STRONG LAW OF LARGE NUMBERS AND MIXING FOR THE INVARIANT DISTRIBUTIONS OF MEASURE-VALUED DIFFUSIONS

Ross G. Pinsky

Department of Mathematics
Technion-Israel Institute of Technology
Haifa, 32000 Israel
email:pinsky@techunix.technion.ac.il

Abstract. Let $\mathcal{M}(R^d)$ denote the space of locally finite measures on R^d and let $\mathcal{M}_1(\mathcal{M}(R^d))$ denote the space of probability measures on $\mathcal{M}(R^d)$. Define the *mean measure* π_ν of $\nu \in \mathcal{M}_1(\mathcal{M}(R^d))$ by

$$\pi_\nu(B) = \int_{\mathcal{M}(R^d)} \eta(B) d\nu(\eta), \text{ for } B \subset R^d.$$

For such a measure ν with locally finite mean measure π_ν, let f be a nonnegative, locally bounded test function satisfying $< f, \pi_\nu >= \infty$. ν is said to satisfy the strong law of large numbers with respect to f if $\frac{<f_n,\eta>}{<f_n,\pi_\nu>}$ converges almost surely to 1 with respect to ν as $n \to \infty$, for any increasing sequence $\{f_n\}$ of compactly supported functions which converges to f. ν is said to be mixing with respect to two sequences of sets $\{A_n\}$ and $\{B_n\}$ if

$$\int_{\mathcal{M}(R^d)} f(\eta(A_n))g(\eta(B_n))d\nu(\eta) - \int_{\mathcal{M}(R^d)} f(\eta(A_n))d\nu(\eta) \int_{\mathcal{M}(R^d)} g(\eta(B_n))d\nu(\eta)$$

converges to 0 as $n \to \infty$ for every pair of functions $f, g \in C_b^1([0,\infty))$. It is known that certain classes of measure-valued diffusion processes possess a family of invariant distributions. These distributions belong to $\mathcal{M}_1(\mathcal{M}(R^d))$ and have locally finite mean measures. We prove the strong law of large numbers and mixing for many such distributions.

In this paper we prove a strong law of large numbers and a mixing result for the invariant probability distributions of certain spatially dependent measure-valued diffusions. Because of the spatial dependence, these distributions will not be stationary. Let $\mathcal{M}(R^d)$ denote the space of locally finite measures on R^d, equipped

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with the vague topology, and let $\mathcal{M}_1(\mathcal{M}(R^d))$ denote the space of Borel probability measures on $\mathcal{M}(R^d)$. The invariant probability distributions we will study will belong to $\mathcal{M}_1(\mathcal{M}(R^d))$. We begin by defining a strong law of large numbers and a mixing condition for probability measures $\nu \in \mathcal{M}_1(\mathcal{M}(R^d))$. The *mean measure* π_ν of $\nu \in \mathcal{M}_1(\mathcal{M}(R^d))$ is defined by

$$\pi_\nu(B) = \int_{\mathcal{M}(R^d)} \eta(B) d\nu(\eta), \text{ for measurable } B \subset R^d.$$

(Note that the mean measure π_ν does not necessarily belong to $\mathcal{M}(R^d)$.) If the mean measure possesses a density with respect to Lebesgue measure, the density will also be denoted by π_ν. We will use the notation $< f, \eta >= \int_{R^d} f d\eta$, for $\eta \in \mathcal{M}(R^d)$. Thus, for $f \geq 0$, we have

$$\int_{\mathcal{M}(R^d)} < f, \eta > d\nu(\eta) =< f, \pi_\nu > .$$

Definition 1. Let $\nu \in \mathcal{M}_1(\mathcal{M}(R^d))$ satisfy $\pi_\nu \in \mathcal{M}(R^d)$, and let f be a measurable, locally bounded, nonnegative function satisfying $< f, \pi_\nu >= \infty$. *ν satisfies the strong law of large numbers with respect to f* if

$$\lim_{n\to\infty} \frac{< f_n, \eta >}{< f_n, \pi_\nu >} = 1, \text{ a.s. } [\nu].$$

for any increasing sequence $\{f_n\}$ of measurable, nonnegative, compactly supported functions converging pointwise to f.

Definition 2. Let $\{A_n\}_{n=1}^\infty$, $\{B_n\}_{n=1}^\infty \subset R^d$ be sequences of measurable sets. The measure $\nu \in \mathcal{M}_1(\mathcal{M}(R^d))$ *is mixing with respect to* $\{A_n\}_{n=1}^\infty$ *and* $\{B_n\}_{n=1}^\infty$ *if*

$$\int_{\mathcal{M}(R^d)} f(\eta(A_n))g(\eta(B_n))d\nu(\eta) - \int_{\mathcal{M}(R^d)} f(\eta(A_n))d\nu(\eta) \int_{\mathcal{M}(R^d)} g(\eta(B_n))d\nu(\eta))$$

converges to 0 as $n \to \infty$, for every pair of functions $f, g \in C_b^1([0,\infty))$.

In defining mixing, one typically works with translations $A + x_n$ and $B + y_n$ of fixed sets A and B. This is not appropriate in our setting because the random measure η under ν is not in general stationary under translation. One situation in which it is stationary is when ν is the invariant distribution of super Brownian motion. As is well-known, for $d \geq 3$ there exists a one-parameter family, $\zeta^{(c)} \in \mathcal{M}_1(\mathcal{M}(R^d))$, $c > 0$, of invariant probability distributions for d-dimensional super Brownian motion, and the mean measure of $\zeta^{(c)}$ possesses a density given by $\pi_{\zeta^{(c)}} = c$. These measures are translation invariant and ergodic [2], from which one can obtain a law of large numbers. (Mixing results also exist for these measures [5].) However, when one considers the invariant distributions of measure-valued diffusions whose underlying motion is a spatially dependent diffusion process rather than Brownian motion, the invariant distributions will not be translation invariant and one can not appeal directly to an ergodic theorem.

We now describe the class of measure-valued diffusion processes whose invariant distributions we will be studying. Consider an operator L_0 on R^d of the form

$$L_0 = \frac{1}{2}\nabla \cdot a\nabla + (a\nabla Q)\cdot\nabla = \frac{1}{2}\exp(-2Q)\nabla\cdot\exp(2Q)a\nabla, \tag{1.1}$$

where $a = \{a_{i,j}\}$ is positive definite and $a_{i,j}, Q \in C^{1,\kappa}(R^d)$ for some $\kappa \in (0,1]$. The operator L_0 is symmetric with respect to the density

$$m_{sym} \equiv \exp(2Q).$$

We will assume that the martingale problem for L_0 is well-posed; that is, the diffusion process corresponding to L_0 is conservative (nonexplosive). Because of the symmetry, the diffusion will be reversible. This conservative, reversible diffusion will serve as the underlying motion for the measure-valued diffusion.

The branching mechanism for the measure-valued diffusion is assumed to be of the form $\Phi(x,z) = \beta(x)z - \alpha(x)z^2$, where β is bounded from above, $\alpha > 0$, and $\alpha, \beta \in C^\kappa(R^d)$ for some $\kappa \in (0,1]$. The coefficients β and α should be thought of respectively as the *mass creation* and *variance* parameters for the measure-valued process. Let $X(t), 0 \leq t < \infty$, denote a canonical path with values in $\mathcal{M}(R^d)$. The measure-valued diffusion is an $\mathcal{M}(R^d)$-valued Markov process $X(t) = X(t,\cdot)$ which is uniquely defined via the following log-Laplace equation:

$$E(\exp(-< f, X(t) >)|X(0) = \eta) = \exp(-< u_f(\cdot,t), \eta >), \tag{1.2}$$

for $f \in C_c^+(R^d)$ and for locally finite initial measures η satisfying an appropriate growth condition [13, condition (1.3)], where u_f is the minimal positive solution to the evolution equation

$$\begin{aligned} u_t &= L_0 u + \beta u - \alpha u^2 \text{ in } R^d \times (0,\infty) \\ u(x,0) &= f(x) \text{ in } R^d. \end{aligned} \tag{1.3}$$

For the existence of a minimal positive solution $u_f \in C^{2,1}(R^d \times (0,\infty)) \cap C(R^d \times [0,\infty))$ to (1.3), see [7, Theorem 1]. The measure-valued process may be obtained as a scaled, high-density limit of the point measure-valued process corresponding to independent branching particles which undergo L_0-diffusion and whose branching mechanism is related to the coefficients α and β above [6].

