \phi_{m,n} \leq \phi^+_{m,n}$ [13]. Clearly, $\phi^-_{m,n}(x) \equiv 0$ is a lower solution and $\phi^+_{m,n}(x) = C$ is an upper solution if C (depending on n) is sufficiently large. By the elliptic maximum principle in Proposition 7, $\phi_{m,n}$ is nondecreasing in n and m. Actually, Proposition 7 does not apply directly since $\hat{V}$ is not nonpositive in all of R. However, recalling how the operator in (6.12) was obtained from the original operator in (6.8), it follows that $\phi_{m,n}$ solves (6.12) if and only if $h\phi_{m,n}$ solves $\alpha w'' - w^p + h^p \psi_n = 0$ for $|x| < m$

and $w(\pm m) = 0$. From this and the fact that Proposition 7 holds for the original operator, it follows that the maximum principle holds for the transformed one.

Using the standard compactness argument, it will follow that $\phi_m \equiv \lim_{n\to\infty} \phi_{m,n}$ is a solution to $aw'' + bw' - w^p = 0$ in $\{m_0 < |x| < m\}$ with $w(\pm m) = 0$ if we show that $\{\phi_{m,n}\}_{n=1}^{\infty}$ is uniformly bounded on $(m_0 + \epsilon, m)$ for each $\epsilon > 0$.

To show the uniform boundedness, let $g(x) = \lambda(|x| - m_0)^{-\frac{2}{p-1}}$. An easy calculation shows that $ag'' + bg' - g^p \leq 0$ on (m_0, m) if $\lambda > 0$ is chosen sufficiently large. Thus, by the elliptic maximum principle in Proposition 7 (recall that $\hat{V} = 0$ in (m_0, m)), it follows that $\phi_{m,n} \leq g(x)$ on (m_0, m), proving the uniform boundedness.

We now prove that $\lim_{|x|\downarrow m_0} \phi_m(x) = \infty$. Let $Z(x) = \lambda(|x| - m + 2\epsilon)^{-\frac{2}{p-1}}$ for $m_0 - \epsilon < |x| < m_0 + \epsilon$. One can check that there exists a $\rho > 0$ such that if $\epsilon, \lambda \in (0, \rho)$, then

$$aZ'' + bZ' + \hat{V}Z - \gamma Z^p \geq 0 \text{ in } \{m_0 - \epsilon < |x| < m_0 + \epsilon\}. \tag{6.13}$$

Choose $\lambda > 0$ even smaller if necessary so that

$$Z(x) \leq \phi_{m,1}(x), \quad \text{for } |x| = m_0 + \epsilon. \tag{6.14}$$

Now extend Z to be smooth and positive on $\{|x| \leq m_0 - \epsilon\}$. Since $\psi_n(x) = n$ for $|x| \leq m_0 - \frac{1}{n}$, it is clear that for sufficiently large n,

$$aZ'' + bZ' + \hat{V}Z - \gamma Z^p + \psi_n \geq 0, \text{ for } |x| \leq m_0 - \epsilon. \tag{6.15}$$

From (6.13)-(6.15) and the maximum principle in Proposition 1, it follows that $Z(x) \leq \phi_{m,n}(x)$ for $|x| \leq m_0 + \epsilon$ and n sufficiently large. Letting $n \to \infty$, we obtain $\liminf_{|x|\downarrow m_0} \phi_m(x) \geq \lambda(2\epsilon)^{-\frac{2}{p-1}}$. As ϵ is arbitrary we conclude that $\lim_{|x|\downarrow m_0} \phi_m(x) = \infty$.

Letting $m \to \infty$ and using the standard compactness argument and the maximum principle, it follows that $\phi \equiv \lim_{m\to\infty} \phi_m$ is a positive solution to (6.11). By the maximum principle, any positive solution w to (6.11) satisfies $w \geq \phi_{n,m}$. Thus $w \geq \phi$, proving that ϕ is minimal.

