TRANSIENCE/RECURRENCE AND GROWTH RATES FOR DIFFUSION PROCESSES IN TIME-DEPENDENT REGIONS

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Abstract. Let $\mathcal{K} \subset R^d$, $d \geq 2$, be a smooth, bounded domain satisfying $0 \in \mathcal{K}$, and let $f(t),\ t \geq 0$, be a smooth, continuous, nondecreasing function satisfying $f(0) > 1$. Define $D_t = f(t)\mathcal{K} \subset R^d$. Consider a diffusion process corresponding to the generator $\frac{1}{2}\Delta + b(x)\nabla$ in the time-dependent region $\bar{D}_t$ with normal reflection at the time-dependent boundary. Consider also the one-dimensional diffusion process corresponding to the generator $\frac{1}{2}\frac{d^2}{dx^2} + B(x)\frac{d}{dx}$ on the time-dependent region $[1, f(t)]$ with reflection at the boundary. We give precise conditions for transience/recurrence of the one-dimensional process in terms of the growth rates of $B(x)$ and $f(t)$. In the recurrent case, we also investigate positive recurrence, and in the transient case, we also consider the asymptotic growth rate of the process. Using the one-dimensional results, we give conditions for transience/recurrence of the multi-dimensional process in terms of the growth rates of $B^+(r)$, $B^-(r)$ and $f(t)$, where $B^+(r) = \max_{|x|=r} b(x) \cdot \frac{x}{|x|}$ and $B^-(r) = \min_{|x|=r} b(x) \cdot \frac{x}{|x|}$.

1. Introduction and Statement of Results

Let $\mathcal{K} \subset R^d$, $d \geq 2$, be a bounded domain with C^3-boundary satisfying $0 \in \mathcal{K}$, and let $f(t),\ t \geq 0$, be a continuous, nondecreasing C^3-function satisfying $f(0) > 1$. Define $D_t = f(t)\mathcal{K} \subset R^d$. It is known that one can define a Brownian motion $X(t)$ with normal reflection at the boundary in the time-dependent region $\{(x,t) : x \in \bar{D}_t, t \geq 0\}$. More precisely, one has

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for $0 \le s < t$,

$$X(t) = X(s) + W(t) - W(s) + \int_s^t 1_{\partial D_u}(X(u))n(u, X(u))d\mathcal{L}_u,$$
$$\mathcal{L}_t = \int_s^t 1_{\partial D_u}(X(u))d\mathcal{L}_u,$$

where $W(\cdot)$ is a Brownian motion, $n(u,x)$ is the unit inward normal to D_u at $x \in \partial D_u$ and $\mathcal{L}_u$ is the local time up to time u of $X(\cdot)$ at the time-dependent boundary. See [1].

The process $X(t)$ is *recurrent* if, with probability one, $X(t) \in \mathcal{K}$ at arbitrarily large times t, and is *transient* if, with probability zero, $X(t) \in \mathcal{K}$ at arbitrarily large times t. As with non-degenerate diffusion processes in unrestricted space, transience is equivalent to $\lim_{t\to\infty} |X(t)| = \infty$ with probability one. It is simple to see that the definitions are independent of the starting point and the starting time of the process. In a recent paper [2], it was shown that for $d \ge 3$, if $\int^\infty \frac{1}{f^d(t)}dt < \infty$, then the process is transient, while if $\int^\infty \frac{1}{f^d(t)}dt = \infty$, and an additional technical condition is fulfilled, then the process is recurrent. The additional technical condition is that either $\mathcal{K}$ is a ball, or that $\int_0^\infty (f')^2(t)dt < \infty$. In particular, this result indicates that if for sufficiently large t, $f(t) = ct^a$, for some $c > 0$, then the process is transient if $a > \frac{1}{d}$ and recurrent if $a \le \frac{1}{d}$. The paper [2] also studies the analogous problem for simple, symmetric random walk in growing domains.

In this paper we study the transience/recurrence dichotomy in the case that the Brownian motion is replaced by a diffusion process; namely, Brownian motion with a locally bounded drift $b(x)$. That is, the generator of the process when it is away from the boundary is $\frac{1}{2}\Delta + b(x)\nabla$ instead of $\frac{1}{2}\Delta$. Using the Cameron-Martin-Girsanov change-of-measure formula, or alternatively in the case of a Lipschitz drift, by a direct construction as in [1], one can show that the diffusion process in the time-dependent region can be defined. We will show how the strength of the radial component,

$b(x) \cdot \frac{x}{|x|}$, of the drift, and the growth rate of the region–via $f(t)$–affect the transience/recurrence dichotomy.

In fact, we will prove a transience/recurrence dichotomy for a one-dimensional process. Our result for the multi-dimensional case will follow readily from the one-dimensional result along with results in [2]. Let $f(t)$ be as in the first paragraph. Consider the diffusion process corresponding to the generator $\frac{1}{2}\frac{d^2}{dx^2} + B(x)\frac{d}{dx}$, where B is locally bounded, in the time-dependent region $[1, f(t)]$ with reflection at the endpoint $x = 1$ (for all times) and at the endpoint $f(t)$ at time t. If $B(x) = \frac{k}{x}$, the process is a Bessel process. When this process is considered on $[1, \infty)$ with reflection at 1, it is recurrent for $k \leq \frac{1}{2}$ and transient for $k > \frac{1}{2}$. In particular, it is the radial part of a d-dimensional Brownian motion when $k = \frac{d-1}{2}$. The result of [2] noted above can presumably be slightly modified to show that for $k > \frac{1}{2}$, the process on the time dependent region $[1, f(t)]$ with reflection at the endpoints is transient or recurrent according to whether $\int^\infty \frac{1}{f^{2k+1}(t)}dt < \infty$ or $\int^\infty \frac{1}{f^{2k+1}(t)}dt = \infty$. In this paper we consider drifts that are on a larger order than $\frac{1}{x}$. We will prove the following theorem concerning transience/recurrence.

Theorem 1. *Consider the diffusion process corresponding to the generator* $\frac{1}{2}\frac{d^2}{dx^2} + B(x)\frac{d}{dx}$ *in the time-dependent region* $[1, f(t)]$*, with reflection at both the fixed endpoint and the time-dependent one. Let* $\gamma > -1$ *and* $b, c > 0$.

i. Assume that

$$B(x) \leq bx^\gamma, \quad \textit{for sufficiently large } x,$$

$$f(t) \leq c(\log t)^{\frac{1}{1+\gamma}}, \textit{ for sufficiently large } t.$$

If

$$\frac{2bc^{1+\gamma}}{1+\gamma} < 1, \qquad \textit{or} \quad \frac{2bc^{1+\gamma}}{1+\gamma} = 1 \textit{ and } \gamma \geq -\frac{1}{2},$$

then the process is recurrent.

ii. Assume that

$$B(x) \geq bx^\gamma, \quad \textit{for sufficiently large } x,$$

$$f(t) \geq c(\log t)^{\frac{1}{1+\gamma}}, \textit{ for sufficiently large } t.$$

If

$$\frac{2bc^{1+\gamma}}{1+\gamma} > 1,$$

then the process is transient.

Remark. We expect that the process is also recurrent in part (i) if $\frac{2bc^{1+\gamma}}{1+\gamma} = 1$ and $\gamma \in (-1, -\frac{1}{2})$.

Using Theorem 1, we will prove the following result for the multi-dimensional process.

Theorem 2. *Consider the diffusion process corresponding to the generator $\frac{1}{2}\Delta + b(x)\nabla$ in the time-dependent region $\bar{D}(t) = f(t)\bar{\mathcal{K}} \subset R^d$, with reflection at the boundary, where $\mathcal{K}$ and f are as in the first paragraph. Let*

$$B^+(r) = \max_{|x|=r} b(x) \cdot \frac{x}{|x|}, \quad B^-(r) = \min_{|x|=r} b(x) \cdot \frac{x}{|x|},$$

and let

$$rad^+(\mathcal{K}) = \max(|x| : x \in \partial\mathcal{K}), \quad rad^-(\mathcal{K}) = \min(|x| : x \in \partial\mathcal{K}).$$

Let $\gamma > -1$ and $b, c > 0$.

i. Assume that

$$\begin{aligned} &B^+(r) \le br^\gamma, \quad \textit{for sufficiently large } r, \\ &f(t) \le \frac{c}{rad^+(\mathcal{K})}(\log t)^{\frac{1}{1+\gamma}}, \quad \textit{for sufficiently large } t. \end{aligned} \tag{1.1}$$

Also assume either that $\mathcal{K}$ is a ball or that $\int_0^\infty (f')^2(t)dt < \infty$.

If

$$\frac{2bc^{1+\gamma}}{1+\gamma} < 1, \qquad or \qquad \frac{2bc^{1+\gamma}}{1+\gamma} = 1, \ d = 2 \ and \ \gamma \ge 0,$$

then the process is recurrent.

ii. Assume that

$$\begin{aligned} &B^-(r) \ge br^\gamma, \quad \textit{for sufficiently large } r, \\ &f(t) \ge \frac{c}{rad^-(\mathcal{K})}(\log t)^{\frac{1}{1+\gamma}}, \quad \textit{for sufficiently large } t. \end{aligned} \tag{1.2}$$

If

$$\frac{2bc^{1+\gamma}}{1+\gamma} > 1,$$

then the process is transient.

Remark 1. We expect that the process is recurrent in part (i) when $\frac{2bc^{1+\gamma}}{1+\gamma} = 1$, for all values of $\gamma > -1$ and $d \geq 2$.

Remark 2. If $f(t) = C(\log t)^{\frac{1}{1+\gamma}}$, for all large t, where $C > 0$ and $\gamma > -1$, then the condition $\int_0^\infty (f')^2(t)dt < \infty$ in part (i) is satisfied.

In the recurrent case, it is natural to consider *positive recurrence*, which we define as follows: the one-dimensional process above is positive recurrent if starting from $x > 1$, the expected value of the first hitting time of 1 is finite, while the multi-dimensional process defined above is positive recurrent if starting from a point $x \notin \bar{\mathcal{K}}$, the expected value of the first hitting time of $\bar{\mathcal{K}}$ is finite. It is simple to see that this definition is independent of the starting point and the starting time of the process. We have the following theorem regarding positive recurrence of the one-dimensional process.

Theorem 3. *Under the conditions of part (i) of Theorem 1 or Theorem 2, the process is positive recurrent if*

$$\frac{2bc^{1+\gamma}}{1+\gamma} < 1.$$

Remark. The proof of Theorem 3 relies heavily on the estimates in the proof of part (i) of Theorem 1. We suspect that in the borderline cases, when $\frac{2bc^{1+\gamma}}{1+\gamma} = 1$, the process is never positive recurrent. However, the estimates in the proof of part (ii) of Theorem 1 don't go quite far enough to prove this.

In the transient case, it is natural to consider the asymptotic growth rate of the process. It is known that the process $X(t)$ corresponding to the generator $\frac{1}{2}\frac{d^2}{dx^2} + bx^\gamma \frac{d}{dx}$ on $[1, \infty)$ with reflection at 1 grows a.s. on the order $t^{\frac{1}{1-\gamma}}$ if $\gamma \in (-1, 1)$. (In fact, the solutions $\hat{x}(t)$ to the differential equation $x' = bx^\gamma$ satisfy $\lim_{t\to\infty} \frac{\hat x(t)}{t^{\frac{1}{1-\gamma}}} = (b(1-\gamma))^{\frac{1}{1-\gamma}}$, and it is not hard to show that $X(t)$ satisfies $\lim_{t\to\infty} \frac{X(t)}{t^{\frac{1}{1-\gamma}}} = (b(1-\gamma)^{\frac{1}{1-\gamma}}$ a.s.) The process grows a.s. exponentially if $\gamma = 1$, and explodes a.s. if $\gamma > 1$ [5]. From this it is clear that the one-dimensional process $X(t)$ with $B(x) = bx^\gamma$ on the time-dependent region $[1, f(t)]$ satisfies

$$X(t) = f(t) \text{ for arbitrarily large } t \text{ a.s.,}$$

and consequently,

$$\limsup_{t\to\infty} \frac{X(t)}{f(t)} = 1 \text{ a.s.,} \tag{1.3}$$

if $f(t) = o(t^{\frac{1}{1-\gamma}})$ and $\gamma \in (-1, 1)$, if $f(t)$ grows sub-exponentially and $\gamma = 1$, and with no restrictions on f if $\gamma > 1$. The next theorem treats the behavior of $\liminf_{t\to\infty} \frac{X(t)}{f(t)}$ in what turns out to be the delicate case that $B(x) = bx^\gamma$ and $f(t) = c(\log t)^{\frac{1}{1+\gamma}}$, with $\frac{2bc^{1+\gamma}}{1+\gamma} > 1$. (Recall from Theorem 1 that if $\frac{2bc^{1+\gamma}}{1+\gamma} < 1$, then the process is recurrent and thus $\liminf_{t\to\infty} X(t) = 1$.) We restrict to $\gamma \in (-1, 1)$ for technical reasons, but we suspect that the following result also holds for $\gamma \geq 1$.