For a probability distribution $\nu \in \mathcal{M}_1(\mathcal{M}(R^d))$ whose support is contained in the set of measures for which (1.2) is well-defined (that is, ν satisfies [13, condition (1.3)]), let P_ν denote the probability measure which corresponds to the Markov process with transition mechanism given by (1.2) and such that $X(0)$ has distribution ν under P_ν. Denote the distribution of $X(t)$ under P_ν by ζ_t, for each $t \geq 0$. A probability measure $\nu \in \mathcal{M}_1(\mathcal{M}(R^d))$ is called an *invariant distribution* if $\zeta_0 = \nu$ implies that $\zeta_t = \nu$, for all $t > 0$.

In order to describe a class of invariant distributions of the above measure-valued diffusions, we need to recall some basic facts concerning the semigroup of the operator

$$L = L_0 + \beta$$

which comprises the linear part of the right hand side of (1.3). Denote by $\mathcal{P}.$ the solution to the martingale problem for L_0 on R^d and denote by $Y(t)$ a canonical diffusion path in $C([0,\infty), R^d)$. Let T_t denote the semigroup corresponding to the operator L on R^d, and let $p(t,x,y)$ denote its transition kernel so that $T_t f(x) = \int_{R^d} p(t,x,y) f(y) dy$, for $f \in C_c^+(R^d)$. (The existence of the density in the case $L = L_0$ follows from [16, chapter 9]; the extension to the case $L = L_0 + \beta$ is easy.) We assume that the operator L on R^d is so-called *subcritical*; that is, that it possesses a Green's function: $G(x,y) = \int_0^\infty p(t,x,y) dt < \infty$, for $x \neq y$. It then follows from the general theory that $\int_B G(x,y)dy < \infty$, for $x \in R^d$ and bounded sets $B \subset R^d$ [12]. The subcriticality condition can be stated probabilistically as follows. The Feynman-Kac formula gives

$$T_t f(x) = \mathcal{E}_x \exp(\int_0^t \beta(Y(s))ds) f(Y(t)),$$

for $f \in C_c^+(R^d)$, where $\mathcal{E}_x$ is the expectation corresponding to $\mathcal{P}_x$. Thus, subcriticality is equivalent to the condition

$$\mathcal{E}_x \int_0^\infty \exp(\int_0^t \beta(Y(s))ds) 1_B(Y(t))dt < \infty, \tag{1.4}$$

for $x \in R^d$ and bounded, open $B \subset R^d$. In particular, if $\beta \equiv 0$, then subcriticality is equivalent to the transience of the diffusion process $Y(t)$.

For later use, we note that it follows from the symmetry property of L_0 that

$$m_{sym}(x)p(t,x,y) = m_{sym}(y)p(t,y,x); \tag{1.5-a}$$

$$m_{sym}(x)G(x,y) = m_{sym}(y)G(y,x). \tag{1.5-b}$$

(We sketch a proof of this at the end of the section.)

A positive function $f \in C^\kappa(R^d)$, for some $\kappa \in (0,1]$, which satisfies $T_t f = f$, for all $t > 0$ is called an *invariant positive function* for the semigroup T_t. If f is invariant, then it is L-harmonic; that is, it satisfies $Lf = 0$. (To see this, multiply both sides of the equality $T_t f = f$ by $\exp(-t)$ and integrate from 0 to ∞ to obtain $\int_{R^d} G_1(x,y)f(y)dy = f(x)$, where G_1 is the Green's function for the operator $L-1$. It then follows essentially from [12, Theorem 4.3.8] that the left hand side above is smooth and satisfies $(L-1)\int_{R^d} G_1(x,y)f(y)dy = -f$. Substituting f for the integrated expression gives $(L-1)f = -f$, and we conclude that $Lf = 0$.)

A positive function $\mu \in C^\kappa(R^d)$, for some $\kappa \in (0,1]$, which satisfies $\mu T_t = \mu$ (i.e., $< \mu, T_t f > = < \mu, f >$, for all bounded, measurable f), for all $t > 0$ is called an *invariant density* for T_t. In the symmetric case under consideration, the connection between invariant densities and invariant positive functions is as follows:

$$\begin{aligned} &\mu \text{ is an invariant density for } T_t \text{ if and only if it is of the form} \\ &\mu = h m_{sym}, \text{ where } h \text{ is an invariant positive function for } T_t. \end{aligned} \tag{1.6}$$

Remark. Note in particular that if $\beta = 0$, then $h = 1$ is an invariant positive function and $\mu = m_{sym}$ is an invariant density (the invariance of $h = 1$ is equivalent to the conservativeness of the diffusion).

We can now state a result concerning invariant distributions of the above class of measure-valued diffusions.

Theorem A[13]. *Assume that $L = L_0 + \beta$ on R^d is subcritical; that is, that it possesses a Green's function. Let $\mu = hm_{sym}$ as in (1.6) be an invariant density for the semigroup T_t corresponding to L. Assume that $\alpha \leq \frac{c}{h}$, for some $c > 0$. Then the measure-valued diffusion satisfying (1.2) possesses an invariant distribution $\zeta^{(\mu)}$ which is uniquely defined by its Laplace-transform:*

(1.7)
$$\int_{\mathcal{M}(R^d)} \exp(-<f,\eta>)d\zeta^{(\mu)}(\eta) = \exp(-<f,\mu> + \int_0^\infty <\alpha u_f^2(\cdot,t),\mu> dt),$$

for $f \in C_c^+(R^d)$. The mean measure of $\zeta^{(\mu)}$ is absolutely continuous with respect to Lebesgue measure and its density is given by $\pi_{\zeta^{(\mu)}} = \mu$. Furthermore, for $f \in C_c^+(R^d)$, the random variables $<T_t f,\eta>$ on $(\mathcal{M}(R^d),\zeta^{(\mu)})$ satisfy

$$\lim_{t\to\infty} <T_t f,\eta> = <f,\mu> \ \ in\ \zeta^{(\mu)} - probability.$$

Remark. In fact, Theorem A was proven under more general conditions; namely, when the operator L_0 on R^d corresponding to a conservative, reversible diffusion is replaced by an operator on an arbitrary domain $D \subset R^d$ with coefficients that are smooth away from the boundary and which corresponds to a reversible but not-necessarily conservative diffusion process with absorption at the boundary ∂D. All the results in this paper go through in this more general context, the only caveat being that the function $h = 1$ is not necessarily invariant when $\beta = 0$ since the diffusion is not necessarily conservative. We have restricted to the case of conservative diffusions on R^d in order to keep the exposition more transparent.

Before stating our first result concerning a law of large numbers for $\zeta^{(\mu)}$, we need to recall a fact from spectral theory. Since β is bounded from above, the operator L acting on smooth compactly supported functions is semibounded and can be extended via the Friedrichs' extension to a unique self-adjoint operator on $L_2(R^d, m_{sym}dx)$ [14] which we will continue to call L. Let $\sigma(L)$ denote the spectrum of L which, by self-adjointness, is contained in the real line. Define

$$\lambda_c(L) = \sup \sigma(L).$$

In fact $\lambda_c(L)$ coincides with the so-called generalized principle eigenvalue for L on R^d, this latter being defined as $\inf\{\lambda : L - \lambda$ possesses a Green's function$\}$ [12]. Thus, in light of (1.4), we have

$$\lambda_c(L) = \inf\{\lambda : \mathcal{E}_x \int_0^\infty \exp(-\lambda t) \exp(\int_0^t \beta(Y(s))ds) 1_B(Y(t))dt < \infty\},$$

for any $x \in R^d$ and any bounded, open $B \subset R^d$. The subcriticality assumption guarantees that $\lambda_c(L) \leq 0$.