For $g > 0$ define

$$A_g = a\frac{d^2}{dx^2} + b\frac{d}{dx} + \hat{V} - \gamma g^{p-1}$$

and recall that $\hat{V} = 0$ in a neighborhood of $\pm\infty$. We will now show that ϕ is a positive solution of minimal growth at $\pm\infty$ for the operator A_ϕ. What this means is that if $W > 0$ and $A_\phi W = 0$ in a neighborhood of $\pm\infty$ then $\phi \leq CW$ in a neighborhood of $\pm\infty$, for some $C > 0$. By the maximum principle and the construction of ϕ, it will follow that ϕ is a positive solution of minimal growth at $+\infty$ for A if we show that for $m > 2m_0$ the solution W_m to $A_\phi W_m = 0$ in $(2m_0, m)$, $W(2m_0) = \phi_m(2m_0)$ and $W_m(m) = 0$ satisfies $\lim_{m\to\infty} W_m = \phi$. An identical argument of course works at $-\infty$. Since ϕ_m satisfies $A_{\phi_m}\phi_m = 0$ in $(2m_0, m)$ and has the same boundary values as W_m, and since $\phi_m \leq \phi$, it follows from the maximum principle that $W_m \leq \phi_m$. Thus letting $W_\infty = \lim_{m\to\infty} W_m$, we have

$$W_\infty \leq \phi, \text{ for } |x| \geq 2m_0. \tag{6.16}$$

Converting $A_{\phi_m}\phi_m = 0$ and $A_\phi W_m = 0$ into integral equations by integrating twice, and using the boundary conditions, and then letting $m \to \infty$ and using the monotone convergence theorem, we obtain

$$\begin{aligned} &W_\infty(x) = \phi(2m_0) - c_{W_\infty}\int_{2m_0}^{x} dy \exp(-\int_{2m_0}^{y} \frac{b}{a}(r)dr) + \\ &\int_{2m_0}^{x} dy \exp(-\int_{2m_0}^{y} \frac{b}{a}(r)dr) \int_{2m_0}^{y} \frac{1}{a(z)} \exp(\int_{2m_0}^{z} \frac{b}{a}(r)dr) \phi^{p-1}(z) W_\infty(z)dz \end{aligned} \tag{6.17}$$

and

$$\phi(x) = \phi(2m_0) - c_\phi \int_{2m_0}^{x} dy \exp(-\int_{2m_0}^{y} \frac{b}{a}(r)dr) + \tag{6.18}$$
$$\int_{2m_0}^{x} dy \exp(-\int_{2m_0}^{y} \frac{b}{a}(r)dr) \int_{2m_0}^{y} \frac{1}{a(z)} \exp(\int_{2m_0}^{z} \frac{b}{a}(r)dr) \phi^{p-1}(z)\phi(z)dz,$$

where

$$c_{W_\infty} = \phi(2m_0) +$$
$$\int_{2m_0}^{\infty} dx \exp(-\int_{2m_0}^{x} \frac{b}{a}(r)dr) \int_{2m_0}^{x} \frac{1}{a(y)} \exp(\int_{2m_0}^{y} \frac{b}{a}(r)dr) \phi^{p-1}(y) W_\infty(y)dy,$$

and c_ϕ is defined by the same formula except that the term W_∞ is replaced by ϕ. By (6.16) it follows that $c_{W_\infty} \le c_\phi$. If it were true that $c_{W_\infty} < c_\phi$, then from (6.17) and (6.18) we would have $W'_\infty(2m_0) > \phi'(2m_0)$. Since $W_\infty(2m_0) = \phi(2m_0)$, this would contradict (6.16). We conclude that $c_{W_\infty} = c_\phi$. Thus, since W_∞ and ϕ and their first derivatives agree at $2m_0$, and since they solve the same second order linear equation, it follows from the uniqueness theorem for ODE's that $W_\infty \equiv \phi$. This completes the proof that ϕ is a positive solution of minimal growth for A_ϕ at $\pm\infty$.