Theorem 4. *Consider the diffusion process corresponding to the generator* $\frac{1}{2}\frac{d^2}{dx^2} + B(x)\frac{d}{dx}$ *in the time-dependent region* $[1, f(t)]$*, with reflection at both the fixed endpoint and the time-dependent one. Let* $\gamma \in (-1, 1)$ *and* $b, c > 0$*. Assume that for sufficiently large* x, t*,*

$$B(x) = bx^\gamma,$$
$$f(t) = c(\log t)^{\frac{1}{1+\gamma}},$$

where

$$\frac{2bc^{1+\gamma}}{1+\gamma} > 1.$$

Then

$$\liminf_{t\to\infty} \frac{X(t)}{f(t)} = \big(1 - \frac{1+\gamma}{2bc^{1+\gamma}}\big)^{\frac{1}{1+\gamma}} \ \textit{a.s.}$$

We now consider the asymptotic growth behavior in the case that $B(x) = x^\gamma$, $\gamma \in (-1,1)$, and that $f(t)$ is on a larger order than $(\log t)^{\frac{1}{1+\gamma}}$, but on a smaller order than $t^{\frac{1}{1-\gamma}}$. (Recall from the paragraph preceding Theorem 4 that this latter order is the order on which the process would grow if it lived on $[1,\infty)$ rather than on the time-dependent region.) For simplicity we will assume that $f(t) = (\log t)^l$, with $l > \frac{1}{1+\gamma}$, or that $f(t) = t^l$, with $l \in (0, \frac{1}{1-\gamma})$. (We have dispensed with the coefficients b and c because here they no longer play a role at the level of asymptotic behavior we investigate.)

Theorem 5. *Consider the diffusion process corresponding to the generator* $\frac{1}{2}\frac{d^2}{dx^2} + B(x)\frac{d}{dx}$ *in the time-dependent region* $[1, f(t)]$, *with reflection at both the fixed endpoint and the time-dependent one. Let* $\gamma \in (-1,1)$. *Assume that*

$$B(x) = x^\gamma.$$

i. Assume that for $t \geq 2$,

$$f(t) = (\log t)^l, \text{ with } l > \frac{1}{1+\gamma}.$$

Then

$$\lim_{t\to\infty} \frac{X(t)}{f(t)} = 1 \text{ a.s.}$$

ii. Assume that

$$f(t) = t^l, \text{ with } l \in (0, \frac{1}{1-\gamma}).$$

Let

$$q_0 = \begin{cases} 0, & \text{if } \gamma \geq 0; \\ -l\gamma, & \text{if } \gamma \in (-1,0). \end{cases}$$

Then

$$\limsup_{t\to\infty} \frac{f(t) - X(t)}{t^q} = 0 \text{ a.s. for } q > q_0, \tag{1.4}$$

and

$$\limsup_{t\to\infty} \frac{f(t) - X(t)}{t^{q_0}} = \infty \text{ a.s., when } \gamma \in (-1,0]. \tag{1.5}$$

In particular (in light of (1.3)*),*

$$\lim_{t\to\infty}\frac{X(t)}{f(t)}=1\ a.s.$$

Remark. Note in particular that for $b(x)=x^\gamma$ and $f(t)=t^l$, if $\gamma\in[0,1)$, then the deviation of $X(t)$ from $f(t)$ as $t\to\infty$ is $o(t^q)$, for any $q>0$, while if $\gamma\in(-1,0)$, then this deviation is $o(t^q)$ for $q>-l\gamma$, but not for $q=-l\gamma$.

In section 2 we prove several auxiliary results which will be needed for the proofs. The proofs of Theorem 1-5 are given in sections 3-7 respectively.

Throughout the paper, the following notation will be employed:
Let $X(t)$ denote a canonical, continuous real-valued path, and let $T_\alpha=\inf\{t\ge 0: X(t)=\alpha\}$. Let

$$L_{bx^\gamma}=\frac{1}{2}\frac{d^2}{dx^2}+bx^\gamma\frac{d}{dx}.$$

Let $P_x^{bx^\gamma;\mathrm{Ref}\leftarrow:\beta}$ and $E_x^{bx^\gamma;\mathrm{Ref}\leftarrow:\beta}$ denote probabilities and expectations for the diffusion process corresponding to L_{bx^γ} on $[1,\beta]$, starting from $x\in[1,\beta]$, with reflection at β and stopped at 1, and let $P_x^{bx^\gamma;\mathrm{Ref}\rightarrow:\alpha}$ and $E_x^{bx^\gamma;\mathrm{Ref}\rightarrow:\alpha}$ denote probabilities and expectations for the diffusion process corresponding to L_{bx^γ} on $[\alpha,\infty)$, starting from $x\in[\alpha,\infty)$, with reflection at α. We note that this latter diffusion is explosive if $\gamma>1$, but we will only be considering it until time T_β for some $\beta>\alpha$. We will sometimes work with a constant drift, which we will denote by D (instead of bx^γ with $\gamma=0$), in which case D will replace bx^γ in all of the above notation.

2. Auxiliary Results

In this section we prove four propositions. The first three of them are used explicitly in the proof of Theorem 1, and implicitly in many of the other theorems, since many of the calculations in the proof of Theorem 1 are used in the proofs of the other theorems. Proposition 4 is used only for the proof of (1.5) in Theorem 5.

Proposition 1. *For $\alpha \in [1,\beta]$,*

$$E_x^{bx^\gamma;Ref\leftarrow:\beta}\exp(\lambda T_\alpha) \le 2, \text{ for } x \in [\alpha,\beta] \text{ and } \lambda \le \hat{\lambda}(\alpha,\beta), \tag{2.1}$$

where

$$\hat{\lambda}(\alpha,\beta) = \exp\Big(-\big(2+2b\max(\alpha^\gamma,\beta^\gamma)\big)(\beta-\alpha)\Big). \tag{2.2}$$

Proof. Of course, it suffices to work with $\lambda \ge 0$. Consider the function

$$u(x) = 2 - \exp(-r(x-\alpha)),\ \alpha \le x \le \beta, \tag{2.3}$$

where $r > 0$. Then

(2.4)
$$\exp(r(x-\alpha))(L_{bx^\gamma}+\lambda)u = -\frac{1}{2}r^2 + rbx^\gamma - \lambda + 2\lambda\exp(r(x-\alpha)),\ x \in [\alpha,\beta].$$

For $\lambda \ge 0$,

$$\sup_{x\in[\alpha,\beta]}\big(-\frac{1}{2}r^2 + rbx^\gamma - \lambda + 2\lambda\exp(r(x-\alpha))\big) \le$$
$$-\frac{1}{2}r^2 + rb\max(\alpha^\gamma,\beta^\gamma) - \lambda + 2\lambda\exp(r(\beta-\alpha)).$$

Thus, we have $(L_{bx^\gamma}+\lambda)u \le 0$ on $[\alpha,\beta]$ if

$$0 \le \lambda \le \frac{r\big(\frac{r}{2} - b\max(\alpha^\gamma,\beta^\gamma)\big)}{2\exp(r(\beta-\alpha))-1}.$$

Choosing

$$r = 2 + 2b\max(\alpha^\gamma,\beta^\gamma),$$

it follows that the right hand side of the above inequality is greater than $\hat{\lambda}(\alpha,\beta)$. We have thus shown that there exists a positive function u on $[\alpha,\beta]$ satisfying $(L_{bx^\gamma}+\hat{\lambda}(\alpha,\beta))u \le 0$ in $[\alpha,\beta]$ and $u'(\beta) \ge 0$. By the criticality theory of second order elliptic operators [6, chapter 4], [4], it follows that the principal eigenvalue for $-L_{bx^\gamma}$ on (α,β) with the Dirichlet boundary condition at α and the Neumann boundary condition at β is larger than $\hat{\lambda}(\alpha,\beta)$. By the Feynman-Kac formula, when λ is less than the aforementioned principal eigenvalue, the function $u_\lambda(x) \equiv E_x^{bx^\gamma;\mathrm{Ref}\leftarrow:\beta}\exp(\lambda T_\alpha)$ satisfies the boundary-value problem $(L_{bx^\gamma}+\lambda)u = 0$ in (α,β), $u(\alpha) = 1$

and $u'(\beta)=0$. Since λ is smaller than the principal eigenvalue, it follows from the generalized maximum principal [6, chapter 3], [4] that $u_\lambda\le u$, if u satisfies $(L+\lambda)u\le 0$ in $[\alpha,\beta]$, $u(\alpha)\ge 1$ and $u'(\beta)\ge 0$. The calculation above showed that u as defined in (2.3), with $r=2+2b\max(\alpha^\gamma,\beta^\gamma)$, satisfies these requirements; thus in particular, (2.1) holds. □

Proposition 2. *For* $1\le x<\beta$,

(2.5)
$$E_x^{D;\mathrm{Ref}\to:1}\exp(\frac{D^2}{2}T_\beta)=\frac{\exp(D(\beta-1))}{1+D(\beta-1)}\Big(1+D(x-1)\Big)\exp\big(-D(x-1)\big).$$

Proof. The function
$$u(x)=\frac{\exp(D(\beta-1))}{1+D(\beta-1)}\Big(1+D(x-1)\Big)\exp\big(-D(x-1)\big)$$
solves the boundary value problem $(L_D+\frac{D^2}{2})u=0$ in $(1,\beta)$ with $u'(1)=0$ and $u(\beta)=1$. Since $u>0$, it follows again from the criticality theory of elliptic operators that the principal eigenvalue of $-L_D$ on $(1,\beta)$ with the Neumann boundary condition at 1 and the Dirichlet boundary condition at β is greater than $\frac{D^2}{2}$. Thus, $E_x^{D;\mathrm{Ref}\to:1}\exp(\frac{D^2}{2}T_\beta)<\infty$ and by the Feynman-Kac formula, this function of $x\in[1,\beta]$ solves the above boundary value problem, and consequently coincides with u. □

Proposition 3. *For* $\lambda>0$ *and* $1<\alpha<\beta$,
$$E_\beta^{D;\mathrm{Ref}\leftarrow:\beta}\exp(-\lambda T_\alpha)=$$
$$\frac{2\sqrt{D^2+2\lambda}\,e^{-2D(\beta-\alpha)}}{(-D+\sqrt{D^2+2\lambda}\,)\,e^{(-D+\sqrt{D^2+2\lambda}\,)(\beta-\alpha)}+(D+\sqrt{D^2+2\lambda}\,)\,e^{(-D-\sqrt{D^2+2\lambda}\,)(\beta-\alpha)}}.$$

Proof. By the Feynman-Kac formula, $E_x^{D;\mathrm{Ref}\leftarrow:\beta}\exp(-\lambda T_\alpha)$, for $x\in[\alpha,\beta]$, solves the boundary value problem $(L_D-\lambda)u=0$ in (α,β), with $u(\alpha)=1$ and $u'(\beta)=0$. The solution of this linear equation is given by
$$u(x)=\frac{r_1e^{-r_1(\beta-\alpha)}e^{r_2(x-\alpha)}+r_2e^{r_2(\beta-\alpha)}e^{-r_1(x-\alpha)}}{r_2e^{r_2(\beta-\alpha)}+r_1e^{-r_1(\beta-\alpha)}},$$
where $r_1=D+\sqrt{D^2+2\lambda}$ and $r_2=-D+\sqrt{D^2+2\lambda}$. Substituting $x=\beta$ completes the proof. □

Proposition 4.

$$E_x^{bx^\gamma;Ref\to:\alpha}\exp(\lambda\tau_\beta)\le 2,\ \textit{for } x\in[\alpha,\beta] \textit{ and } \lambda\le\bar\lambda, \tag{2.6}$$

where $\bar\lambda=\frac{b\min(\alpha^\gamma,\beta^\gamma)}{(2e-1)(\beta-\alpha)}$.

Proof. The proof is similar to that of Proposition 1. By the Feynman-Kac formula, when λ is less than the principal eigenvalue for the operator L_{bx^γ} on (α,β) with the Neumann boundary condition at α and the Dirichlet boundary condition at β, the function $u_\lambda(x)\equiv E_x^{bx^\gamma;\mathrm{Ref}\to:\alpha}\exp(\lambda\tau_\beta)$ solves the equation $(L_{bx^\gamma}+\lambda)u=0$ in (α,β), $u'(\alpha)=0$ and $u(\beta)=1$. Also, if $u>0$ satisfies $(L_{bx^\gamma}+\lambda)u\le 0$ in (α,β), $u'(\alpha)\le 0$ and $u(\beta)\ge 1$, then λ is smaller than the principal eigenvalue and $u_\lambda\le u$. We look for such a function u in the form $u(x)=2-\exp\big(-r(\beta-x)\big)$, where $r>0$. Note then that $u(\beta)=1$, $u'(\alpha)\le 0$ and $1\le u\le 2$ on $[\alpha,\beta]$. We have

$$\exp\big(r(\beta-x)\big)\big(L_{bx^\gamma}+\lambda\big)u=(-\frac{1}{2}r^2-bx^\gamma r-\lambda)+2\lambda\exp\big(r(\beta-x)\big).$$

It follows readily that if

$$\lambda\le\frac{\frac{1}{2}r^2+br\min(\alpha^\gamma,\beta^\gamma)}{2\exp(r(\beta-\alpha))-1}, \tag{2.7}$$

then $(L_{bx^\gamma}+\lambda)u\le 0$ on $[\alpha,\beta]$. With the choice $r=\frac{1}{\beta-\alpha}$ in (2.7), it is clear that $\bar\lambda$ in the statement of the proposition is smaller than the right hand side of (2.7). Thus, $u_\lambda(x)\le u(x)\le 2$, for $\lambda\le\bar\lambda$. □

3. Proof of Theorem 1

We will denote probabilities for the process staring from 1 at time 0 by P_1. Let $\mathcal{F}_t=\sigma(X(s),0\le s\le t)$ denote the standard filtration on real-valued continuous paths $X(t)$. By standard comparison results and the fact that the transience/recurrence dichotomy is not affected by a bounded change in the drift over a compact set, we may assume that

(3.1)
$$B(x)=bx^\gamma,\ \text{for all } x\ge 1,\qquad f(t)=\begin{cases}2,\ t\in[0,\exp\big((\frac{2}{c})^{1+\gamma}\big)];\\ c(\log t)^{\frac{1}{1+\gamma}},\ t>\exp\big((\frac{2}{c})^{1+\gamma}\big).\end{cases}$$

Proof of (i). Let $j_0=[(\frac{2}{c})^{1+\gamma}]+1$. Let $t_j=e^j$. Then $f(t_j)=cj^{\frac{1}{1+\gamma}}$, for $j\ge j_0$. For $j\ge j_0$, let A_{j+1} denote the event that the process hits 1 at some time $t\in[t_j,t_{j+1}]$. The conditional version of the Borel-Cantelli lemma [3] shows that if

$$\sum_{j=j_0}^{\infty}P_1(A_{j+1}|\mathcal{F}_{t_j})=\infty,\ \text{a.s.,}\tag{3.2}$$

then $P_1(A_j\ \text{i.o.})=1$, and thus the process is recurrent. Thus, to show recurrence, it suffices to show (3.2).