Remark. If $\lambda_c(L) > 0$, then the operator L is called *supercritical* and T_t possesses no invariant density. If the mean measure of an invariant distribution is locally finite, that mean measure must be an invariant density for T_t; thus, if $\lambda_c(L) > 0$, the measure-valued diffusion does not possess any invariant distribution with locally finite mean measure [13].

Theorem 1. *Let $\mu = hm_{sym}$ be an invariant density as in (1.6) and let $\zeta^{(\mu)}$ be the invariant distribution defined in Theorem A and whose Laplace transform is given by (1.7). Assume that $\lambda_c(L) < 0$. Then $\zeta^{(\mu)}$ satisfies the strong law of large numbers with respect to any nonnegative f for which $\frac{f}{h}$ is bounded and $< f, \mu >= \infty$; that is, if $\{f_n\}$ is an increasing sequence of compactly supported functions converging to f, then*

$$\lim_{n\to\infty} \frac{< f_n, \eta >}{< f_n, \mu >} = 1 \ a.s. \ [\zeta^{(\mu)}].$$

Remark. As is explained below in the paragraph preceding Theorem 2, it is always true that $\int_{R^d} h^2 m_{sym} dx = \infty$; thus, the requirements on f in Theorem 1 will be satisfied in particular if $c_1 h \leq f \leq c_2 h$, for positive constants c_1, c_2. Also note that if h is bounded away from 0 (which occurs in particular if $\beta = 0$ and one chooses $h = 1$), then $f = 1_A$ is admissible if $A \subset R^d$ satisfies $\mu(A) = \int_A hm_{sym}dx = \infty$. Thus, choosing bounded sets A_n which increase to A, it follows from Theorem 1 that $\lim_{n\to\infty} \frac{\eta(A_n)}{\mu(A_n)} = 1$ a.s. $[\zeta^{(\mu)}]$.

We now present briefly a couple of multi-dimensional examples where $\lambda_c(L) < 0$ so that Theorem 1 applies and where furthermore the invariant positive harmonic functions h can be calculated explicitly so that the invariant distributions $\zeta^{(\mu)}$ may be exhibited explicitly via Theorem A.

Example 1. Let $L_0 = \frac{1}{2}\Delta + kx \cdot \nabla$ for some $k > 0$ be a "transient Ornstein Uhlenbeck process." Note that $m_{sym} = \exp(k|x|^2)$. One can show that $\lambda_c(L_0) = -kd$, and thus $\lambda_c(L) < 0$ if $\sup \beta < kd$. When β is constant, the cone of positive harmonic functions is generated by a collection of minimal elements which are in one to one correspondence with the sphere S^{d-1}, and all of these positive harmonic functions are invariant positive functions. In particular, in the case that $\beta = 0$, this collection of minimal invariant positive functions, $\{h_v\}_{v\in S^{d-1}}$, may be represented explicitly (as can be deduced from [4]): $h_v(x) = \int_{-\infty}^{\infty} \exp(-k|x - ve^{kt}|^2)\exp(2kt)dt$.

Example 2. Let $L_0 = \frac{1}{2}\Delta$ in $R^d, d \geq 1$, and let $\beta = -c$, for some constant $c > 0$. Then $\lambda_c(L) = -c$. The cone of positive harmonic functions is again generated by a collection of minimal elements which are in one to one correspondence with the sphere S^{d-1}, and all of these positive harmonic functions are invariant positive functions [12]. These minimal invariant positive functions, $\{h_v\}_{v\in S^{d-1}}$, can be represented explicitly as follows: $h_v(x) = \exp((2c)^{\frac{1}{2}} v \cdot x)$.

The second example above treated the case of *subcritical* super-Brownian motion. The classical *critical* super-Brownian motion ($L_0 = \frac{1}{2}\Delta, \beta = 0, \alpha = 1$) is not covered by Theorem 1 since $\lambda_c(L) = 0$ in this case. We now present an alternative sufficiency condition for a strong law of large numbers to hold as well as a sufficiency condition for a weak law to hold. These alternative conditions cover, in particular, the critical super-Brownian motion.

Before stating the theorem, we recall a few basic facts about h-transforms. Let

h be invariant for the semigroup T_t and let L^h denote the h-transform of L: $L^h f = \frac{1}{h}L(hf)$; equivalently,

$$L^h = L_0 + a\frac{\nabla h}{h} \cdot \nabla.$$

The semigroup T_t^h corresponding to L^h satisfies $T_t^h f = \frac{1}{h}T_t hf$ and the measure m_{sym}^h with respect to which L^h is symmetric is given by

$$m_{sym}^h = h^2 m_{sym}.$$

Since h is invariant for T_t, the function 1 is invariant for T_t^h; thus, the operator L^h corresponds to a conservative diffusion process $P_\cdot^h$. Subcriticality is preserved under h-transforms; thus L^h on R^d is subcritical, that is, the diffusion process $P_\cdot^h$ is transient. From this it follows that $\int_{R^d} h^2 m_{sym} dx = \infty$. Indeed, $m_{sym}^h = h^2 m_{sym}$ is an invariant density for the conservative diffusion process L^h. If it were integrable, then the diffusion process would be positive recurrent. The Green's function $G^h(x, y)$ for L^h satisfies

$$G^h(x, y) = G(x, y)\frac{h(y)}{h(x)}.$$

Theorem 2. *Let $\mu = hm_{sym}$ be an invariant density as in (1.6) and let $\zeta^{(\mu)}$ be the invariant distribution defined in Theorem A and whose Laplace transform is given by (1.7). Assume that it is possible to choose for each $x \in R^d$, a sequence $\{D_n(x)\}_{n=1}^\infty$ of sets such that*

$$\gamma_n \equiv \sup_{x\in R^d} \sup_{y\in D_n(x)} \frac{G^h(x, y)}{m_{sym}^h(y)} \text{ satisfies } \lim_{n\to\infty} \gamma_n = 0,$$

and such that

$$l_n \equiv \sup_{x\in R^d} \int_{y\in R^d - D_n(x)} G^h(x, y)dy < \infty, \text{ for some } n = 1, 2, ...$$

Then the invariant distribution $\zeta^{(\mu)}$ satisfies the weak law of large numbers with respect to any nonnegative f for which $\frac{f}{h}$ is bounded and $< f, \mu >= \infty$; that is, if $\{f_n\}$ is an increasing sequence of compactly supported functions converging to f, then

$$\frac{< f_n, \eta >}{< f_n, \mu >} \text{ converges in } \zeta^{(\mu)} - \text{probability to } 1.$$

If one can choose the sequence $\{D_n(x)\}$ such that $\sum_{n=1}^{\infty}\gamma_n < \infty$, and such that $l_n < \infty$ for all n and $\limsup_{n\to\infty}\frac{\hat{l}_{n+1}}{\hat{l}_n} < \infty$, for a nondecreasing sequence $\{\hat{l}_n\}$ satisfying $\hat{l}_n \ge l_n$, then $\zeta^{(\mu)}$ satisfies the strong law of large numbers with respect to $\{f_n\}$; that is,

$$\lim_{n\to\infty}\frac{< f_n,\eta >}{< f_n,\mu >} = 1 \ a.s.\ [\zeta^{(\mu)}].$$

Remark 1. The remark after Theorem 1 concerning the class of admissible functions f and the possibility of choosing $f = 1_A$ holds just as well, of course, for Theorem 2.

Remark 2. As will be seen in the proofs of Corollaries 1 and 2 below, in typical cases the sets $D_n(x)$ will be decreasing in n and satisfy $\cap_{n=1}^{\infty}D_n(x) = \emptyset$.

Remark 3. In Theorem 1, where the strict negativity condition on the spectrum is assumed, the requirements in order that a strong law of large numbers hold for a function f and an invariant distribution $\zeta^{(\mu)}$ depended on the particular distribution in question—that is, on $\mu = hm_{sym}$— only through the requirement that $\frac{f}{h}$ be bounded and that $< f,\mu >= \infty$, this latter condition of course being necessary in order to even consider a law of large numbers. In Theorem 2 however, where the strict negativity condition on the spectrum is no longer assumed, the requirements depend more heavily on the particular distribution via the function h.