Let Z be a solution of minimal growth at $\pm\infty$ for A_{w^*}. Since $w^* \le \phi$ and since ϕ is a solution of minimal growth at $\pm\infty$ for A_ϕ, it follows from the maximum principle and the above method of construction of solutions of minimal growth that $\phi \le CZ$ in a neighborhood of $\pm\infty$, for some $C > 0$. Thus, we have $w^* \le CZ$ in a neighborhood of ∞, where w^* solves $A_{w^*}w^* = 0$ in all of R and Z is a positive solution of minimal growth at $\pm\infty$ for A_{w^*}.

We will show that the operator A_{w^*} is so-called *subcritical* and that for a subcritical operator, it is impossible for a positive solution in the whole space to be dominated at $\pm\infty$ by a solution of minimal growth; thus we will conclude that $w^* = 0$. For an exposition on criticality theory for elliptic operators, see [12, chapter 4], and for the result we have just mentioned, see [12, Theorem 7.3.9]. However, since we are dealing with the one-dimensional case in which it is possible to keep everything self-contained without too much work, we will derive everything we need below.

An elliptic operator of the form $A = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$ is called subcritical if there exists a function $f > 0$ satisfying $Af \lneq 0$ in R. If $h > 0$ and we define the so-called h-transformed operator A^h by $A^h f = \frac{1}{h}A(hf)$, then clearly A^h is subcritcal if and only if A is. Similarly, if $\rho > 0$, then the operator ρA is subcritical if and only if A is. Finally we note that in the case $c = 0$, the operator is subcritical if and only if

$$\int_{-\infty}^{\infty} dx \exp(-\int_0^x \frac{b}{a}(y)dy) < \infty. \tag{6.19}$$

To see this, first assume that $f > 0$ satisfies $Af \equiv -g \lneq 0$. Solving $Af = -g$ directly via two integrations reveals that $f > 0$ is impossible if (6.19) does not hold. On the other hand, for any compactly supported $g \geq 0$, if one solves $Af = -g$ for f, one finds that a positive solution f does exist if (6.19) holds.

Assume now that $w^* \not\equiv 0$. Then by the strong maximum principle, $w^* > 0$. The operator $a\frac{d^2}{dx^2} + b\frac{d}{dx} + \hat{V}$ was obtained from the original operator $L = \alpha\frac{d^2}{dx^2}$ by an h-transform followed by multiplication by the scalar $\frac{1}{h^{p-1}}$. Thus the operator A_{w^*} is obtained via h-transform and scalar multiplication from the operator $L - \gamma(w^*)^{p-1}h^{p-1}$. The operator $L - \gamma(w^*)^{p-1}h^{p-1}$ is subcritical since $(L - \gamma(w^*)^{p-1}h^{p-1})1 < 0$. It then follows that A_{w^*} is subcritical.

Since $w^* > 0$, we can make an h-transform with $h = w^*$. Using the fact that $A_{w^*}w^* = 0$, we obtain $A_{w^*}^{w^*} = a\frac{d^2}{dx^2} + B\frac{d}{dx}$, where $B = b + 2a\frac{(w^*)'}{w^*}$. Recalling that Z is a solution of minimal growth at $\pm\infty$ for A_{w^*} and that $w^* \leq CZ$ in a neighborhood of $\pm\infty$, we conclude that $Y \equiv \frac{Z}{w^*}$ is a solution of minimal growth at $\pm\infty$ for $A_{w^*}^{w^*}$ and that $Y \geq C_1$ in a neighborhood of $\pm\infty$, where $C_1 > 0$. Now $A_{w^*}^{w^*}$ is subcritical since A_{w^*} is, and therefore (6.19) holds with b replaced by B. Thus, we can define the function

$$M(x)=\begin{cases}\int_x^\infty dy\exp(-\int_0^y \frac{B}{a}(z)dz), & x>1\\ \int_{-\infty}^x dy\exp(-\int_0^y \frac{B}{a}(z)dz), & x<-1.\end{cases}$$