Since up to time t_j, the largest the process can be is $f(t_j)$, and since up to time t_{j+1} the time-dependent region is contained in $[1,f(t_{j+1})]$, it follows by comparison that

$$P_1(A_{j+1}|\mathcal{F}_{t_j})\ge P_{f(t_j)}^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}(T_1\le t_{j+1}-t_j)\ \text{a.s.}\tag{3.3}$$

We estimate the right hand side of (3.3). Let $\sigma_0^{(j)}=0$, $\kappa_i^{(j)}=\inf\{t\ge\sigma_{i-1}^{(j)}:X(t)=f(t_{j+1})\}$ and $\sigma_i^{(j)}=\inf\{t>\kappa_i^{(j)}:X(t)=f(t_j)\}$, $j\ge j_0,\ i=1,2,\ldots$. For any $l_j\in\mathbb{N}$,

$$\{T_1<\sigma_{l_j}^{(j)}\}-\{\sigma_{l_j}^{(j)}>t_{j+1}-t_j\}\subset\{T_1\le t_{j+1}-t_j\}.$$

Also, it follows by the strong Markov property that

$$P_{f(t_j)}^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}(T_1<\sigma_{l_j}^{(j)})=1-\big(P_{f(t_j)}^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})}<T_1)\big)^{l_j}.$$

Thus

(3.4)
$$P_{f(t_j)}^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}(T_1\le t_{j+1}-t_j)\ge 1-\big(P_{f(t_j)}^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})}<T_1)\big)^{l_j}-P_{f(t_j)}^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}(\sigma_{l_j}^{(j)}>t_{j+1}-t_j).$$

From (3.2)-(3.4), we will obtain $P_1(A_j \text{ i.o.}) = 1$, and thus recurrence, if we can select $\{l_j\}_{j=1}^{\infty}$ such that

$$\sum_{j=j_0}^{\infty}\Big(1-\big(P_{f(t_j)}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})}<T_1)\big)^{l_j}\Big)=\infty, \tag{3.5}$$

and

$$\sum_{j=j_0}^{\infty} P_{f(t_j)}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(t_{j+1})}(\sigma_{l_j}^{(j)}>t_{j+1}-t_j)<\infty. \tag{3.6}$$

Let

$$\phi(x)=\int_x^\infty \exp(-\int_0^t 2bs^\gamma ds)dt=\int_x^\infty \exp(-\frac{2bt^{1+\gamma}}{1+\gamma})dt,\ \ x\ge 1. \tag{3.7}$$

Since $L\phi=0$, it follows by standard probabilistic potential theory [6, chapter 5] that

(3.8)
$$P_{f(t_j)}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})}<T_1)=\frac{\phi(1)-\phi(f(t_j))}{\phi(1)-\phi(f(t_{j+1}))}=1-\frac{\phi(f(t_j))-\phi(f(t_{j+1}))}{\phi(1)-\phi(f(t_{j+1}))}.$$

Applying L'Hôpital's rule shows that

$$\lim_{x\to\infty}\frac{\int_x^\infty \exp(-\frac{2bt^{1+\gamma}}{1+\gamma})dt}{x^{-\gamma}\exp(-\frac{2bx^{1+\gamma}}{1+\gamma})}=\frac{1}{2b};$$

thus,

$$\phi(x)\sim\frac{1}{2b}x^{-\gamma}\exp(-\frac{2bx^{1+\gamma}}{1+\gamma}),\ \text{ as } x\to\infty. \tag{3.9}$$

Using the fact that $(1-t)^l\le\exp(-lt)\le 1-lt+\frac{1}{2}(lt)^2\le 1-\frac{1}{2}lt$, if $l,t\ge 0$ and $lt\le 1$, along with (3.8), we have

$$\begin{aligned}&1-\big(P_{f(t_j)}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})}<T_1)\big)^{l_j}\ge\frac{1}{2}l_j\frac{\phi(f(t_j))-\phi(f(t_{j+1}))}{\phi(1)-\phi(f(t_{j+1}))},\\ &\text{for sufficiently large } j,\ \text{ if } \lim_{j\to\infty} l_j\phi(f(t_j))=0.\end{aligned} \tag{3.10}$$

Using (3.9) along with the facts that $f(x)=c(\log x)^{\frac{1}{1+\gamma}}$ and $t_j=e^j$, it follows that there exists a $K_0\in(0,1)$ such that $\phi(f(t_{j+1}))\le K_0\phi(f(t_j))$ for all large j. Thus,

$$\frac{\phi(f(t_j))-\phi(f(t_{j+1}))}{\phi(1)-\phi(f(t_{j+1}))}\ge K_1\phi(f(t_j))\ge K_2\, j^{-\frac{\gamma}{1+\gamma}}\exp(-\frac{2bc^{1+\gamma}}{1+\gamma}j), \tag{3.11}$$
for sufficiently large j,

for constants $K_1,K_2>0$. From (3.10) and (3.11), it follows that (3.5) will hold if we define $l_j\in\mathbb{N}$ by

$$l_j=[\frac{1}{j^{\frac{1}{1+\gamma}}\log j}\exp(\frac{2bc^{1+\gamma}}{1+\gamma}j)], \tag{3.12}$$

since then the general term, $1-\big(P^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}(T_{f(t_{j+1})}<T_1)\big)^{l_j}$, in (3.5) will be on the order at least $\frac{1}{j\log j}$.

With l_j chosen as above, we now analyze $P^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}(\sigma^{(j)}_{l_j}>t_{j+1}-t_j)$ and show that (3.6) holds. By the strong Markov property, $\sigma^{(j)}_{l_j}=\sum_{i=1}^{l_j}X_i+\sum_{i=1}^{l_j}Y_i$, where $\{X_i\}_{i=1}^\infty$ is an IID sequence distributed according to $T_{f(t_{j+1})}$ under $P^{bx^\gamma;\text{Ref}\to:1}_{f(t_j)}$, $\{Y_i\}_{i=1}^\infty$ is an IID sequence distributed according to $T_{f(t_j)}$ under $P^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}_{f(t_{j+1})}$, and the two IID sequences are independent of one another. By Markov's inequality,

$$\begin{aligned}&P^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}(\sigma^{(j)}_{l_j}>t)\le\exp(-\lambda t)E^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}\exp(\lambda\sigma^{(j)}_{l_j})=\\&\exp(-\lambda t)\big(E^{bx^\gamma;\text{Ref}\to:1}_{f(t_j)}\exp(\lambda T_{f(t_{j+1})})\big)^{l_j}\big(E^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}_{f(t_{j+1})}\exp(\lambda T_{f(t_j)})\big)^{l_j},\end{aligned} \tag{3.13}$$

for any $\lambda>0$.

By Proposition 1,

$$E^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}_{f(t_{j+1})}\exp(\lambda T_{f(t_j)})\le 2,\ \text{for } \lambda\le\hat\lambda(f(t_j),f(t_{j+1})), \tag{3.14}$$

where $\hat\lambda(\cdot,\cdot)$ is as in (2.2). Using the fact that $f(t_j)=cj^{\frac{1}{1+\gamma}}$, it is easy to check that there exists a $\hat\lambda_0>0$ such that

$$\hat\lambda(f(t_j),f(t_{j+1}))\ge\hat\lambda_0,\ \text{for all } j\ge j_0. \tag{3.15}$$

By comparison,

$$E^{bx^\gamma;\text{Ref}\to:1}_{f(t_j)}\exp(\lambda T_{f(t_{j+1})})\le E^{D_j;\text{Ref}\to:1}_{f(t_j)}\exp(\lambda T_{f(t_{j+1})}), \tag{3.16}$$

if

$$D_j \le \min_{x\in[1,f(t_{j+1})]} bx^\gamma.$$

If $\gamma \ge 0$, choose $D_j = \min(b, \sqrt{2\hat\lambda_0}\,)$, for all $j \ge j_0$; thus, $\frac{D_j^2}{2} \le \hat\lambda_0$. If $\gamma \in (-1,0)$, choose $D_j = b(f(t_{j+1}))^\gamma = bc^\gamma(j+1)^{\frac{\gamma}{1+\gamma}}$. With these choices of D_j, we have for all $\gamma > -1$,

$$\frac{D_j^2}{2} \le \hat\lambda_0, \quad \text{for sufficiently large } j. \tag{3.17}$$

It is easy to check that if one substitutes $D = D_j$, $x = f(t_j) = c(\log j)^{\frac{1}{1+\gamma}}$ and $\beta = f(t_{j+1}) = c(\log(j+1))^{\frac{1}{1+\gamma}}$ in the expression on the right hand side of (2.5) in Proposition 2, the resulting expression is bounded in j. Letting $M > 1$ be an upper bound, it follows that

$$E_{f(t_j)}^{D_j;\text{Ref}\to:1} \exp(\frac{D_j^2}{2} T_{f(t_{j+1})}) \le M. \tag{3.18}$$

Noting that $t_{j+1} - t_j = e^{j+1} - e^j \ge e^j$, and choosing $\lambda = \frac{D_j^2}{2}$ in (3.13), it follows from (3.13)-(3.18) that

(3.19)

$$P_{f(t_j)}^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}(\sigma_{l_j}^{(j)} > t_{j+1} - t_j) \le \exp(-\frac{D_j^2}{2} e^j)(2M)^{l_j}, \text{ for sufficiently large } j.$$

Recalling l_j from (3.12), we conclude from (3.19) that

(3.20)

$$P_{f(t_j)}^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}(\sigma_{l_j}^{(j)} > t_{j+1} - t_j) \le \exp(-\frac{D_j^2}{2} e^j)(2M)^{j^{-\frac{1}{1+\gamma}}(\log j)^{-1}\exp(\frac{2bc^{1+\gamma}}{1+\gamma}j)} =$$

$$\exp(-\frac{D_j^2}{2} e^j) \exp\Big(j^{-\frac{1}{1+\gamma}}(\log j)^{-1} e^{\frac{2bc^{1+\gamma}}{1+\gamma}j} \log 2M\Big), \text{ for sufficiently large } j.$$

Recalling that D_j is equal to a positive constant, if $\gamma \ge 0$, and that D_j is on the order $j^{\frac{\gamma}{1+\gamma}}$, if $\gamma < 0$, it follows that the right hand side of (3.20) is summable in j if $\frac{2bc^{1+\gamma}}{1+\gamma} < 1$, or if $\frac{2bc^{1+\gamma}}{1+\gamma} = 1$ and $\gamma \ge -\frac{1}{2}$. Thus (3.6) holds for this range of b, c and γ. This completes the proof of (i).

Proof of (ii). Let $j_1 = [\exp\big((\frac{2}{c})^{1+\gamma}\big)] + 1$. Then $f(j) = c(\log j)^{\frac{1}{1+\gamma}}$, for $j \ge j_1$. For $j \ge j_1$, let B_j be the event that the process hits 1 sometime between the first time it hits $f(j)$ and the first time it hits $f(j+1)$: $B_j = \{X(t) = 1 \text{ for some } t \in (T_{f(j)}, T_{f(j+1)})\}$. If we show that

$$\sum_{j=j_1}^{\infty} P_1(B_j) < \infty, \tag{3.21}$$

then by the Borel-Cantelli lemma it will follow that $P_1(B_j \text{ i.o.}) = 0$, and consequently the process is transient.