We now apply Theorem 2 to two different classes of measure-valued diffusions. The first class contains the critical super-Brownian motion as a particular case. The operator $L_0 = \frac{1}{2}\nabla\cdot a\nabla$ is *uniformly elliptic* if there exists a constant $c > 0$ such that $c|v|^2 \le \sum_{i,j=1}^{d} a_{i,j}v_iv_j \le \frac{1}{c}|v|^2$, for all $v \in R^d$. If L_0 is uniformly elliptic and $\beta = 0$, then $L = L_0$ is subcritical if $d \ge 3$. The invariant positive functions are $h = c$, where $c > 0$ is a constant. A proof of this can be found after (2.17). The symmetric density is $m_{sym} = 1$. Thus, from (1.6), it follows that the invariant densities are of the form $\mu = hm_{sym} = c$. Let $\zeta^{(c)}$ denote the corresponding invariant distribution as described in Theorem A.

Corollary 1. *Assume that $\beta = 0$ and that $L = L_0 = \frac{1}{2}\nabla \cdot a\nabla$ is uniformly elliptic in R^d, $d \geq 3$. Let $\zeta^{(c)}$, $c > 0$, be the invariant distributions as in Theorem A. Then $\zeta^{(c)}$ satisfies the strong law of large numbers for any nonnegative, bounded f for which $\int_{R^d} f dx = \infty$; that is, if $\{f_n\}$ is an increasing sequence of compactly supported functions converging to f, then*

$$\lim_{n\to\infty} \frac{< f_n, \eta >}{\int_{R^d} f_n dx} = c \ a.s. \ [\zeta^{(c)}].$$

The second corollary of Theorem 2 treats one-dimensional measure-valued diffusions. Before stating the result, we need to make a few remarks about one-dimensional diffusion processes. Let

$$L_0 = \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \tag{1.8}$$

with $0 < a \in C^{1,\kappa}(R)$ and $b \in C^{\kappa}(R)$, for $\kappa \in (0, 1]$. Every such one-dimensional operator may be put into divergence form ((1.1)) as follows: $L = \frac{1}{2}\frac{1}{m_{sym}(x)}\frac{d}{dx}m_{sym}(x)a(x)\frac{d}{dx}$, where $m_{sym}(x) = \frac{1}{a(x)}\exp(\int_0^x \frac{2b}{a}(y)dy)$. We continue to assume that the diffusion process corresponding to L_0 is conservative (a condition which in the one-dimensional case at hand can be checked by subjecting the coefficients a and b to Feller's integral test [12, chapter 5]). The operator L_0 is subcritical (i.e., possesses a Green's function) if and only if it corresponds to a transient diffusion process, and this will occur if and only if $\exp(-\int_0^x \frac{2b}{a}(y)dy)$ is integrable at either $+\infty$ or $-\infty$. If it is integrable only at $+\infty$ $(-\infty)$, then the diffusion process will run off to $+\infty$ $(-\infty)$ with probability one and we will say that the diffusion is transient to $+\infty$ $(-\infty)$. On the other hand, if it is integrable at both $+\infty$ and $-\infty$, then the diffusion process will have a positive probability of running off to $+\infty$ and a positive probability of running off to $-\infty$, and we will say that the diffusion is transient to both $+\infty$ and $-\infty$. (See [12, chapter 5] for the above facts.)

Now let L_0 be as above (conservative, but not necessarily subcritical) and let $L = L_0 + \beta$. Assume that L is subcritical and that h is an invariant positive function for the corresponding semigroup T_t. Then the h-transformed operator is given by $L^h = L_0 + \frac{ah'}{h}\frac{d}{dx}$. This new operator is of the same form as L_0— the diffusion coefficient is still a but the drift is now $b + \frac{ah'}{h}$ instead of b. Since subcriticality is preserved under h-transforms, the operator L^h will correspond to a transient diffusion and, substituting $b + \frac{ah'}{h}$ for b above, it follows that it will be transient to $+\infty$(resp. to $-\infty$, resp. to both $+\infty$ and $-\infty$) if and only if $\frac{1}{h^2(x)}\exp(-\int_0^x \frac{2b}{a}(y)dy)$ is integrable only at $+\infty$ (resp. only at $-\infty$, resp. at both $+\infty$ and $-\infty$).

Corollary 2. *Let $L = L_0 + \beta$ be subcritical, where L_0 is a one-dimensional operator as in (1.8). Let $\mu = hm_{sym}$ be an invariant density as in (1.6) and let $\zeta^{(\mu)}$ be the invariant distribution defined in Theorem A. Let $f \geq 0$ and assume that $\frac{f}{h}$ is bounded and $< f, \mu >= \infty$. Define*

$$H(x) = \int_0^x dr \frac{1}{h^2(r)} \exp(-\int_0^r \frac{2b}{a}(s)ds)$$

Recall that the subcriticality assumption guarantees that at least one of $H(-\infty)$ and $H(\infty)$ is finite.

Case i. Assume that both $H(-\infty)$ and $H(\infty)$ are finite. Then $\zeta^{(\mu)}$ satisfies the weak law of large numbers with respect to f; that is, if $\{f_n\}$ is an increasing sequence of compactly supported functions converging to f, then

$$\frac{< f_n, \eta >}{< f_n, \mu >} \text{ converges in } \zeta^{(\mu)} - \text{probability to } 1. \tag{1.9}$$

If in addition,

$$\limsup_{n\to\infty} \frac{\int_{-z_{n+1}}^{z_{n+1}} m^h_{sym}(y)dy}{\int_{-z_n}^{z_n} m^h_{sym}(y)dy < \infty} < \infty,$$

for some increasing, positive sequence $\{z_n\}_{n=1}^\infty$ satisfying

$$\sum_{n=1}^\infty (H(\infty) - H(z_n)) + (H(-z_n) - H(-\infty)) < \infty,$$

then $\zeta^{(\mu)}$ satisfies the strong law of large numbers with respect to $\{f_n\}$; that is,

$$\lim_{n\to\infty} \frac{< f_n, \eta >}{< f_n, \mu >} = 1 \ a.s. \ [\zeta^{(\mu)}]. \tag{1.10}$$

Case ii. Assume that $H(\infty)=\infty$ (resp. $H(-\infty)=-\infty$). Assume in addition that f is supported away from $+\infty$(resp. $-\infty$). Then $\zeta^{(\mu)}$ satisfies the weak law of large numbers with respect to f; that is, (1.9) occurs. If in addition,

$$\limsup_{n\to\infty}\frac{\int_{-z_{n+1}}^{0} m_{sym}^h(y)dy}{\int_{-z_n}^{0} m_{sym}^h(y)dy}<\infty,\ (\limsup_{n\to\infty}\frac{\int_0^{z_{n+1}} m_{sym}^h(y)dy}{\int_0^{z_n} m_{sym}^h(y)dy}<\infty),$$

for some increasing, positive sequence $\{z_n\}_{n=1}^{\infty}$ satisfying

$$\sum_{n=1}^{\infty}(H(-z_n)-H(-\infty))<\infty\ (resp. \sum_{n=1}^{\infty}(H(\infty)-H(z_n))<\infty),$$

then $\zeta^{(\mu)}$ satisfies the strong law of large numbers with respect to $\{f_n\}$; that is, (1.10) occurs.

Remark. If case (i) of the corollary holds for the operator $L=L_0+\beta$, then we obtain a law of large numbers for functions f supported on the entire line, while if case (ii) holds, then we require that f be supported on a half-line. We are not sure whether this restriction is essential, or whether it is just a result of the technical limitations of the method of proof. One might wonder whether Theorem 1 could be used to extend case (ii) to f supported on the entire line. However, it cannot because one can show that in case (ii), $\sigma(L)$ is always equal to 0.

We now turn to the result on mixing, for which we will need the following assumption.

Assumption 1. The two sequences $\{A_n^{(i)}\}_{n=1}^{\infty}$, $i=1,2$, of measurable sets in R^d satisfy the following condition:

$$\lim_{n\to\infty}\int_{A_n^{(1)}}\int_{A_n^{(2)}} G(x,y)m_{sym}(x)dxdy=0.$$

Remark. Since $m_{sym}(x)G(x,y)=m_{sym}(y)G(y,x)$, Assumption 1 is symmetric in $\{A_n^{(1)}\}_{n=1}^{\infty}$ and $\{A_n^{(2)}\}_{n=1}^{\infty}$.