The fact that M solves $A^{w^*}_{w^*}M = 0$ in a neighborhood of $\pm\infty$ and satisfies $\lim_{|x|\to\infty} M(x) = 0$ contradicts the fact that Y is a solution of minimal growth bounded away from zero. Thus, we conclude that $w^* = 0$.

ii-b. We will prove the claim under the assumption that the condition on α holds for $x > 0$. Under the assumption on the coefficients, the right hand side of (4.3) equal 0 and thus $w^* = 0$. Therefore, by Theorem 4, it suffices to show that there exists a nontrivial, nonnegative solution to the elliptic equation

$$\alpha w'' - w^p = 0 \text{ in } R. \tag{6.20}$$

By the argument following (6.1), we may assume that $\alpha(x) = C|x|^{1+p+\epsilon}$, for $|x| \geq m_0$, where $\epsilon, C > 0$. The maximal, nonnegative solution w_{max} of (6.20) is obtained as $w_{max} = \lim_{k\to\infty}\lim_{m\to\infty} w_{m,k}$ where $w_{m,k}$ satisfies $\alpha w'' - w^p = 0$ in $(-k, k)$ with $w(\pm k) = m$.

Let $W(x) = c(x-m_0)^{1+\frac{\epsilon}{p-1}}$ for $x \geq m_0$ and $W(x) = 0$ for $x < m_0$. Then W is a C^2 function except at $x = m_0$. It is easy to check that if $c > 0$ is sufficiently small, then $\alpha W'' - W^p \geq 0$ for $x \in R - \{m_0\}$. One can easily check that the maximum principle in Proposition 7 goes through in the present case even though W is not twice differentiable at m_0. Thus, $w_{max} \geq W$. □

References

1. Brezis, H., *Semilinear equations in R^n without condition at infinity*, Appl. Math. Opt. **12** (1984), 271-282.
2. Engländer, J., *Criteria for the existence of positive solutions to the equation $\rho\Delta u = u^2$ in R^d for all $d \geq 1$– a new probabilistic approach*, Positivity **4** (2000), 327-337.
3. Engländer, J. and Pinsky, R., *On the construction and support properties of measure-valued diffusions on $D \subset R^d$ with spatially dependent branching*, Ann. of Probab. **27** (1999), 684-730.

4. Fife, P.C. and McLeod, J.B., *A phase plane discussion of convergence to travelling fronts for nonlinear diffusion*, Archive for Rat. Mech. and Anal. **75** (1981), 281-314.
5. Gilbarg, D. and Trudinger, N.S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag Berlin, 1983.
6. Ishige, K. and Murata, M., *An intrinsic metric approach to uniqueness of the positive Cauchy problem for parabolic equations*, Math. Z. **227** (1998), 313-335.
7. Kenig, C. and Ni, W. M., *in exterior Dirichlet problem with applications to some nonlinear equations arising in geometry*, Amer. J. Math. **106** (1984), 689-702.
8. Lieberman, G. M., *Second Order Parabolic Differential Operators*, World Scientific Publishing Co. Singapore, 1996.
9. Lin, F., *On the elliptic equation* $D_i[a_{i,j}(x)D_jU] - k(x)U + K(x)U^p = 0$, Proc. Amer. Math. Soc. (1985), 219-226.
10. Marcus, M. and Veron, L., *Initial trace of positive solutions of some nonlinear parabolic equations.*, Comm. in Partial Diff. Equa. (1999), 1445-1499.
11. Ni, W. M., *On the elliptic equation* $\Delta U + KU^{\frac{n+2}{n-2}} = 0$*, its generalizations and application in geometry*, Indiana J. Math. **4** (1982).
12. Pinsky, R.G., *Positive Harmonic Functions and Diffusions*, Cambridge Univ. Press, 1995.
13. Sattinger, D. H., *Topics in Stability and Bifurcation Theory*, Lecture Notes in Math. 309 (1973), Springer-Verlag.