To prove (3.21), we need to use different methods depending on whether $\gamma \le 0$ or $\gamma > 0$. We begin with the case $\gamma \le 0$. To consider whether or not the event B_j occurs, we first wait until time $T_{f(j)}$. Of course, necessarily, $T_{f(j)} \ge j$, since $f(j)$ is not accessible to the process before time j. Since we may have $T_{f(j)} < j+1$, the point $f(j+1)$ may not be accessible to the process at time $T_{f(j)}$, however, if we wait one unit of time, then after that, the point $f(j+1)$ certainly will be accessible, since $T_{f(j)} + 1 \ge j+1$. Let $M_j < f(j) - 1$. Now if in that one unit of time, the process never got to the level $f(j) - M_j$, then by comparison, the probability of B_j occurring is no more than $P_{f(j)-M_j}^{bx^\gamma;\text{Ref}\leftarrow:f(j+1)}(T_1 < T_{f(j+1)})$ (because after this one unit of time the process will be at a position greater than or equal to $f(j) - M_j$). By comparison with the process that is reflected at the fixed point $f(j)$, the probability that the process got to the level $f(j) - M_j$ in that one unit of time is bounded from above by $P_{f(j)}^{bx^\gamma;\text{Ref}\leftarrow:f(j)}(T_{f(j)-M_j} \le 1)$. From these considerations, we conclude that

$$P_1(B_j) \le P_{f(j)-M_j}^{bx^\gamma;\text{Ref}\leftarrow:f(j+1)}(T_1 < T_{f(j+1)}) + P_{f(j)}^{bx^\gamma;\text{Ref}\leftarrow:f(j)}(T_{f(j)-M_j} \le 1). \tag{3.22}$$

Similar to (3.8), we have

$$P_{f(j)-M_j}^{bx^\gamma;\text{Ref}\leftarrow:f(j+1)}(T_1 < T_{f(j+1)}) = \frac{\phi(f(j) - M_j) - \phi(f(j+1))}{\phi(1) - \phi(f(j+1))}. \tag{3.23}$$

For $\epsilon \in (0,1)$ to be chosen later sufficiently small, choose $M_j = \epsilon f(j)$. Recall that $f(j) = c(\log j)^{\frac{1}{1+\gamma}}$. Then from (3.9) we have

$$\begin{aligned} &\phi(f(j) - M_j) = \phi\big(c(1-\epsilon)(\log j)^{\frac{1}{1+\gamma}}\big) \sim \\ &\frac{1}{2b}\big(c(1-\epsilon)(\log j)^{\frac{1}{1+\gamma}}\big)^{-\gamma} \exp\big(-\frac{2b(c(1-\epsilon))^{1+\gamma}\log j}{1+\gamma}\big) = \\ &\frac{1}{2b}\big(c(1-\epsilon)(\log j)^{\frac{1}{1+\gamma}}\big)^{-\gamma} j^{-\frac{2b(c(1-\epsilon))^{1+\gamma}}{1+\gamma}}. \end{aligned} \tag{3.24}$$

Since by assumption, $\frac{2bc^{1+\gamma}}{1+\gamma} > 1$, we can select $\epsilon \in (0,1)$ such that $\frac{2b(c(1-\epsilon))^{1+\gamma}}{1+\gamma} > 1$. With such a choice of ϵ, it follows from (3.23) and (3.24) that

$$\sum_{j=j_1}^{\infty} P_{f(j)-M_j}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j+1)}(T_1 < T_{f(j+1)}) < \infty. \tag{3.25}$$

We now estimate $P_{f(j)}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j)}(T_{f(j)-M_j} \le 1)$, where $M_j = \epsilon f(j)$, with ϵ as above. By comparison, we have

$$P_{f(j)}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j)}(T_{f(j)-M_j} \le 1) \le P_{f(j)}^{D_j;\mathrm{Ref}\leftarrow,f(j)}(T_{f(j)-M_j} \le 1), \tag{3.26}$$

where D_j is equal to the minimum of the original drift on the interval $[f(j) - M_j, f(j)]$; that is,

$$D_j = bc^\gamma(\log j)^{\frac{\gamma}{1+\gamma}}.$$

By Markov's inequality, we have for $\lambda > 0$,

$$P_{f(j)}^{D_j;\mathrm{Ref}\leftarrow,f(j)}(T_{f(j)-M_j} \le 1) \le \exp(\lambda) E_{f(j)}^{D_j;\mathrm{Ref}\leftarrow,f(j)} \exp(-\lambda T_{f(j)-M_j}). \tag{3.27}$$

Using Proposition 3 with $\alpha = f(j) - M_j$, $\beta = f(j)$ and $D = D_j$, we have

$$\begin{aligned} &E_{f(j)}^{D_j;\mathrm{Ref}\leftarrow:f(j)} \exp(-\lambda T_{f(j)-M_j}) = \\ &\frac{2\sqrt{D_j^2+2\lambda}\, e^{-2D_jM_j}}{(-D_j + \sqrt{D_j^2+2\lambda})\, e^{(-D_j+\sqrt{D_j^2+2\lambda})M_j} + (D_j + \sqrt{D_j^2+2\lambda})\, e^{(-D_j-\sqrt{D_j^2+2\lambda})M_j}}. \end{aligned} \tag{3.28}$$

If $\gamma < 0$, then $\lim_{j\to\infty} D_j = 0$ and $M_j \to \infty$, and it follows from (3.28) that

$$E_{f(j)}^{D_j;\mathrm{Ref}\leftarrow:f(j)} \exp(-\lambda T_{f(j)-M_j}) \le K \exp(-\sqrt{2\lambda}\, M_j), \tag{3.29}$$

for some $K>0$. If $\gamma=0$, then $D_j=b$, for all j, and we have from (3.28),

(3.30)
$$E^{D_j;\text{Ref}\leftarrow:f(j)}_{f(j)}\exp(-\lambda T_{f(j)-M_j})\sim\frac{2\sqrt{b^2+2\lambda}}{-b+\sqrt{b^2+2\lambda}}\exp\big(-(b+(\sqrt{b^2+2\lambda}\,)M_j\big),$$

as $j\to\infty$.

Since $M_j=\epsilon c(\log j)^{\frac{1}{1+\gamma}}$, it follows from (3.29) and (3.30) that

$$\sum_{j=j_1}^{\infty}E^{D_j;\text{Ref}\leftarrow:f(j)}_{f(j)}\exp(-\lambda T_{f(j)-M_j})<\infty, \tag{3.31}$$

for all choices of $\lambda>0$ in the case $\gamma<0$, and for sufficiently large λ in the case $\gamma=0$. Thus, we conclude from (3.31) and (3.27) that

$$\sum_{j=j_1}^{\infty}P^{D_j;\text{Ref}\leftarrow,f(j)}_{f(j)}(T_{f(j)-M_j}\le 1)<\infty. \tag{3.32}$$

Now (3.21) follows from (3.22), (3.25) and (3.32).

We now turn to the case that $\gamma>0$. Let $\zeta_{j+1}=\inf\{t\ge j+1: X(t)\ge f(j)\}$. Since the process cannot reach $f(j+1)$ before time $j+1$, it follows that $T_{f(j)}\le\zeta_{j+1}\le T_{f(j+1)}$. Let $C_j=\{X(t)=1 \text{ for some } t\in(T_{f(j)},\zeta_{j+1})\}$, and let $G_j=\{X(t)=1\text{ for some } t\in(\zeta_{j+1},T_{f(j+1)})\}$. Then $B_j=C_j\cup G_j$; thus,

$$P_1(B_j)\le P_1(C_j)+P_1(G_j). \tag{3.33}$$

Since the right hand endpoint of the region is larger than or equal to $f(t_{j+1})$ at all times $t\ge\zeta_{j+1}$, it follows by comparison that $P_1(G_j)\le P^{bx^\gamma;\text{Ref}\leftarrow:f(j+1)}_{f(j)}(T_1<T_{f(j+1)})$. Thus, similar to (3.8) we have

$$P_1(G_j)\le\frac{\phi(f(j))-\phi(f(j+1))}{\phi(1)-\phi(f(j+1))}. \tag{3.34}$$

As in (3.24), but with $\epsilon=0$, we have

$$\phi(f(j))\sim\frac{1}{2b}\big(c(\log j)^{\frac{1}{1+\gamma}}\big)^{-\gamma}j^{-\frac{2bc^{1+\gamma}}{1+\gamma}}. \tag{3.35}$$

From (3.34), (3.35) and the fact that $\frac{2bc^{1+\gamma}}{1+\gamma} > 1$, it follows that

$$\sum_{j=j_1}^{\infty} P_1(G_j) < \infty. \tag{3.36}$$

For any s_j, we have the estimate

$$P_1(C_j) \le P_{f(j)}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j)}(T_1 \le s_j + 1) + P_1^{b;\mathrm{Ref}\rightarrow:1}(T_{f(j)} > s_j). \tag{3.37}$$

Here is the explanation for the above estimate. To check whether or not the event C_j occurs, one waits until time $T_{f(j)}$, at which time the process has first reached $f(j)$. Of course $T_{f(j)} \ge j$. If in fact, $T_{f(j)} \ge j+1$, then $\zeta_{j+1} = T_{f(j)}$ and C_j does not occur. Otherwise, one watches the process between time $T_{f(j)}$ and time $j+1$. If the process hit 1 in this time interval, whose length is no more than 1, then C_j occurs. (Note that during this interval of time, the right hand boundary for reflection is always at least $f(j)$.) Otherwise, C_j has not yet occurred, but one continues to watch the process after time $j+1$ until the first time the process is again greater than or equal to $f(j)$. If the process reaches 1 in this interval, then C_j occurs, while if not, then we conclude that C_j did not occur. (Note that if $X(j+1) \ge f(j)$, then the length of this final time interval is 0.) The random variable denoting the length of this final time interval is stochastically dominated by the random variable $T_{f(j)}$ under $P_1^{b;\mathrm{Ref}\rightarrow:1}$, since the actually drift is always larger than or equal to b everywhere, and the actual starting point of the process at the beginning of this final time interval is certainly greater than or equal to 1. In the estimate (3.37), one should think of s_j as a possible value for the length of this final time interval.

We first estimate $P_1^{b;\mathrm{Ref}\rightarrow:1}(T_{f(j)} > s_j)$, the second term on the right hand side of (3.37). By Markov's inequality, for any $\lambda > 0$,

$$P_1^{b;\mathrm{Ref}\rightarrow:1}(T_{f(j)} > s_j) \le \exp(-\lambda s_j) E_1^{b;\mathrm{Ref}\rightarrow:1} \exp(\lambda T_{f(j)}). \tag{3.38}$$

Applying Proposition 2 with $D=b$, $x=1$ and $\beta=f(j)=c(\log j)^{\frac{1}{1+\gamma}}$, we have

$$E_1^{b;\mathrm{Ref}\rightarrow:1}\exp(\frac{b^2}{2}T_{f(j)})=\frac{\exp\Big(b\big(c(\log j)^{\frac{1}{1+\gamma}}-1\big)\Big)}{1+b(c(\log j)^{\frac{1}{1+\gamma}}-1)}. \tag{3.39}$$

Letting

$$s_j=\frac{4}{b^2}\log j, \tag{3.40}$$

it follows from (3.38) with $\lambda=\frac{b^2}{2}$, (3.39) and the fact that $\gamma>0$ that

$$\sum_{j=j_1}^{\infty}P_1^{b;\mathrm{Ref}\rightarrow:1}(T_{f(j)}>s_j)<\infty. \tag{3.41}$$

We now estimate $P_{f(j)}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j)}(T_1\le s_j+1)$, the first term on the right hand side of (3.37), where s_j has now been defined in (3.40). Note that by the strong Markov property, $T_1=T_{[f(t_j)]}+\sum_{i=2}^{[f(t_j)]}(T_i-T_{i-1})$, where $\{T_i-T_{i-1}\}_{i=2}^{[f(t_j)]}$ and $T_{[f(t_j)]}$ are independent random variables under $P_{f(j)}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j)}$, and T_i-T_{i-1} is distributed as T_{i-1} under $P_i^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j)}$. Let $\{X_i\}_{i=2}^{[f(j)]}$ be independent random variables with X_i distributed as T_1 under $P_2^{D_i;\mathrm{Ref}\leftarrow:2}$, where

$$D_i=b(i-1)^\gamma. \tag{3.42}$$

We will use the generic P and E for calculating probabilities and expectations for the X_i. Note that D_i is the minimum of the original drift on the interval $[i-1,i]$. Also note that when one considers T_{i-1} under $P_i^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j)}$, the process gets reflected at $f(j)$, which is to the right of the starting point i, while when one considers T_1 under $P_2^{D_i;\mathrm{Ref}\leftarrow:2}$, the process gets reflected at its starting point. Thus, by comparison, it follows that the distribution of T_i-T_{i-1} under $P_i^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j)}$ dominates the distribution of X_i, and consequently, the distribution of T_1 under $P_{f(j)}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j)}$ dominates the distribution of $\sum_{i=2}^{[f(j)]}X_i$. Thus, we have

$$P_{f(j)}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j)}(T_1\le s_j+1)\le P(\sum_{i=2}^{[f(j)]}X_i\le s_j+1). \tag{3.43}$$

By Markov's inequality, we have for any $\lambda > 0$,

$$P(\sum_{i=2}^{[f(j)]} X_i \le s_j + 1) \le \exp(\lambda(s_j+1)) E \exp(-\lambda \sum_{i=2}^{[f(j)]} X_i) = \exp(\lambda(s_j+1)) \prod_{i=2}^{[f(j)]} E_2^{D_i;\mathrm{Ref}\leftarrow:2} \exp(-\lambda T_1). \tag{3.44}$$