Theorem 3. *Let $\mu=hm_{sym}$ be an invariant density as in (1.6) and let $\zeta^{(\mu)}$ be the invariant distribution defined in Theorem A and whose Laplace transform is* *given by (1.7). Then $\zeta^{(\mu)}$ is mixing with respect to any pair of sequences $\{A_n^{(i)}\}_{n=1}^{\infty}$, $i = 1, 2$, of measurable sets in R^d which satisfy Assumption 1.*

We apply Theorem 3 to the class of uniformly elliptic operators that appeared in Corollary 1. Let $|A|$ denote the Lebesgue measure of $A \subset R^d$.

Corollary 3. *Under the conditions of Corollary 1, the invariant distributions $\zeta^{(c)}$, $c > 0$, are mixing with respect to any pair of sequences $\{A_n^{(i)}\}_{n=1}^{\infty}$, $i = 1, 2$, of measurable sets satisfying*

$$\lim_{n\to\infty} \frac{|A_n^{(1)}||A_n^{(2)}|}{(dist(A_n^{(1)}, A_n^{(2)}))^{d-2}} = 0. \tag{1.11}$$

In particular therefore, $\zeta^{(c)}$ is mixing with respect to $\{A_n^{(1)}\}_{n=1}^{\infty}$ and $\{A_n^{(2)}\}_{n=1}^{\infty}$ if the sequences $\{|A_n^{(i)}|\}_{n=1}^{\infty}$, $i = 1, 2$, are bounded and $\lim_{n\to\infty} dist(A_n^{(1)}, A_n^{(2)}) = \infty$.

Remark. Corollary 3 also holds for all the invariant distributions $\zeta^{(\mu)}$ as in Theorem A in the case that L_0 is uniformly elliptic, $d \geq 3$ and $\beta \lneq 0$. Indeed, the semigroup $p(t, x, y)$ and thus also $G(x, y)$ in the case $\beta \lneq 0$ is smaller than in the case $\beta \equiv 0$; thus, Assumption 1 holds a fortiori in the case $\beta \leq 0$.

For the proofs we will need an estimate on the covariance of $\zeta^{(\mu)}$. We define the second moment operator $M_\nu^{(2)}(\cdot,\cdot)$, the covariance operator $Cov_\nu(\cdot,\cdot)$ and the variance operator $Var_\nu(\cdot)$ of a probability measure $\nu \in \mathcal{M}_1(\mathcal{M}(D))$ as follows:

$$\begin{aligned}
&M_\nu^{(2)}(f, g) = \int_{\mathcal{M}(D)} < f, \eta >< g, \eta > d\nu(\eta), \text{ for } f, g \in C_c(D);\\
&Cov_\nu(f, g) = M_\nu^{(2)}(f, g) - < f, \pi_\nu >< g, \pi_\nu >;\\
&Var_\nu(f) = Cov_\nu(f, f).
\end{aligned}$$

Theorem B. The covariance of the invariant distribution $\zeta^{(\mu)}$ defined in Theorem A satisfies

$$Cov_{\zeta_\infty^{(\mu)}}(f, g) = 2\int_0^\infty < \alpha(T_t f)(T_t g), \mu > dt. \tag{1.12}$$

Theorem B in the case of the variance, that is the case that $f = g$, was proved in [13]. For the covariance, one uses the formula $Cov(f, g) = \frac{1}{2}(Var(f, g) - Var(f) - Var(g))$.

We now sketch a proof of (1.5). It is well known that the Green's function is symmetric in x and y in the case of an operator of the form $\frac{1}{2}\nabla \cdot \hat{a}\nabla + \hat{\beta}$. See for example [8] which treats the Laplacian; essentially the same proof works for the more general operator above. It is also known that if $\hat{G}$ denotes the Green's function for an operator $\hat{L}$, then for a positive function γ, $\frac{G(x,y)}{\gamma(y)}$ is the Green's function for the operator γL. This is almost an immediate consequence of [12, Theorem 4.3.8], for example. We apply these facts as follows. Using the notation above, let $\hat{G}(x, y)$ denote the Green's function in the case $\hat{a} = a \exp(2Q)$ and $\hat{\beta} = \beta \exp(2Q)$. Then it follows that $\hat{G}(x, y)$ is symmetric and that the Green's function G for $L = \frac{1}{2} \exp(-2Q)\nabla \cdot \exp(2Q)a\nabla + \beta$ is given by $G(x, y) = \hat{G}(x, y) \exp(2Q(y))$. Recalling that $m_{sym} = \exp(2Q)$, (1.5-b) follows. Now (1.5-a) follows from (1.5-b) and the following resolvent equation for self-adjoint operators: $T_t = \lim_{n\to\infty}(1 - \frac{tL}{n})^{-n} = \lim_{n\to\infty} \frac{t}{n}^n G^n_{\frac{n}{t}}$, where G_λ denotes the Green's function for $L - \lambda$.

We prove Theorems 1 and 2 and Corollaries 1 and 2 in section 2. The proofs of Theorem 3 and Corollary 3 are given in section 3.

2. Proofs of Theorems 1 and 2 and Corollaries 1 and 2.

Proof of Theorem 1. Let f be as in the theorem and let $\{f_n\}$ be an increasing sequence of compactly supported functions converging pointwise to f. Replacing both f and g by $\frac{f_n}{<f_n,\mu>}$ in (1.12), we have

$$Var_{\zeta^{(\mu)}_\infty}\left(\frac{f_n}{< f_n, \mu >}\right) = \frac{2\int_0^\infty < \alpha(T_t f_n)^2, \mu > dt}{< f_n, \mu >^2}. \tag{2.1}$$

Recall that $p(t, x, y)$ denotes the transition kernel of the semigroup T_t so that $Tf(x) = \int_{R^d} p(t, x, y)f(y)dy$. By the subcriticality assumption, the Green's function $G(x, y) = \int_0^\infty p(t, x, y)dt$ exists. By the symmetry assumption, we have $m_{sym}(x)p(t, x, y) = m_{sym}(y)p(t, y, x)$. Recall from Theorem A that $\mu = hm_{sym}$

and $\alpha \leq \frac{c}{h}$. Thus, substituting for μ and α, and using the symmetry, we have

$$
\begin{aligned}
&\int_0^\infty <\alpha(T_t f_n)^2, \mu> dt \\
&\leq c\int_0^\infty dt \int_{R^d} dx \int_{R^d} dy \int_{R^d} dz \ p(t,x,y) f_n(y) p(t,x,z) f_n(z) m_{sym}(x) \\
&= c\int_0^\infty dt \int_{R^d} dx \int_{R^d} dy \int_{R^d} dz \ p(t,y,x) f_n(y) p(t,x,z) f_n(z) m_{sym}(y) \\
&= c\int_0^\infty dt \int_{R^d} dy \int_{R^d} dz \ p(2t,y,z) f_n(y) f_n(z) m_{sym}(y) \\
&= \frac{c}{2}\int_{R^d}\int_{R^d} G(x,y) f_n(x) f_n(y) m_{sym}(x) dx dy.
\end{aligned} \tag{2.2}
$$

Since $\lambda_c(L) = \sup\sigma(L) < 0$ by assumption, the integral operator G defined by $Gf(x) = \int_{R^d} G(x,y)f(y)dy$ on $L_2(R^d, m_{sym}dx)$ is the inverse of the positive operator $-L$ on $L_2(R^d, m_{sym}dx)$ obtained via the Friedrichs' extension. (See [12, Theorem 4.3.8] where it is shown that G is a generalized inverse operator in the case that $\lambda_c(L) \leq 0$.) Thus, using the assumption in the theorem that $\lambda_c(L) < 0$, we have

$$
\sup\sigma(G) = \frac{1}{\inf\sigma(-L)} = -\frac{1}{\sup\sigma(L)} = -\frac{1}{\lambda_c(L)} \in (0,\infty). \tag{2.3}
$$

By the Rayleigh-Ritz variational formula [15], we have

$$
\sup\sigma(G) = \sup_{g\in L_2(R^d, m_{sym}dx)} \frac{\int_{R^d} g(x)Gg(x)m_{sym}(x)dx}{\int_{R^d} g^2(x)m_{sym}(x)dx}. \tag{2.4}
$$

By assumption, f_n is compactly supported and locally bounded; thus, $f_n \in L_2(R^d, m_{sym}dx)$ and it follows from (2.3) and (2.4) that there exists a constant $c_1 > 0$ such that

(2.5)

$$
\frac{c}{2}\int_{R^d}\int_{R^d} G(x,y)f_n(x)f_n(y)m_{sym}(x)dxdy \leq c_1\int_{R^d} f_n^2(x)m_{sym}(x)dx, \ n = 1,2,...
$$

By assumption, $\frac{f}{h}$ is bounded; thus there exists a constant $c_2 > 0$ such that $f_n \leq c_2 h$, for $n = 1,2,....$ Using this along with (2.2) and (2.5), we have

$$
\begin{aligned}
&\frac{\int_0^\infty <\alpha(T_t f_n)^2,\mu> dt}{<f_n,\mu>^2} \leq \frac{c_1\int_{R^d} f_n^2(x)m_{sym}(x)dx}{<f_n,\mu>^2} \\
&\leq \frac{c_1c_2\int_{R^d} f_n(x)h(x)m_{sym}(x)dx}{(\int_{R^d} f_n(x)h(x)m_{sym}(x)dx)^2} = \frac{c_1c_2}{<f_n,\mu>}.
\end{aligned} \tag{2.6}
$$

Now (2.1) and (2.6) give

$$(2.7)\qquad Var_{\zeta_\infty^{(\mu)}}\left(\frac{f_n}{< f_n,\mu >}\right) \le \frac{2c_1c_2}{< f_n,\mu >}.$$

Let $\rho > 1$. By adding terms to the sequence if necessary and preserving the monotonicity, there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ such that $< f_{n_k},\mu >\in [\rho^k,\rho^{k+1})$. Thus, by Chebyshev's inequality and (2.7) we have for $\epsilon > 0$,

(2.8)

$$\zeta^{(\mu)}\left(\left|\frac{< f_{n_k},\eta >}{< f_{n_k},\mu >}-1\right| > \epsilon\right) \le \frac{1}{\epsilon^2}Var_{\zeta_\infty^{(\mu)}}\left(\frac{f_{n_k}}{< f_{n_k},\mu >}\right) \le \frac{2c_1c_2}{\epsilon^2 < f_{n_k},\mu >} \le \frac{2c_1c_2}{\epsilon^2\rho^k}.$$

It then follows from (2.8) and the lemma of Borel-Cantelli that

$$(2.9)\qquad \lim_{k\to\infty}\frac{< f_{n_k},\eta >}{< f_{n_k},\mu >} = 1 \text{ a.s.}.$$

By monotonicity, there exists a nondecreasing sequence $\{k_m\}_{m=1}^\infty$ satisfying $\lim_{m\to\infty} k_m = \infty$ and such that $f_{n_{k_m}} \le f_m \le f_{n_{k_{m+1}}}$. From this and the defining property of the subsequence $\{n_k\}$ we obtain

(2.10)

$$\frac{1}{\rho^2}\frac{< f_{n_{k_m}},\eta >}{< f_{n_{k_m}},\mu >} \le \frac{< f_{n_{k_m}},\eta >}{< f_{n_{k_{m+1}}},\mu >} \le \frac{< f_m,\eta >}{< f_m,\mu >} \le \frac{< f_{n_{k_{m+1}}},\eta >}{< f_{n_{k_m}},\mu >} \le \rho^2\frac{< f_{n_{k_{m+1}}},\eta >}{< f_{n_{k_{m+1}}},\mu >}.$$

From (2.9) and (2.10) we conclude that

$$\frac{1}{\rho^2} \le \liminf_{m\to\infty}\frac{< f_m,\eta >}{< f_m,\mu >} \le \limsup_{m\to\infty}\frac{< f_m,\eta >}{< f_m,\mu >} \le \rho^2 \text{ a.s.}.$$

Since $\rho > 1$ is arbitrary, we conclude that $\lim_{m\to\infty}\frac{<f_m,\eta>}{<f_m,\mu>} = 1$ a.s.. $\square$

Proof of Theorem 2. Following the proof of Theorem 1 through (2.2), we now give an alternative estimate for the right hand side of (2.2):

$$(2.11)\qquad \begin{aligned}&\int_{R^d}\int_{R^d} G(x,y)f_n(x)f_n(y)m_{sym}(x)dxdy\\ &\le \left(\int_{R^d} f_n(x)h(x)m_{sym}(x)dx\right)\left(\sup_{x\in R^d}\int_{R^d}\frac{G(x,y)}{h(x)}f_n(y)dy\right)\\ &=< f_n,\mu > \left(\sup_{x\in R^d}\int_{R^d} G^h(x,y)\frac{f_n}{h}(y)dy\right).\end{aligned}$$

Thus, from (2.2) and (2.11) we obtain

$$\begin{aligned}(2.12)\qquad &\frac{\int_0^\infty < \alpha(T_t f_n)^2, \mu > dt}{< f_n, \mu >^2} \le \frac{c}{2}\frac{\sup_{x\in R^d}\int_{R^d} G^h(x,y)\frac{f_n}{h}(y)dy}{< f_n,\mu >}\\ &= \frac{c}{2}\frac{\sup_{x\in R^d}\int_{R^d}\frac{G^h(x,y)}{m^h_{sym}(y)}\frac{f_n}{h}(y)m^h_{sym}(y)dy}{\int_{R^d}\frac{f_n}{h}(y)m^h_{sym}(y)dy}.\end{aligned}$$

Recalling the definitions of γ_n and l_n in the statement of the theorem, and fixing a positive integer j, we have

$$\begin{aligned}&\int_{R^d}\frac{G^h(x,y)}{m^h_{sym}(y)}\frac{f_n}{h}(y)m^h_{sym}(y)dy\\ (2.13)\qquad &= \int_{D_j(x)}\frac{G^h(x,y)}{m^h_{sym}(y)}\frac{f_n}{h}(y)m^h_{sym}(y)dy + \int_{R^d-D_j(x)} G^h(x,y)\frac{f_n}{h}(y)dy\\ &\le \gamma_j\int_{R^d}\frac{f_n}{h}(y)m^h_{sym}(y)dy + l_j\sup\frac{f_n}{h},\ \text{for all } x\in R^d.\end{aligned}$$

From (2.12) and (2.13), we have

$$(2.14)\qquad \frac{\int_0^\infty < \alpha(T_t f_n)^2,\mu > dt}{< f_n,\mu >^2} \le \frac{c\gamma_j}{2} + \frac{cl_j\sup\frac{f_n}{h}}{2 < f_n,\mu >},\ \text{for } j=1,2,...$$

Using (2.14), (1.12) and Chebyshev's inequality gives

$$(2.15)\qquad \zeta^{(\mu)}(|\frac{< f_n,\eta >}{< f_n,\mu >}-1| > \epsilon) \le \frac{1}{\epsilon^2}(\frac{c\gamma_j}{2}+\frac{cl_j\sup\frac{f_n}{h}}{2 < f_n,\mu >}),\ \text{for } j=1,2,....$$

Since $\lim_{n\to\infty} < f_n,\mu > = < f,\mu > = \infty$ and $\sup\frac{f_n}{h}\le\sup\frac{f}{h} < \infty$, letting $n\to\infty$ in (2.15) gives

$$\limsup_{n\to\infty}\zeta^{(\mu)}(|\frac{< f_n,\eta >}{< f_n,\mu >}-1| > \epsilon) \le \frac{c\gamma_j}{2\epsilon^2},\ \text{for each } j=1,2,...$$

The weak law of large numbers now follows since $\lim_{j\to\infty}\gamma_j = 0$.