Applying Proposition 3 with $\alpha = 1$, $\beta = 2$ and $D = D_i$, we have

$$E_2^{D_i;\mathrm{Ref}\leftarrow:2} \exp(-\lambda T_1) = \frac{2\sqrt{D_i^2+2\lambda}\, e^{-2D_i}}{(-D_i + \sqrt{D_i^2+2\lambda})\, e^{(-D_i+\sqrt{D_i^2+2\lambda})} + (D_i + \sqrt{D_i^2+2\lambda})\, e^{(-D_i-\sqrt{D_i^2+2\lambda})}}. \tag{3.45}$$

For fixed $\lambda > 0$, $-D_i + \sqrt{D_i^2+2\lambda} \sim \frac{\lambda}{D_i}$, as $D_i \to \infty$. Thus, (3.45) yields

$$E_2^{D_i;\mathrm{Ref}\leftarrow:2} \exp(-\lambda T_1) \sim \frac{2D_i^2}{\lambda} \exp(-2D_i), \text{ as } D_i \to \infty. \tag{3.46}$$

From (3.42) and (3.46), it follows that there exists a $K_0 > 0$ such that

$$\prod_{i=2}^{[f(j)]} E_2^{D_i;\mathrm{Ref}\leftarrow:2} \exp(-\lambda T_1) \le \prod_{i=2}^{[f(j)]} \frac{2D_i^2 K_0}{\lambda} \exp(-2D_i) = \prod_{i=1}^{[f(j)]-1} \frac{2K_0 b^2 i^{2\gamma}}{\lambda} \exp(-2bi^\gamma). \tag{3.47}$$

We have

$$\prod_{i=1}^{[f(j)]-1} i^{2\gamma} \le (f(j))^{2\gamma f(j)} = \left(c(\log j)^{\frac{1}{1+\gamma}}\right)^{2\gamma c(\log j)^{\frac{1}{1+\gamma}}}. \tag{3.48}$$

Also, for some $C_\gamma > 0$,

$$\sum_{i=1}^{[f(j)]-1} i^\gamma \ge \frac{(f(j))^{1+\gamma}}{1+\gamma} - C_\gamma (f(j))^\gamma = \frac{c^{1+\gamma}\log j}{1+\gamma} - C_\gamma c^\gamma (\log j)^{\frac{\gamma}{1+\gamma}};$$

thus,

$$\prod_{i=1}^{[f(j)]-1} \exp(-2bi^\gamma) \le \exp\left(2bC_\gamma c^\gamma (\log j)^{\frac{\gamma}{1+\gamma}}\right) j^{-\frac{2bc^{1+\gamma}}{1+\gamma}}. \tag{3.49}$$

Then from (3.43), (3.44), and (3.47)-(3.49), we have

(3.50)

$$P^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j)}_{f(j)}(T_1\le s_j+1)\le\exp(\lambda(s_j+1))\times$$
$$(1\vee\frac{2K_0b^2}{\lambda})^{c(\log j)^{\frac{1}{1+\gamma}}}\big(c(\log j)^{\frac{1}{1+\gamma}}\big)^{2\gamma c(\log j)^{\frac{1}{1+\gamma}}}\exp\big(2bC_\gamma c^\gamma(\log j)^{\frac{\gamma}{1+\gamma}}\big)\,j^{-\frac{2bc^{1+\gamma}}{1+\gamma}}.$$

From (3.40), $s_j=\frac{4}{b^2}\log j$; so $\exp(\lambda(s_j+1))=e^\lambda j^{\frac{4\lambda}{b^2}}$. By assumption, $\frac{2bc^{1+\gamma}}{1+\gamma}>1$. Thus, choosing $\lambda>0$ sufficiently small so that $\frac{4}{b^2}\lambda-\frac{2bc^{1+\gamma}}{1+\gamma}<-1$, and recalling that $\gamma>0$, it follows from (3.50) that

$$\sum_{j=j_1}^\infty P^{bx^\gamma;\mathrm{Ref}\leftarrow:f(j)}_{f(j)}(T_1\le s_j+1)<\infty. \tag{3.51}$$

(To see this easily, it is useful to convert the long expression on the right hand side of (3.50) to exponential form, similar to what was done in the equality in (3.20).) From (3.37), (3.41) and (3.51) we conclude that

$$\sum_{j=j_1}^\infty P_1(C_j)<\infty. \tag{3.52}$$

Now (3.33), (3.36) and (3.52) give (3.21) and complete the proof of the theorem. □

4. Proof of Theorem 2

First we prove Theorem 2 in the case that $\mathcal{K}$ is a ball. The part of the operator $\frac12\Delta+b\cdot\nabla$ involving radial derivatives is $\frac12\frac{d^2}{dr^2}+(\frac{d-1}{2r}+b(x)\cdot\frac{x}{|x|})\frac{d}{dr}$. Of course, in general, $b(x)\cdot\frac{x}{|x|}$ depends not only on the radial component $r=|x|$ of x, but also on the spherical component $\frac{x}{|x|}$. Let $B^+(r)=\max_{|x|=r}b(x)\cdot\frac{x}{|x|}$ and $B^-(r)=\min_{|x|=r}b(x)\cdot\frac{x}{|x|}$. Then by comparison, if the multi-dimensional process with radial drift $B^+(|x|)\cdot\frac{x}{|x|}$ is recurrent, so is the one with drift $b(x)$, and if the multi-dimensional process with radial drift $B^-(|x|)\cdot\frac{x}{|x|}$ is transient, so is the one with drift $b(x)$. In the case of a radial drift $B(|x|)\cdot\frac{x}{|x|}$, with $\mathcal{K}$ a ball, so that $D_t=f(t)\mathcal{K}$ is a ball, the question of transience/recurrence is equivalent to the question of transience/recurrence considered in Theorem 1 with drift $B(x)+\frac{d-1}{2x}$

and with $D_t = \big(1, \mathrm{rad}(\mathcal{K})\, f(t)\big)$, where $\mathrm{rad}(\mathcal{K})$ is the radius of $\mathcal{K}$. Thus, if $B(r) \equiv B^+(r)$ and $f(t)$ satisfy the inequalities (1.1) in part (i) of Theorem 2 with $\frac{2bc^{1+\gamma}}{1+\gamma} < 1$, then the multi-dimensional process is recurrent, while if $B(r) \equiv B^-(r)$ and $f(t)$ satisfy the inequalities (1.2) in part (ii) of Theorem 2 with $\frac{2bc^{1+\gamma}}{1+\gamma} > 1$, then the multi-dimensional process is transient. (Of course, since $\mathcal{K}$ is a ball, $\mathrm{rad}^{\pm}(\mathcal{K})$ appearing in Theorem 1 are equal to $\mathrm{rad}(\mathcal{K})$.)

Now consider the case that $B(r) \equiv B^+(r)$ and $f(t)$ satisfy the inequalities (1.1) in part (i) of Theorem 2 with $\frac{2bc^{1+\gamma}}{1+\gamma} = 1$. To show recurrence, we need to show recurrence for the one dimensional case when $B(x) = bx^\gamma + \frac{d-1}{2x}$, for large x, and $f(t) = c(\log t)^{\frac{1}{1+\gamma}}$, for large t, with $\frac{2bc^{1+\gamma}}{1+\gamma} = 1$. Thus, the function ϕ appearing in (3.7) must be replaced by

$$\phi(x) = \int_x^\infty \exp\Big(-\int_1^t (2bs^\gamma + \frac{d-1}{s})ds\Big) = C\int_x^\infty t^{1-d}\exp\Big(-\frac{2bt^{1+\gamma}}{1+\gamma}\Big)dt.$$

(Here C is the appropriate constant. In (3.7) we integrated over s starting from 0 for convenience in order to prevent such a constant from entering, however in the present case we can't do this because of the term $\frac{d-1}{s}$.) In place of (3.9), we will now have

$$\phi(x) \sim \frac{C}{2b}x^{-\gamma+1-d}\exp\Big(-\frac{2bx^{1+\gamma}}{1+\gamma}\Big).$$

This causes the term $j^{-\frac{\gamma}{1+\gamma}}$ on the right hand side of (3.11) to be replaced by $j^{-\frac{\gamma+d-1}{1+\gamma}}$, which in turn causes l_j in (3.12) to be changed to $l_j = [\frac{j^{\frac{d-2}{1+\gamma}}}{\log j}\exp(\frac{2bc^{1+\gamma}}{1+\gamma}j)]$. Finally, this causes the term on the right hand side of (3.20) to be changed to $\exp(-\frac{D_j^2}{2}e^j)\exp\Big(j^{\frac{d-2}{1+\gamma}}(\log j)^{-1}e^{\frac{2bc^{1+\gamma}}{1+\gamma}j}\log 2M\Big)$. Recalling that D_j is equal to a positive constant, if $\gamma \geq 0$, and D_j is on the order $j^{\frac{\gamma}{1+\gamma}}$, if $\gamma < 0$, we conclude that if $\frac{2bc^{1+\gamma}}{1+\gamma} = 1$, then the above expression is summable in j if $d = 2$ and $\gamma \geq 0$. This proves recurrence when $\frac{2bc^{1+\gamma}}{1+\gamma} = 1$, $d = 2$ and $\gamma \geq 0$.

We now extend from the radial case to the case of general $\mathcal{K}$. In [2], the proof of a condition for transience was first given for the radial case. The extension to the case of general $\mathcal{K}$, which appears as step III in the proof of Theorem 1.15 in that paper, followed by Lemma 2.1 in that paper. This lemma implies that if one considers two such processes, one corresponding to $\mathcal{K}_1$ and one corresponding to $\mathcal{K}_2$, where $\mathcal{K}_1$ is a ball and $\mathcal{K}_2 \supset \bar{\mathcal{K}}_1$, then the process corresponding to $\mathcal{K}_2$ is transient if the one corresponding to $\mathcal{K}_1$ is transient. Lemma 2.1 goes through just as well when the Brownian motion is replaced by our Brownian motion with drift. This extends our proof of transience to the case of general $\mathcal{K}$.

In [2], the proof of the condition for recurrence also was first given in the radial case. The extension to the general case, which is more involved than in the case of transience, and which requires the additional condition $\int_0^\infty (f')^2(t)dt < \infty$, appears in step V in the proof of Theorem 1.15 in that paper. The analysis in that step also go through when Brownian motion is replaced by our Brownian motion with drift. This extends the proof of recurrence to the case of general $\mathcal{K}$.

□

5. Proof of Theorem 3

We will prove the theorem for the one-dimensional case. The proof for the multi-dimensional case follows from the proof of the one-dimensional case, similar to the way the proof of Theorem 2 follows from the proof of Theorem 1. Let P_2 and E_2 denote probabilities and expectations for the process starting from $x = 2$ at time 0.

Let $t_j = e^j$ as in the proof of part (i) of Theorem 1. We have

$$E_2T_1 \le t_1 + \sum_{j=1}^{\infty} t_{j+1}P_2(T_1 \ge t_j) = e + \sum_{j=1}^{\infty} e^{j+1}P_2(T_1 \ge t_j). \tag{5.1}$$

Recall the definition of j_0 and of A_{j+1} from the beginning of the proof of part (i) of Theorem 1. From (3.3) we have for $j \ge j_0 + 1$,

(5.2)

$$P_2(T_1 \ge t_j) \le P_2(\cap_{i=j_0}^{j-1} A_{i+1}^c) \le \prod_{i=j_0}^{j-1} \Big(1 - P_{f(t_i)}^{bx^\gamma;\mathrm{Ref}\leftarrow:f(t_{i+1})}(T_1 \le t_{i+1} - t_i)\Big).$$

If we show that

$$\lim_{j\to\infty} P^{bx^\gamma;\mathrm{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}(T_1 \le t_{j+1}-t_j) = 1, \tag{5.3}$$

then it will certainly follow from (5.1) and (5.2) that $E_2T_1 < \infty$, proving positive recurrence. In order to prove (5.3), it suffices from (3.4) to prove that for some choice of positive integers $\{l_j\}_{j=j_0}^\infty$,

$$\lim_{j\to\infty} \big(P^{bx^\gamma;\mathrm{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}(T_{f(t_{j+1})} < T_1)\big)^{l_j} = 0 \tag{5.4}$$

and

$$\lim_{j\to\infty} P^{bx^\gamma;\mathrm{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}(\sigma^{(j)}_{l_j} > t_{j+1}-t_j) = 0. \tag{5.5}$$

From (3.8), (3.11) and the fact that $\lim_{y\to\infty}(1-\frac{1}{y})^{yg(y)} = 0$, if $\lim_{y\to\infty} g(y) = \infty$, it follows that (5.4) holds if we choose

$$l_j = [j^{\frac{\gamma}{1+\gamma}}(\log j)\exp(\frac{2bc^{1+\gamma}}{1+\gamma}j)]. \tag{5.6}$$

With this choice of l_j, we have from (3.19),

(5.7)

$$P^{bx^\gamma;\mathrm{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}(\sigma^{(j)}_{l_j} > t_{j+1}-t_j) \le \exp(-\frac{D_j^2}{2}e^j)\exp\big(j^{\frac{\gamma}{1+\gamma}}(\log j)e^{\frac{2bc^{1+\gamma}}{1+\gamma}j}\log 2M\big),$$

where, as noted after (3.20), D_j is equal to a postive constant if $\gamma \ge 0$, and D_j is on the order $j^{\frac{\gamma}{1+\gamma}}$, if $\gamma \in (-1,0)$. Thus, (5.5) follows from (5.7) if $\frac{2bc^{1+\gamma}}{1+\gamma} < 1$ □