Assume now that $\sum_{n=1}\gamma_n < \infty$ and $\limsup_{n\to\infty}\frac{\hat l_{n+1}}{\hat l_n} < \infty$, where $\{\hat l_n\}$ is non-decreasing and $\hat l_n \ge l_n$. Let $\rho > 1$. By adding terms to the series if necessary and preserving the monotonicity, there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ such that $< f_{n_k},\mu > \in [\hat l_k\rho^k, \hat l_k\rho^{k+1})$. Then setting $n=n_k$ and $j=k$ in (2.15), we have

$$\zeta^{(\mu)}(|\frac{< f_{n_k},\eta >}{< f_{n_k},\mu >}-1| > \epsilon) \le \frac{1}{\epsilon^2}(\frac{c\gamma_k}{2}+\frac{c\sup\frac{f_{n_k}}{h}}{2\rho^k}).$$

Since $\sup \frac{f_{n_k}}{h}$ is uniformly bounded in k, it follows from the lemma of Borel-Cantelli that

$$\lim_{k\to\infty} \frac{< f_{n_k}, \eta >}{< f_{n_k}, \mu >} = 1 \text{ a.s.}.$$

Now an argument similar to (2.10) gives the strong law of large numbers. □

We now turn to the proofs of Corollaries 1 and 2.

Proof of Corollary 1. For uniformly elliptic operators, it is known [1] that there exist constants $c_i > 0, i = 1, ...4$, such that

$$\frac{c_1}{t^{\frac{d}{2}}} \exp(-\frac{|y-x|^2}{c_2 t}) \le p(t,x,y) \le \frac{c_3}{t^{\frac{d}{2}}} \exp(-\frac{|y-x|^2}{c_4 t}). \tag{2.16}$$

From this it follows in particular that there exists a constant $C > 1$ such that

$$\frac{1}{C}|y-x|^{2-d} \le G(x,y) \le C|y-x|^{2-d}, \text{ for } d \ge 3. \tag{2.17}$$

We use (2.17) to show that the invariant positive harmonic functions are the positive constants. By the nonexplosion assumption, the constants are indeed invariant. As was noted when invariant positive functions were defined, any such function h must be L-harmonic; that is, satisfy $Lh = 0$. By the Martin boundary theory [12, chapter 9], every positive harmonic function h is of the form $h(x) = \lim_{n\to\infty} \frac{G(x,y_n)}{G(0,y_n)}$ for some sequence $\{y_n\}$ satisfying $\lim_{n\to\infty} |y_n| = \infty$. Using this with (2.17) shows that every positive harmonic function is bounded and bounded away from 0. Fix a positive harmonic function h. Choose M larger than the supremum of h and let $\epsilon_0 > 0$ be the supremum of those ϵ for which $M - \epsilon h$ is nonnegative. Let $u = M - \epsilon_0 h$. By the maximum principle, either $u \equiv 0$ or $u > 0$. In the latter case, u is a positive L-harmonic function and thus it must be bounded away from 0. Since h is bounded, this contradicts the maximality of ϵ_0. Thus, we conclude that $u \equiv 0$ in which case h is constant.

Recall from the paragraph preceeding Corollary 1 that $m_{sym} = 1$. Since h is constant, we have $G^h(x,y) = G(x,y)$. Define $D_n(x) = \{y \in R^d : |y - x| > n^{\frac{2}{d-2}}\}$. It then follows from (2.17) and the definition of γ_n in Theorem 2 that there exists a $C_1 > 0$ such that

$$\gamma_n \leq \frac{C_1}{n^2}. \tag{2.18}$$

From (2.17) and the definition of l_n in Theorem 2, we obtain for some $C_2 > 1$,

$$\frac{1}{C_2} n^{\frac{4}{d-2}} \leq \frac{1}{C} \int_{|y| \leq n^{\frac{2}{d-2}}} \frac{1}{|y|^{d-2}} dy \leq l_n \leq C \int_{|y| \leq n^{\frac{2}{d-2}}} \frac{1}{|y|^{d-2}} dy \leq C_2 n^{\frac{4}{d-2}}. \tag{2.19}$$

In light of (2.18) and (2.19), the strong law of large numbers follows from Theorem 2. □

Proof of Corollary 2. The operator L^h is given by

$$L^h = \frac{1}{2} a \frac{d^2}{dx^2} + (b + a\frac{h'}{h}) \frac{d}{dx}.$$

We have

$$m^h_{sym}(y) = h^2(y) m_{sym}(y) = \frac{h^2(y)}{a(y)} \exp(\int_0^y \frac{2b}{a}(z) dz).$$

We must consider two cases separately. First consider the case that both $H(-\infty)$ and $H(\infty)$ are finite. In this case, the Green's function is given by

$$G^h(x,y) = \frac{2m^h_{sym}(y)}{H(\infty) - H(-\infty)} (H(\infty) - H(y \vee x))(H(y \wedge x) - H(-\infty)). \tag{2.20}$$

(To calculate this, one solves for u_n in the equation $L^h u_n = -f$ on $(-n, n)$ with $u_n(-n) = u_n(n) = 0$, then lets $n \to \infty$ and substitutes $f(z) = \delta_y(z)$. The resulting quantity is $G^h(x,y)$. In solving the equation, it's convenient to write $L^h u = -f$ in the form $(\exp(\int_0^x \frac{2b}{a}(y)dy) h^2(x) u'(x))' = -\frac{2}{a(x)} \exp(\int_0^x \frac{2b}{a}(y)dy) h^2(x) f(x)$ and integrate twice, using $-n$ as the lower limit of integration.) It is easy to see that there exists a constant $C > 0$ such that

$$\begin{aligned} &\frac{1}{H(\infty) - H(-\infty)} (H(\infty) - H(y \vee x))(H(y \wedge x) - H(-\infty)) \\ &\leq C \min(H(\infty) - H(y), H(y) - H(-\infty)), \text{ for all } x \in R. \end{aligned} \tag{2.21}$$

Let $\{z_n\}$ be a positive sequence converging to ∞ and satisfying

$$\sum_{n=1}^{\infty} (H(\infty) - H(z_n)) + (H(-z_n) - H(-\infty)) < \infty. \tag{2.22}$$

Define $D_n(x)$ independent of x by $D_n = (-\infty, -z_n) \cup (z_n, \infty)$, and recall from Theorem 2 that

$$\gamma_n = \sup_{x \in R^d} \sup_{y \in D_n(x)} \frac{G^h(x,y)}{m^h_{sym}(y)}$$

Then it follows from (2.20)-(2.22) and the monotonicity of H that

$$\sum_{n=1}^{\infty} \gamma_n < \infty. \tag{2.23}$$

From (2.20) there exists a $C > 0$ such that

$$\sup_{x \in R} G^h(x,y) \leq C m^h_{sym}(y). \tag{2.24}$$

It follows from (2.24) that

$$\sup_{x \in R} \int_{y \in R - D_n(x)} G^h(x,y)dy \leq C \int_{-z_n}^{z_n} m^h_{sym}(y)dy < \infty, \text{ for } n = 1, 2, ... \tag{2.25}$$

Letting $\hat{l}_n = C \int_{-z_n}^{z_n} m^h_{sym}(y)dy$, the corollary now follows from (2.22), (2.23), (2.25) and Theorem 2.