6. Proof of Theorem 4

As in the proof of Theorem 1, we can assume that b and f satisfy (3.1). We will first show that

$$\liminf_{t\to\infty}\frac{X(t)}{f(t)} \le \rho \ \text{ a.s., for any } \rho > \big(1-\frac{1+\gamma}{2bc^{1+\gamma}}\big)^{\frac{1}{1+\gamma}}. \tag{6.1}$$

The proof of (6.1) is just a small variant of the proof of recurrence in Theorem 1; that is, part (i) of Theorem 1. As in that proof, let $t_j = e^j$. Recalling the definition of j_0 appearing at the very beginning of the proof of part (i) of Theorem 1, it follows from (3.1) that $f(t_j) = cj^{\frac{1}{1+\gamma}}$, for $j \ge j_0$. In that proof, for $j\ge j_0$, A_{j+1} was defined as the event that the process hits 1 at some time $t\in[t_j,t_{j+1}]$. For the present proof, we define instead, for each $\rho\in(0,1)$, the event $A^{(\rho)}_{j+1}$ that the process $X(t)$ satisfies $X(t)\le\rho f(t_j)$ for some $t\in[t_j,t_{j+1}]$. We mimic the proof of Theorem 1-i up through (3.9), using $A^{(\rho)}_{j+1}$ in place of A_{j+1}, replacing the stopping time T_1 by the stopping time $T_{\rho f(t_j)}$, and replacing $\phi(1)$ by $\phi(\rho f(t_j))$. Instead of (3.10), we obtain

(6.2)
$$1-\big(P^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}(T_{f(t_{j+1})}<T_{\rho f(t_j)})\big)^{l_j}\ge\frac12 l_j\frac{\phi(f(t_j))-\phi(f(t_{j+1}))}{\phi(\rho f(t_j))-\phi(f(t_{j+1}))},$$
$$\text{for sufficiently large } j,\ \text{if}\ \lim_{j\to\infty}l_j\frac{\phi(f(t_j))}{\phi(\rho f(t_j))}=0.$$

Instead of (3.11), we have

(6.3)
$$\frac{\phi(f(t_j))-\phi(f(t_{j+1}))}{\phi(\rho f(t_j))-\phi(f(t_{j+1}))}\ge K_1\frac{\phi(f(t_j))}{\phi(\rho f(t_j))}\ge K_2\exp\big(-\frac{2bc^{1+\gamma}}{1+\gamma}(1-\rho^{1+\gamma})j\big),$$
$$\text{for sufficiently large } j,$$

for constants $K_1,K_2>0$. From (6.2) and (6.3), it follows that (3.5) with T_1 replaced by $T_{\rho f(t_j)}$ will hold if we define $l_j\in\mathbb{N}$ by

$$l_j=[\frac{1}{j\log j}\exp\big(\frac{2bc^{1+\gamma}}{1+\gamma}(1-\rho^{1+\gamma})j\big)],\tag{6.4}$$

since then the general term, $1-\big(P^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}(T_{f(t_{j+1})}<T_{\rho f(t_j)})\big)^{l_j}$, will be on the order at least $\frac{1}{j\log j}$.

We now continue to mimic the proof of Theorem 1-i, starting from the paragraph after (3.12) and up through (3.19). We then insert the present l_j from (6.4) in (3.19) to obtain

(6.5)
$$P^{bx^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}(\sigma^{(j)}_{l_j}>t_{j+1}-t_j)\le\exp(-\frac{D_j^2}{2}e^j)(2M)^{\frac{1}{j\log j}\exp\big(\frac{2bc^{1+\gamma}}{1+\gamma}(1-\rho^{1+\gamma})j\big)}=$$
$$\exp(-\frac{D_j^2}{2}e^j)\exp\Big(\frac{1}{j\log j}e^{\frac{2bc^{1+\gamma}}{1+\gamma}(1-\rho^{1+\gamma})j}\log 2M\Big),\ \text{for sufficiently large } j.$$

Recalling that D_j is equal to a positive constant, if $\gamma\ge0$, and that D_j is on the order $j^{\frac{\gamma}{1+\gamma}}$, if $\gamma<0$, it follows that the right hand side of (6.5) is summable in j if $\frac{2bc^{1+\gamma}}{1+\gamma}(1-\rho^{1+\gamma}) < 1$, or equivalently, if $\rho > \big(1-\frac{1+\gamma}{2bc^{1+\gamma}}\big)^{\frac{1}{1+\gamma}}$. Analogous to the proof of Theorem 1, we conclude then that $P_1(A_j^{(\rho)} \text{ i.o.}) = 1$ for ρ as above. From the definition of $A_j^{(\rho)}$ and the fact that f is increasing, we conclude that (6.1) holds.

To complete the proof of Theorem 4, we will prove that

$$\liminf_{t\to\infty} \frac{X(t)}{f(t)} \ge \rho \text{ a.s., for any } \rho < \big(1-\frac{1+\gamma}{2bc^{1+\gamma}}\big)^{\frac{1}{1+\gamma}}. \tag{6.6}$$

For this direction, we will need some new ingredients. Recalling again the definition of j_0 appearing at the very beginning of the proof of part (i) of Theorem 1, it follows from (3.1) that $f(t) = c(\log t)^{\frac{1}{1+\gamma}}$ for $t \ge e^{j_0}$. Let $\tau_1 = \inf\{t \ge e^{j_0} : X(t) = f(t)\}$, and for $j \ge 2$, let $\tau_j = \inf\{t \ge \tau_{j-1} + 1 : X(t) = f(t)\}$. By the remarks in the paragraph preceding Theorem 4, it follows that $\tau_j < \infty$ a.s. $[P_1]$, for all j. By construction, we have

$$\tau_j > j, \text{ for all } j \ge 1. \tag{6.7}$$

Let $\epsilon \in (0,1)$ and let $\rho \in (0,1)$. Define $s = s(t)$ by

$$\rho(1-2\epsilon)f(s) = \rho(1-\epsilon)f(t), \text{ for } t \ge e^{j_0}. \tag{6.8}$$

Since $f(t) = c(\log t)^{\frac{1}{1+\gamma}}$, we have

$$s(t) = t^{(\frac{1-\epsilon}{1-2\epsilon})^{1+\gamma}}. \tag{6.9}$$

Of course, $f(s(t)) = \frac{1-\epsilon}{1-2\epsilon} f(t)$. For $j \ge j_0$, define B_j to be the event that the following three inequalities hold:

i. $X(t) \ge (1-\epsilon)f(\tau_j)$, $\tau_j \le t \le \tau_j + 1$;

ii. $X(t) \ge \rho(1-\epsilon)f(\tau_j)$, $\tau_j + 1 \le t \le \tau_{j+1}$;

iii. $\tau_{j+1} \le s(\tau_j)$.

(We have suppressed the dependence of B_j on ϵ and ρ.) It follows from (6.8) that on the event B_j one has $X(t) \ge (1-2\epsilon)\rho f(t)$, for all $t \in [\tau_j, \tau_{j+1}]$. Thus, for any N, on the event $\cap_{j=N}^{\infty} B_j$, one has $\liminf_{t\to\infty} \frac{X(t)}{f(t)} \ge (1-2\epsilon)\rho$.

We will complete the proof of (6.6) by showing that

$$\lim_{M\to\infty} P_1(\cap_{j=M}^{\infty} B_j) = 1, \tag{6.10}$$

for all $\rho < \left(1 - \frac{1+\gamma}{2bc^{1+\gamma}}\right)^{\frac{1}{1+\gamma}}$ and all sufficiently small ϵ (depending on ρ).

We write

$$P_1(\cap_{j=M}^{N} B_j) = \prod_{j=M}^{N} P_1(B_j | \cap_{i=M}^{j-1} B_i), \tag{6.11}$$

where $\cap_{i=M}^{M-1} B_i$ denotes the entire probability space. Let

$$C_j = \{X(t) \ge (1-\epsilon) f(\tau_j),\ \tau_j \le t \le \tau_j + 1\}.$$

(Note that C_j depends on the random variable τ_j.) Let $P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}$ denote probabilities for the diffusion process corresponding to L_{bx^γ} without reflection, starting from $(1-\epsilon)f(\tau_j)$. Noting that if $\tau_{j+1} \le s(\tau_j)$, then $X(\tau_{j+1}) = f(\tau_{j+1}) \le f(s(\tau_j))$, it follows by the strong Markov property and comparison that

(6.12)

$$P_1(B_j | \cap_{i=M}^{j-1} B_i, \tau_j) \ge$$

$$P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{\rho(1-\epsilon)f(\tau_j)} > T_{f(s(\tau_j))},\ T_{f(s(\tau_j))} \le s(\tau_j) - \tau_j - 1) - P_1(C_j^c | \tau_j).$$

Also,

(6.13)

$$P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{\rho(1-\epsilon)f(\tau_j)} > T_{f(s(\tau_j))},\ T_{f(s(\tau_j))} \le s(\tau_j) - \tau_j - 1) =$$

$$P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{\rho(1-\epsilon)f(\tau_j)} > T_{f(s(\tau_j))},\ T_{f(s(\tau_j))} \wedge T_{\rho(1-\epsilon)f(\tau_j)} \le s(\tau_j) - \tau_j - 1) \ge$$

$$P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{\rho(1-\epsilon)f(\tau_j)} > T_{f(s(\tau_j))}) -$$

$$P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{f(s(\tau_j))} \wedge T_{\rho(1-\epsilon)f(\tau_j)} \ge s(\tau_j) - \tau_j - 1).$$

In order to get a lower bound on $P_1(B_j | \cap_{i=M}^{j-1} B_i, \tau_j)$, we will bound $P_1(C_j^c|\tau_j)$ and $P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{f(s(\tau_j))} \wedge T_{\rho(1-\epsilon)f(\tau_j)} \ge s(\tau_j) - \tau_j - 1)$ from above, and we will calculate the asymptotic behavior of $P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{\rho(1-\epsilon)f(\tau_j)} > T_{f(s(\tau_j))})$.

We start with $P_1(C_j^c|\tau_j)$. Let P_0^{BM} denote probabilities for a standard Brownian motion starting from 0, and let $\hat{T}_x = \min(T_x, T_{-x})$, for $x > 0$. By the strong Markov property and comparison we clearly have

$$P_1(C_j^c|\tau_j) \le P_0^{\mathrm{BM}}(\hat{T}_{\epsilon f(\tau_j)} \le 1). \tag{6.14}$$

From [6, Theorem 2.2.2], we have $P_0^{\mathrm{BM}}(\hat{T}_x \le t) \le 2\exp(-\frac{x^2}{2t})$. Thus from (6.14) we obtain

$$P_1(C_j^c|\tau_j) \le 2\exp\big(-\frac{1}{2}\epsilon^2 c^2 (\log \tau_j)^{\frac{2}{1+\gamma}}\big). \tag{6.15}$$

We now turn to $P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{f(s(\tau_j))} \wedge T_{\rho(1-\epsilon)f(\tau_j)} \ge s(\tau_j) - \tau_j - 1)$. Denote by $Y(t)$ the diffusion corresponding to the operator $\frac{1}{2}\frac{d^2}{dx^2} + bx^\gamma \frac{d}{dx}$ and the measure $P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}$, and denote by $\mathcal{W}(t)$ standard Brownian motion on $\big(\rho(1-\epsilon)f(\tau_j), \infty)$ with reflection at the endpoint. Denote probabilities for this Brownian motion starting from x by $P_x^{0;\mathrm{Ref}\to:\rho(1-\epsilon)f(\tau_j)}$. (The superscript 0 signifies 0 drift.) Since the drift of the Y diffusion is positive, we can couple $Y(t)$ and $\mathcal{W}(t)$ so that $\mathcal{W}(t) \le Y(t)$, for all $t \in [0, T_{\rho(1-\epsilon)f(\tau_j)} \wedge T_{f(s(\tau_j))}]$, where $T_{\rho(1-\epsilon)f(\tau_j)}$ and $T_{f(s(\tau_j))}$ refer to the hitting times for the Y process. (Note that we have been using the generic T_a for the hitting time of a for any process, the process in question being inferred from the probability measure which appears with it.) Thus, for any $t > 0$,

(6.16)
$$P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{f(s(\tau_j))} \wedge T_{\rho(1-\epsilon)f(\tau_j)} \ge t) \le P^{0;\mathrm{Ref}\to:\rho(1-\epsilon)f(\tau_j)}_{(1-\epsilon)f(\tau_j)}(T_{f(s(\tau_j))} \ge t).$$

For ease of notation, in the analysis below, we let $L_1 = \rho(1-\epsilon)f(\tau_j)$, $L_2 = (1-\epsilon)f(\tau_j)$ and $L_3 = f(s(\tau_j))$. Let P_x^{BM} denote probabilities for a standard Brownian motion starting from x. By the isotropy of Brownian motion and the fact that a reflected Brownian motion can be realized as the absolute value of a Brownian motion, we have

$$P^{0;\mathrm{Ref}\to:\rho(1-\epsilon)f(\tau_j)}_{(1-\epsilon)f(\tau_j)}(T_{f(s(\tau_j))} \ge t) = P^{\mathrm{BM}}_{L_2-L_1}(T_{L_3-L_1} \wedge T_{-(L_3-L_1)} \ge t). \tag{6.17}$$

Using Brownian scaling for the first inequality and symmetry for the second one, we have

$$P^{\mathrm{BM}}_{L_2-L_1}(T_{L_3-L_1}\wedge T_{-(L_3-L_1)}\ge t)=P^{\mathrm{BM}}_{\frac{L_2-L_1}{L_3-L_1}}(T_1\ge \frac{t}{(L_3-L_1)^2})\le P^{\mathrm{BM}}_0(T_1\ge \frac{t}{(L_3-L_1)^2}). \tag{6.18}$$

As is well-known, there exist $\kappa,\lambda>0$ such that $P^{\mathrm{BM}}_0(T_1\ge t)\le \kappa e^{-\lambda t}$, for all $t\ge 0$. Thus, from (6.16)-(6.18), choosing $t=s(\tau_j)-\tau_j-1$, we conclude that

$$P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{f(s(\tau_j))}\wedge T_{\rho(1-\epsilon)f(\tau_j)}\ge s(\tau_j)-\tau_j-1)\le \kappa\exp\big(-\frac{\lambda(s(\tau_j)-\tau_j-1)}{(L_3-L_1)^2}\big). \tag{6.19}$$

We now calculate the asymptotic behavior of $P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{\rho(1-\epsilon)f(\tau_j)}>T_{f(s(\tau_j))})$ via that of $P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{\rho(1-\epsilon)f(\tau_j)}<T_{f(s(\tau_j))})$. Similar to (3.8), we have

$$P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{\rho(1-\epsilon)f(\tau_j)}<T_{f(s(\tau_j))})=\frac{\phi(f(s(\tau_j)))-\phi((1-\epsilon)f(\tau_j))}{\phi(f(s(\tau_j)))-\phi(\rho(1-\epsilon)f(\tau_j))}. \tag{6.20}$$

In light of (6.9) and (3.9), it follows from (6.20) that

$$P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{\rho(1-\epsilon)f(\tau_j)}<T_{f(s(\tau_j))})\sim\frac{\phi((1-\epsilon)f(\tau_j))}{\phi(\rho(1-\epsilon)f(\tau_j))},\ \text{as}\ \tau_j\to\infty. \tag{6.21}$$

From (3.9) and the fact that $f(t)=c(\log t)^{\frac{1}{1+\gamma}}$, we have

$$\frac{\phi((1-\epsilon)f(\tau_j))}{\phi(\rho(1-\epsilon)f(\tau_j))}\sim\rho^\gamma\tau_j^{-\frac{2bc^{1+\gamma}(1-\epsilon)^{1+\gamma}}{1+\gamma}(1-\rho^{1+\gamma})},\ \text{as}\ \tau_j\to\infty. \tag{6.22}$$

Thus, from (6.22) and (6.21),

$$P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{\rho(1-\epsilon)f(\tau_j)}<T_{f(s(\tau_j))})\sim\rho^\gamma\tau_j^{-\frac{2bc^{1+\gamma}(1-\epsilon)^{1+\gamma}}{1+\gamma}(1-\rho^{1+\gamma})},\ \text{as}\ \tau_j\to\infty. \tag{6.23}$$

From (6.12), (6.13), (6.15), (6.19), (6.23) and (6.7) we have

$$P_1(B_j|\cap_{i=M}^{j-1}B_i,\tau_j)\ge 1-2\rho^\gamma\tau_j^{-\frac{2bc^{1+\gamma}(1-\epsilon)^{1+\gamma}}{1+\gamma}(1-\rho^{1+\gamma})}-$$

$$2\exp\big(-\frac{1}{2}\epsilon^2c^2(\log\tau_j)^{\frac{2}{1+\gamma}}\big)-\kappa\exp\big(-\frac{\lambda(s(\tau_j)-\tau_j-1)}{(L_3-L_1)^2}\big), \tag{6.24}$$

for sufficiently large j.

Since

$$\frac{s(\tau_j)-\tau_j-1}{(L_3-L_1)^2}=\frac{\tau_j^{(\frac{1-\epsilon}{1-2\epsilon})^{1+\gamma}}-\tau_j-1}{(\frac{1}{1-2\epsilon}-\rho)^2(1-\epsilon)^2c^2(\log\tau_j)^{\frac{2}{1+\gamma}}},$$

and since $\tau_j>j$ by (6.7), it follows that (6.24) holds with τ_j replaced by j on the right hand side of the inequality. Then taking the conditional expectation with respect to $\cap_{i=M}^{j-1}B_i$ on the left hand side, to remove τ_j from the conditioning there, we conclude that

(6.25)

$$P_1(B_j|\cap_{i=M}^{j-1}B_i)\ge 1-2\rho^\gamma j^{-\frac{2bc^{1+\gamma}(1-\epsilon)^{1+\gamma}}{1+\gamma}(1-\rho^{1+\gamma})}-$$

$$2\exp\big(-\frac{1}{2}\epsilon^2c^2(\log j)^{\frac{2}{1+\gamma}}\big)-\kappa\exp\big(-\frac{j^{(\frac{1-\epsilon}{1-2\epsilon})^{1+\gamma}}-j-1}{(\frac{1}{1-2\epsilon}-\rho)^2(1-\epsilon)^2c^2(\log j)^{\frac{2}{1+\gamma}}}\big),$$

for sufficiently large j.

Clearly, $\sum_{j=j_0}^\infty\exp\big(-\frac{j^{(\frac{1-\epsilon}{1-2\epsilon})^{1+\gamma}}-j-1}{(\frac{1}{1-2\epsilon}-\rho)^2(1-\epsilon)^2c^2(\log j)^{\frac{2}{1+\gamma}}}\big)<\infty$. Since by assumption $\gamma<1$, it follows that $\sum_{j=j_0}^\infty\exp\big(-\frac{1}{2}\epsilon^2c^2(\log j)^{\frac{2}{1+\gamma}}\big)<\infty$. Since $\frac{2bc^{1+\gamma}}{1+\gamma}(1-\rho^{1+\gamma})>1$ is equivalent to $\rho<\big(1-\frac{1+\gamma}{2bc^{1+\gamma}}\big)^{\frac{1}{1+\gamma}}$, we also have $\sum_{j=j_0}^\infty j^{-\frac{2bc^{1+\gamma}(1-\epsilon)^{1+\gamma}}{1+\gamma}(1-\rho^{1+\gamma})}<\infty$, for all $\rho<\big(1-\frac{1+\gamma}{2bc^{1+\gamma}}\big)^{\frac{1}{1+\gamma}}$ and sufficiently small ϵ (depending on ρ). Using these facts with (6.25) and (6.11), we conclude that

$$\lim_{M\to\infty}\lim_{N\to\infty}P_1(\cap_{j=M}^NB_j)=1,$$

which gives (6.10) for all $\rho<\big(1-\frac{1+\gamma}{2bc^{1+\gamma}}\big)^{\frac{1}{1+\gamma}}$ and all sufficiently small ϵ (depending on ρ). □

7. Proof of Theorem 5

Proof of (i). The proof is almost exactly the same as the proof of Theorem 4 starting from (6.6), using $(\log t)^l$ instead of $(\log t)^{\frac{1}{1+\gamma}}$ (and with $b = c = 1$). The one place in the proof where this results in a meaningful difference is in the estimate on $P^{x^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{\rho(1-\epsilon)f(\tau_j)} < T_{f(s(\tau_j))})$. We have (6.21) as in the proof of Theorem 4. From (3.9) and the fact that $f(t) = (\log t)^l$, we have, instead of (6.22),

$$\frac{\phi((1-\epsilon)f(\tau_j))}{\phi(\rho(1-\epsilon)f(\tau_j))} \sim \rho^\gamma \exp\Big(-\frac{2}{1+\gamma}(1-\epsilon)^{1+\gamma}(1-\rho^{1+\gamma})(\log\tau_j)^{l(1+\gamma)}\Big).$$

Using this with (6.21) gives, instead of (6.23),

$$P^{bx^\gamma}_{(1-\epsilon)f(\tau_j)}(T_{\rho(1-\epsilon)f(\tau_j)} < T_{f(s(\tau_j))}) \sim \\ \rho^\gamma \exp\Big(-\frac{2}{1+\gamma}(1-\epsilon)^{1+\gamma}(1-\rho^{1+\gamma})(\log\tau_j)^{l(1+\gamma)}\Big), \text{ as } \tau_j \to \infty.$$

By (6.7), $\tau_j > j$. Since $l(1+\gamma) > 1$, the right hand side above is summable for all $\rho \in (0,1)$. The proof of Theorem 4 then gives $\liminf_{t\to\infty} \frac{X(t)}{f(t)} \geq \rho$ a.s., for all $\rho \in (0,1)$; thus, $\liminf_{t\to\infty} \frac{X(t)}{f(t)} \geq 1$ a.s. Since $X(t) \leq f(t)$, we conclude that $\lim_{t\to\infty} \frac{X(t)}{f(t)} = 1$ a.s.

Proof of (ii). We first prove (1.4). We follow the same kind of strategy used to prove (6.6) in the proof of Theorem 4. Let $\{\tau_j\}_{j=1}^\infty$ be defined as it is following (6.6). Let $\epsilon \in (0,1)$ and $q \in (0,l)$. If we were to define $s(t)$, for $t \geq 1$, by

$$f(s) - (1+\epsilon)s^q = f(t) - t^q \text{ (equivalently, } s^l - (1+\epsilon)s^q = t^l - t^q),$$

we would have $s^l = t^l + (1+\epsilon)s^q - t^q \geq t^l + \epsilon t^q$. In light of this, we define for simplicity $s = s(t)$ by

$$s^l = t^l + \epsilon t^q.$$

Then

$$\begin{aligned} &f(s(t)) = (s(t))^l = t^l + \epsilon t^q; \\ &s(t) = t(1+\epsilon t^{q-l})^{\frac{1}{l}} = t + \epsilon t^{1+q-l} + \text{lower order terms}, \text{ as } t \to \infty. \end{aligned} \tag{7.1}$$

and

$$f(s(t))-(1+\epsilon)(s(t))^q=(s(t))^l-(1+\epsilon)(s(t))^q=$$
(7.2)
$$t^l+\epsilon t^q-(1+\epsilon)(s(t))^q\le t^l-t^q=f(t)-t^q.$$

Let B_j be the event that

i. $X(t)\ge f(\tau_j)-\epsilon\tau_j^q,\ \tau_j\le t\le\tau_j+1;$

ii. $X(t)\ge f(\tau_j)-\tau_j^q,\ \tau_j+1\le t\le\tau_{j+1};$

iii. $\tau_{j+1}\le s(\tau_j)$.

(We have suppressed the dependence of B_j on ϵ and q.) It follows from (7.2) that on the event B_j one has $X(t)\ge f(t)-(1+\epsilon)t^q$, for all $t\in[\tau_j,\tau_{j+1}]$. Thus, for any N, on the event $\cap_{j=N}^\infty B_j$, one has

$$\limsup_{t\to\infty}\big(f(t)-X(t)-(1+\epsilon)t^q\big)\le 0.$$

Therefore, the proof of (1.4) will be completed when we show that

$$\lim_{M\to\infty}P_1(\cap_{j=M}^\infty B_j)=1, \tag{7.3}$$

for some $\epsilon\in(0,1)$ and all $q>q_0$.

We write

$$P_1(\cap_{j=M}^N B_j)=\prod_{j=M}^N P_1(B_j|\cap_{i=M}^{j-1}B_i), \tag{7.4}$$

where $\cap_{i=M}^{M-1}B_i$ denotes the entire probability space. Let

$$C_j=\{X(t)\ge f(\tau_j)-\epsilon\tau_j^q,\ \tau_j\le t\le\tau_j+1\}.$$

(Note that C_j depends on the random variable τ_j.) Let $P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}$ denote probabilities for the diffusion process corresponding to L_{x^γ} without reflection, starting from $f(\tau_j)-\epsilon\tau_j^q$. Noting that if $\tau_{j+1}\le s(\tau_j)$, then $X(\tau_{j+1})=f(\tau_{j+1})\le f(s(\tau_j))$, it follows by the strong Markov property and comparison that

(7.5)
$$P_1(B_j|\cap_{i=M}^{j-1}B_i,\tau_j)\ge$$
$$P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(\tau_j)-\tau_j^q}>T_{f(s(\tau_j))},\ T_{f(s(\tau_j))}\le s(\tau_j)-\tau_j-1)-P_1(C_j^c|\tau_j).$$

Also,

(7.6)

$$P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(\tau_j)-\tau_j^q}>T_{f(s(\tau_j))},\ T_{f(s(\tau_j))}\le s(\tau_j)-\tau_j-1)=$$
$$P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(\tau_j)-\tau_j^q}>T_{f(s(\tau_j))},\ T_{f(s(\tau_j))}\wedge T_{f(\tau_j)-\tau_j^q}\le s(\tau_j)-\tau_j-1)\ge$$
$$P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(\tau_j)-\tau_j^q}>T_{f(s(\tau_j))})-$$
$$P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(s(\tau_j))}\wedge T_{f(\tau_j)-\tau_j^q}\ge s(\tau_j)-\tau_j-1).$$

In order to get a lower bound on $P_1(B_j|\cap_{i=M}^{j-1}B_i,\tau_j)$, we will bound $P_1(C_j^c|\tau_j)$ and $P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(s(\tau_j))}\wedge T_{f(\tau_j)-\tau_j^q}\ge s(\tau_j)-\tau_j-1)$ from above, and we will calculate the asymptotic behavior of $P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(\tau_j)-\tau_j^q}>T_{f(s(\tau_j))})$.