We now turn to the case in which $H(-\infty)$ is finite but $H(\infty)$ is not (the opposite case being treated similarly). In this case the Green's function is given by

$$G^h(x,y) = 2m^h_{sym}(y)(H(x \wedge y) - H(-\infty)). \tag{2.26}$$

(One calculates the Green's function by the method noted parenthetically after (2.20), except that this time it is better to let 0 be the lower limit of integration.) Define $D_n(x)$ independent of x by $D_n = (-\infty, -z_n)$, where $\{z_n\}$ is a sequence increasing to ∞ and chosen so that

$$\sum_{n=1}^{\infty} (H(-z_n) - H(-\infty)) < \infty. \tag{2.27}$$

Recall that an assumption of the corollary in this case is that the support of f is bounded away from $+\infty$. From (2.26), (2.27) and the choice of $D_n(x)$, it follows that

$$\sum_{n=1}^{\infty} \gamma_n < \infty. \tag{2.28}$$

Let $z_0 = \sup supp(f)$. Define

$$\hat{l}_n = C \int_{-z_n}^{z_0} 2m^h_{sym}(y)dy, \text{ for } n = 1, 2, ..., \tag{2.29}$$

where $C > 0$. If C is sufficiently large, then from (2.26) we have

$$\hat{l}_n \geq \sup_{x \in supp(f)} \int_{y \in supp(f) \cap \{R - D_n(x)\}} G^h(x, y)dy. \tag{2.30}$$

A look at the proof of Theorem 2 reveals immediately that everything works just as well if the requirement $\hat{l}_n \geq l_n \equiv \sup_{x \in R^d} \int_{y \in R^d - D_n(x)} G^h(x, y)dy$ is replaced by the requirement $\hat{l}_n \geq \sup_{x \in supp(f)} \int_{y \in supp(f) \cap \{R^d - D_n(x)\}} G^h(x, y)dy$. Thus, the corollary now follows from (2.28)-(2.30) and Theorem 2. $\square$

3. Proofs of Theorem 3 and Corollary 3.

Proof of Theorem 3. The space $\mathcal{M}(R^d)$ is equipped with the natural partial ordering defined by $\eta_1 \leq \eta_2$ if $\eta_1(A) \leq \eta_2(A)$ for all Borel sets $A \subset R^d$. A function $f : \mathcal{M}(R^d) \to R$ is called nondecreasing if $f(\eta_1) \leq f(\eta_2)$ whenever $\eta_1 \leq \eta_2$. A probability measure $\nu \in \mathcal{M}_1(\mathcal{M}(R^d))$ is called *associated* if for each pair of nondecreasing functions $f, g : \mathcal{M}(R^d) \to R$, one has

$$Cov(f(\nu), g(\nu)) \equiv \int_{\mathcal{M}(R^d)} f(\eta)g(\eta)d\nu(\eta) - \int_{\mathcal{M}(R^d)} f(\eta)d\nu(\eta) \int_{\mathcal{M}(R^d)} g(\eta)d\nu(\eta) \geq 0.$$

Since the invariant distribution $\zeta^{(\mu)}$ is infinitely divisible, it follows from [3,9] that $\zeta^{(\mu)}$ is associated. It is known [10, 11] that for associated random variables, independence is equivalent to uncorrelatedness, and we will show below that the asymptotic independence of $\eta(A_n^{(1)})$ and $\eta(A_n^{(2)})$ under $\zeta^{(\mu)}$, which we call mixing (Definition 2), is equivalent to the following asymptotic uncorrelatedness of $\eta(A_n^{(1)})$ and $\eta(A_n^{(2)})$ under $\zeta^{(\mu)}$:

(3.1)

$$\lim_{n \to \infty} \Big(\int_{\mathcal{M}(R^d)} \eta(A_n^{(1)})\eta(A_n^{(2)})d\zeta^{(\mu)}(\eta) - \int_{\mathcal{M}(R^d)} \eta(A_n^{(1)})d\zeta^{(\mu)}(\eta) \int_{\mathcal{M}(R^d)} \eta(A_n^{(2)})d\zeta^{(\mu)}(\eta) \Big) = 0.$$

We first prove that (3.1) holds whenever Assumption 1 is in effect, and then we show that (3.1) implies mixing.

Recalling the definition of $Cov_\nu(f,g)$, defined before (1.12), note that the expression in the parentheses on the left hand side of (3.1) is just $Cov_{\zeta^{(\mu)}}(1_{A_n^{(1)}}, 1_{A_n^{(2)}})$. Thus, by (1.12), the expression in the parentheses on the left hand side of (3.1) is equal to $2\int_0^\infty <\alpha(T_t 1_{A_n^{(1)}})(T_t 1_{A_n^{(2)}}), \mu > dt$. We have $T_t 1_{A_n^{(i)}}(x) = \int_{A_n^{(i)}} p(t,x,y)dy$. Thus, using the symmetry assumption $m_{sym}(x)p(t,x,y) = m_{sym}(y)p(t,y,x)$, along with the fact that $\mu(x) = h(x)m_{sym}(x)$ and along with the underlying assumption in Theorem A that $\alpha \le \frac{c}{h}$, we have

$$
\begin{aligned}
&2\int_0^\infty <\alpha(T_t 1_{A_n^{(1)}})(T_t 1_{A_n^{(2)}}), \mu > dt = \\
&2\int_0^\infty dt \int_{R^d} dx \int_{A_n^{(1)}} dy \int_{A_n^{(2)}} dz\ \alpha(x)p(t,x,y)p(t,x,z)\mu(x) \\
&\le 2c\int_0^\infty dt \int_{R^d} dx \int_{A_n^{(1)}} dy \int_{A_n^{(2)}} dz\ p(t,x,y)p(t,x,z)m_{sym}(x) \\
&= 2c\int_0^\infty dt \int_{R^d} dx \int_{A_n^{(1)}} dy \int_{A_n^{(2)}} dz\ p(t,y,x)p(t,x,z)m_{sym}(y) \\
&= 2c\int_0^\infty dt \int_{A_n^{(1)}} dy \int_{A_n^{(2)}} dz\ p(2t,y,z)m_{sym}(y) \\
&= c\int_{A_n^{(1)}} dy \int_{A_n^{(2)}} dz\ G(y,z)m_{sym}(y).
\end{aligned}
\tag{3.2}
$$

From (3.2), Assumption 1 and the fact that the left hand side of (3.2) coincides with the expression in the parentheses on the left hand side of (3.1), we obtain (3.1).

We now show that (3.1) implies mixing. Let $f, g \in C_b^1([0,\infty))$. Let

$$
H_n^{(1,2)}(x,y) = \zeta^{(\mu)}(\eta(A_n^{(1)}) > x, \eta(A_n^{(2)}) > y) - \zeta^{(\mu)}(\eta(A_n^{(1)}) > x)\cdot\zeta^{(\mu)}(\eta(A_n^{(2)}) > y).
$$

By associativity, $H_n^{(1,2)}(x,y) \ge 0$. Integration by parts along with the finiteness of the covariance shows that

$$
\begin{aligned}
&\int_{\mathcal{M}(R^d)} \eta(A_n^{(1)})\eta(A_n^{(2)})d\zeta^{(\mu)}(\eta) - \int_{\mathcal{M}(R^d)} \eta(A_n^{(1)})d\zeta^{(\mu)}(\eta)\int_{\mathcal{M}(R^d)} \eta(A_n^{(2)})d\zeta^{(\mu)}(\eta) \\
&= \int_{-\infty}^\infty \int_{-\infty}^\infty H_n^{(1,2)}(x,y)dxdy,
\end{aligned}
\tag{3.3}
$$

and that

(3.4)

$$\int_{\mathcal{M}(R^d)} f(\eta(A_n^{(1)}))g(\eta(A_n^{(2)}))d\zeta^{(\mu)}(\eta) - \int_{\mathcal{M}(R^d)} f(\eta(A_n^{(1)}))d\zeta^{(\mu)}(\eta) \int_{\mathcal{M}(R^d)} g(\eta(A_n^{(2)}))d\zeta^{(\mu)}(\eta)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f'(x)g'(y)H_n^{(1,2)}(x,y)dxdy.$$

By (3.1) and (3.3) we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_n^{(1,2)}(x,y)dxdy = 0. \tag{3.5}$$

Since f' and g' are bounded and since $H^{(1,2)} \geq 0$, it follows from (3.4) and (3.5) that mixing holds. □

Proof of Corollary 3. By Theorem 3, we must verify that Assumption 1 holds when $\{A_n^{(1)}\}_{n=1}^{\infty}$ and $\{A_n^{(2)}\}_{n=1}^{\infty}$ are as in the statement of the corollary. By (2.17),

$$G(x,y) \leq C|x-y|^{2-d}.$$

Also, we have $m_{sym} \equiv 1$. Thus,

$$\int_{A_n^{(1)}}\int_{A_n^{(2)}} G(x,y)m_{sym}(x)dxdy \leq C\frac{|A_n^{(1)}||A_n^{(2)}|}{(dist(A_n^{(1)},A_n^{(2)}))^{d-2}}.$$

Therefore, by the condition on $\{A_n^{(1)}\}_{n=1}^{\infty}$ and $\{A_n^{(2)}\}_{n=1}^{\infty}$ in the corollary, Assumption 1 holds. □

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