We start with $P_1(C_j^c|\tau_j)$. We mimic the paragraph containing (6.14), the only change being that $\epsilon f(\tau_j)$ is replaced by $\epsilon\tau_j^q$. Thus, similar to (6.15), we obtain

$$P_1(C_j^c|\tau_j)\le 2\exp\big(-\frac12\epsilon^2\tau_j^{2q}\big). \tag{7.7}$$

We now turn to $P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(s(\tau_j))}\wedge T_{f(\tau_j)-\tau_j^q}\ge s(\tau_j)-\tau_j-1)$. We mimic the paragraph following (6.15), the only changes being that $(1-\epsilon)f(\tau_j)$ is replaced by $f(\tau_j)-\epsilon\tau_j^q$, $\rho(1-\epsilon)f(\tau_j)$ is replaced by $f(\tau_j)-\tau_j^q$ and b is set to 1. Similar to (6.19), we obtain,

(7.8)

$$P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(s(\tau_j))}\wedge T_{f(\tau_j)-\tau_j^q}\ge s(\tau_j)-\tau_j-1)\le\kappa\exp\big(-\frac{\lambda(s(\tau_j)-\tau_j-1)}{(L_3-L_1)^2}\big),$$

where $L_3=f(s(\tau_j))$ and $L_1=f(\tau_j)-\tau_j^q$.

We now calculate the asymptotic behavior of $P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(\tau_j)-\tau_j^q}>T_{f(s(\tau_j))})$ via that of $P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(\tau_j)-\tau_j^q}<T_{f(s(\tau_j))})$. Similar to (3.8), we have

$$P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(\tau_j)-\tau_j^q}<T_{f(s(\tau_j))})=\frac{\phi(f(s(\tau_j)))-\phi(f(\tau_j)-\epsilon\tau_j^q)}{\phi(f(s(\tau_j)))-\phi(f(\tau_j)-\tau_j^q)}. \tag{7.9}$$

For any $\lambda\in\mathbb{R}$, we have

$$(t^l+\lambda t^q)^{1+\gamma}=t^{l(1+\gamma)}+\lambda(1+\gamma)t^{l\gamma+q}+\text{ lower order terms as } t\to\infty. \tag{7.10}$$

We now make the assumption, as in the statement of the theorem, that $q > q_0$. Thus, $l\gamma + q > 0$. Using this in (7.10), along with (3.9) (with $b = 1$) and (7.9), and recalling that $f(\tau_j) = \tau_j^l$ and that, from (7.1), $f(s(\tau_j)) = \tau_j^l + \epsilon\tau_j^q$, we conclude that

$$(7.11)\quad P^{x^\gamma}_{f(\tau_j)-\epsilon\tau_j^q}(T_{f(\tau_j)-\tau_j^q} < T_{f(s(\tau_j))}) \sim \exp(-2(1-\epsilon)\tau_j^{l\gamma+q}), \text{ as } \tau_j \to \infty.$$

From (7.5)-(7.8) and (7.11), we have

$$P_1(B_j | \cap_{i=M}^{j-1} B_i, \tau_j) \geq 1 - 2\exp(-2(1-\epsilon)\tau_j^{l\gamma+q}) -$$
$$2\exp\big(-\frac{1}{2}\epsilon^2\tau_j^{2q}\big) - \kappa\exp\big(-\frac{\lambda(s(\tau_j) - \tau_j - 1)}{(L_3 - L_1)^2}\big), \text{ for sufficiently large } \tau_j.$$

From (7.1), we have $s(\tau_j) - \tau_j - 1 \geq \frac{\epsilon}{2}\tau_j^{1+q-l}$ for large τ_j. From (7.8) and (7.1), we have $L_3 - L_1 = f(s(\tau_j)) - f(\tau_j) + \tau_j^q = (1+\epsilon)\tau_j^q$. Thus, for large τ_j,

$$\frac{s(\tau_j) - \tau_j - 1}{(L_3 - L_1)^2} \geq \frac{\epsilon}{2(1+\epsilon)^2}\tau_j^{1-q-l}.$$

If $1 - q - l > 0$, then we can complete the proof just like we completed the proof of Theorem 4 and conclude that (7.3) holds, and thus that (1.4) holds. Note that in order to come to this conclusion, we have needed to assume that $q > q_0 = \max(0, -l\gamma)$ and that $1 - q - l > 0$; that is, we need $\max(0, -l\gamma) < 1 - l$ and $q \in (\max(0, -l\gamma), 1 - l)$. A fundamental assumption in the theorem is that $l \in (0, \frac{1}{1-\gamma})$. For these values of l, the above inequality always holds. Thus, (7.3) holds for those q which are larger than q_0 and sufficiently close to q_0. Consequently, (1.4) holds for all q which are larger than q_0 and sufficiently close to q_0. However, if (1.4) holds for some q, then clearly it also holds for all larger q. Thus, (1.4) holds for all $q > q_0$.

We now turn to the proof of (1.5). We have $\gamma \in (-1, 0]$ and $q_0 = -\gamma l \in [0, l)$. Let $t_j = j^k$, for $j \geq 1$ and some $k > 1$ to be fixed later. For $M > 0$, let A^M_{j+1} be the event that $X(t) \leq f(t_j) - Mt_j^{q_0}$ for some $t \in [t_j, t_{j+1}]$. Clearly, $\limsup_{t\to\infty} \frac{f(t)-X(t)}{t^{q_0}} \geq M$ on the event $\{A^M_j \text{ i.o.}\}$. The conditional version of the Borel-Cantelli lemma [3] shows that if

$$\sum_{j=1}^{\infty} P_1(A^M_{j+1}|\mathcal{F}_{t_j}) = \infty, \tag{7.12}$$

then $P_1(A^M_j \text{ i.o.}) = 1$ Thus, to prove (1.5), it suffices to show that (7.12) holds for all $M > 0$.

Since up to time t_j, the largest the process can be is $f(t_j)$, and since up to time t_{j+1} the time-dependent region is contained in $[1, f(t_{j+1})]$, it follows by comparison that

$$P_1(A^M_{j+1}|\mathcal{F}_{t_j}) \ge P^{x^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}(T_{f(t_j)-Mt_j^{q_0}} \le t_{j+1} - t_j). \tag{7.13}$$

Clearly,

$$\begin{aligned} &P^{x^\gamma;\text{Ref}\leftarrow:f(t_{j+1})}_{f(t_j)}(T_{f(t_j)-Mt_j^{q_0}} \le t_{j+1} - t_j) = \\ &P^{x^\gamma;\text{Ref}\leftrightarrow:f(t_j)-Mt_j^{q_0},f(t_{j+1})}_{f(t_j)}(T_{f(t_j)-Mt_j^{q_0}} \le t_{j+1} - t_j), \end{aligned} \tag{7.14}$$

where $P^{x^\gamma;\text{Ref}\leftrightarrow:f(t_j)-Mt_j^{q_0},f(t_{j+1})}_{f(t_j)}$ corresponds to the L_{x^γ} diffusion with reflection at both $f(t_j) - Mt_j^{q_0}$ and $f(t_{j+1})$.

We estimate the right hand side of (7.14). We have

$$\{T_{f(t_j)-Mt_j^{q_0}} < T_{f(t_{j+1})}\} - \{T_{f(t_{j+1})} > t_{j+1} - t_j\} \subset \{T_{f(t_j)-Mt_j^{q_0}} \le t_{j+1} - t_j\}.$$

Thus,

$$\begin{aligned} &P^{x^\gamma;\text{Ref}\leftrightarrow:f(t_j)-Mt_j^{q_0},f(t_{j+1})}_{f(t_j)}(T_{f(t_j)-Mt_j^{q_0}} \le t_{j+1} - t_j) \ge \\ &P^{x^\gamma;\text{Ref}\leftrightarrow:f(t_j)-Mt_j^{q_0},f(t_{j+1})}_{f(t_j)}(T_{f(t_j)-Mt_j^{q_0}} < T_{f(t_{j+1})}) - \\ &P^{x^\gamma;\text{Ref}\leftrightarrow:f(t_j)-Mt_j^{q_0},f(t_{j+1})}_{f(t_j)}(T_{f(t_{j+1})} > t_{j+1} - t_j) = \\ &P^{x^\gamma}_{f(t_j)}(T_{f(t_j)-Mt_j^{q_0}} < T_{f(t_{j+1})}) - P^{x^\gamma;\text{Ref}\rightarrow:f(t_j)-Mt_j^{q_0}}_{f(t_j)}(T_{f(t_{j+1})} > t_{j+1} - t_j), \end{aligned} \tag{7.15}$$

where $P^{x^\gamma}_{f(t_j)}$ corresponds to the L_{x^γ} diffusion without reflection.

Similar to (3.8), we have

$$P^{x^\gamma}_{f(t_j)}(T_{f(t_j)-Mt_j^{q_0}} < T_{f(t_{j+1})}) = \frac{\phi(f(t_j)) - \phi(f(t_{j+1}))}{\phi(f(t_j) - Mt_j^{q_0}) - \phi(f(t_{j+1}))}, \tag{7.16}$$

where ϕ is as in (3.7) with $b = 1$. We now choose k so that $kl(1+\gamma) > 1$. Then since

$$(f(t_{j+1}))^{1+\gamma} - (f(t_j))^{1+\gamma} = (j+1)^{kl(1+\gamma)} - j^{kl(1+\gamma)}$$

is on the order $j^{kl(1+\gamma)-1}$, it follows from (7.16) and (3.9) that

$$P^{x^\gamma}_{f(t_j)}(T_{f(t_j)-Mt_j^{q_0}} < T_{f(t_{j+1})}) \sim \frac{\phi(f(t_j))}{\phi(f(t_j) - Mt_j^{q_0})}. \tag{7.17}$$

Now

(7.18)

$$(f(t_j))^{1+\gamma} = j^{kl(1+\gamma)};$$

$$(f(t_j) - Mt_j^{q_0})^{1+\gamma} = (j^{kl} - Mj^{-kl\gamma})^{1+\gamma} = j^{kl(1+\gamma)} - M(1+\gamma) + o(1), \text{ as } j \to \infty.$$

Using (7.18) and (3.9), we conclude from (7.17) that

$$\liminf_{j\to\infty} P^{x^\gamma}_{f(t_j)}(T_{f(t_j)-Mt_j^{q_0}} < T_{f(t_{j+1})}) > 0. \tag{7.19}$$

We now consider $P^{x^\gamma;\mathrm{Ref}\to:f(t_j)-Mt_j^{q_0}}_{f(t_j)}(T_{f(t_{j+1})} > t_{j+1} - t_j)$. By Markov's inequality, we have for $\lambda > 0$,

$$\begin{aligned} &P^{x^\gamma;\mathrm{Ref}\to:f(t_j)-Mt_j^{q_0}}_{f(t_j)}(T_{f(t_{j+1})} > t_{j+1} - t_j) \le \\ &\exp\big(-\lambda(t_{j+1}-t_j)\big)E^{x^\gamma;\mathrm{Ref}\to:f(t_j)-Mt_j^{q_0}}_{f(t_j)}\exp(\lambda T_{f(t_{j+1})}). \end{aligned} \tag{7.20}$$

We apply Proposition 4 to the expectation on the right hand side of (7.20), with $\alpha = f(t_j) - Mt_j^{q_0} = j^{kl} - Mj^{-kl\gamma}$, $\beta = f(t_{j+1}) = (j+1)^{kl}$, $b = 1$ and $\lambda = \bar\lambda$, where $\bar\lambda$ is as in the statement of the proposition. Then

$$E^{x^\gamma;\mathrm{Ref}\to:f(t_j)-Mt_j^{q_0}}_{f(t_j)}\exp(\bar\lambda T_{f(t_{j+1})}) \le 2. \tag{7.21}$$

Since $\gamma \le 0$, we have $\min(\alpha^\gamma, \beta^\gamma) = \beta^\gamma$; thus,

$$\bar\lambda = \frac{(j+1)^{kl\gamma}}{(2e-1)\big((j+1)^{kl} - j^{kl} + Mj^{-kl\gamma}\big)}.$$

Now $(j+1)^{kl} - j^{kl}$ is on the order j^{kl-1}. Recall from the previous paragraph that we have chosen k so that $kl(1+\gamma) > 1$. Thus, the denominator on the right hand side above is on the order j^{kl-1}, and thus $\bar\lambda$ is on the order $j^{kl\gamma-kl+1}$. Since $t_{j+1} - t_j = (j+1)^k - j^k$ is on the order j^{k-1}, the expression

$(t_{j+1}-t_j)\bar\lambda$ is on the order $j^{k\big(1-l(1-\gamma)\big)}$. By assumption, $l(1-\gamma)<1$; thus $\lim_{j\to\infty}(t_{j+1}-t_j)\bar\lambda=\infty$. Using this with (7.21), and substituting $\lambda=\bar\lambda$ in (7.20), we conclude that

$$\lim_{j\to\infty} P_{f(t_j)}^{x^\gamma;\text{Ref}\to:f(t_j)-Mt_j^{q_0}}(T_{f(t_{j+1})}>t_{j+1}-t_j)=0. \tag{7.22}$$

From (7.13)-(7.15), (7.19) and (7.22) we conclude that (7.12) holds for any $M>0$. □

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