AN IDENTITY INVOLVING STIRLING NUMBERS OF BOTH KINDS AND ITS CONNECTION TO RIGHT-TO-LEFT MINIMA OF CERTAIN SET PARTITIONS

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ABSTRACT. For $1 \leq k \leq n, n \in \mathbb{N}$, let $S(n, k)$ denote the Stirling numbers of the second kind and let $s(n, k) = (-1)^{n-k}|s(n, k)|$ denote the Stirling numbers of the first kind, where $|s(n, k)|$ denote the unsigned Stirling numbers of the first kind. Extend the definitions to all $n, k \in \mathbb{N}$ by defining all of these Stirling numbers to be equal to zero if $k > n$. Let $S = \{S(n, k)\}$ (respectively, $s = \{s(n, k)\}$, $|s| = \{|s(n, k)|\}$) denote the infinite matrix whose $nk$-th entry is $S(n, k)$ (respectively $s(n, k), |s(n, k)|$). A classical result states that $S$ and $s$ are inverses of one another: $Ss = sS = Id$. In this note, as a byproduct of a result concerning the statistics of right-to-left minima in certain set partitions, we calculate explicitly $|s|S$, via a completely combinatorial method.

1. Introduction and Statement of Results

Let $S_n$ denote the set of permutations of $[n]$. For a permutation $\sigma = (\sigma_1, \cdots, \sigma_n) \in S_n$, the entry $\sigma_i$ is called a right-to-left minimum if $\sigma_i < \sigma_j$, for all $j = i + 1, \cdots n$. Similarly, one can define a right-to-left maximum and a left-to-right minimum or maximum. These concepts play an important role in the study of pattern-avoiding permutations; see for example [2], [3]. For $j \in [n]$, the number of permutations in $S_n$ with exactly $j$ right-to-left minima is equal to the unsigned Stirling number of the first kind $|s(n, j)|$ [1]. (Recall that $|s(n, j)|$ is also equal to the number of permutations in $S_n$ containing exactly $j$ cycles.)

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One may think of a permutation $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ as a list of the numbers in $[n]$. Let $k \in [n]$. Consider now placing the elements of $[n]$ into $k$ nonempty lists. (Of course, for $k = 1$, what we obtain is a permutation of $S_n$.) We will refer to such an object as a set of lists, and we denote the collection of sets of lists for a particular $n$ and $k$ by $\mathcal{SL}(n, k)$.

It is easy to calculate the number of elements in $\mathcal{SL}(n, k)$. To do so, we first define the collection of lists of lists $\mathcal{LL}(n, k)$. An element of $\mathcal{LL}(n, k)$ is simply an element of $\mathcal{SL}(n, k)$ with the order of the $k$ sets taken into account. Thus, for example, there are 12 elements in $\mathcal{LL}(3, 2)$: $1,2/3; 1,3/2; 1/2,3; 2,1/3; 3,1/2; 1/3,2; 3/1,2; 2/1,3; 2,3/1; 3/2,1; 3,2/1$. While there are 6 elements in $\mathcal{SL}(3, 2)$: $1,2/3; 1,3/2; 1/2,3; 2,1/3; 3,1/2; 1/3,2$.

See [6] for more on sets of lists and lists of lists.

**Proposition 1.** $|\mathcal{SL}(n, k)| = \frac{n!}{k!} \binom{n-1}{k-1}$.

**Proof.** To prove the proposition, it suffices to prove that $|\mathcal{LL}(n, k)| = n! \binom{n-1}{k-1}$.

Choose a permutation $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ of $[n]$. Choose $k - 1$ numbers $\{j_i\}_{i=1}^{k-1}$ from $[n-1]$. This generates an element of $\mathcal{LL}(n, k)$; namely $\sigma_1, \cdots, \sigma_j_1 / \sigma_{j_1+1}, \cdots, \sigma_j_2 / \cdots / \sigma_{j_{k-1}+1}, \cdots, \sigma_n$. Clearly, as one runs over all permutations $\sigma$ of $[n]$ and all subsets $\{j_i\}_{i=1}^{k-1}$ of length $k - 1$ from $[n-1]$, one generates every element of $\mathcal{LL}(n, k)$ exactly once. $\square$

**Remark.** The numbers $\frac{n!}{k!} \binom{n-1}{k-1}$ are called (unsigned) Lah numbers. See entry A048854 in the OEIS (On-line Encyclopedia of Integers sequences).

Consider an element $\nu \in \mathcal{SL}(n, k)$. We say that the number $j \in [n]$ is a right-to-left minimum for $\nu$ if $j$ is a right-to-left minimum with respect to the numbers in the list to which it belongs. For example, consider the element $\nu_0 \in \mathcal{SL}(9, 4)$ given by $4,1/2/7,3,6,5/8,9$. Then the numbers $1,2,3,5,8,9$ are right-to-left minima for $\nu$. For $\nu \in \mathcal{SL}(n, k)$, define $M_{R/L}(\nu)$ to be the number of right-to-left minima it possesses; thus, for example, $M_{R/L}(\nu_0) = 6$, where $\nu_0$ is as above. Clearly, $M_{R/L}(\nu) \in \{k, k+1, \cdots, n\}$, for $\nu \in \mathcal{SL}(n, k)$. Let

$$\mathcal{SL}_j(n, k) = \{ \nu \in \mathcal{SL}(n, k) : M_{R/L}(\nu) = j \}, \text{ for } j = k, \cdots, n.$$
In this note we calculate $|SL_j(n,k)|$ in a completely combinatorially fashion. Let $S(n,k)$ denote the Stirling number of the second kind, which counts the number of ways to partition the set $[n]$ into $k$ non-empty, disjoint subsets.

**Theorem 1.**

(1.1) $|SL_j(n,k)| = |s(n,j)|S(j,k), \ j = k, \cdots n.$

From the theorem and Proposition 1 we obtain the following corollary. Define $|s(n,k)| = S(n,k) = 0$, for $k > n$. Let $S = \{S(n,k)\}$ ($|s| = \{|s(n,k)|\}$) denote the infinite matrix whose $nk$-th entry is $S(n,k)$ ($|s(n,k)|$).

**Corollary 1.**

(1.2) $|s|S = \left\{ \frac{n!}{k!} \left( \frac{n-1}{k-1} \right) ^{1_{n \geq k}} \right\}$; that is, 

$$
\sum_{j=1}^{\infty} |s(n,j)|S(j,k) = \frac{n!}{k!} \left( \frac{n-1}{k-1} \right), \ 1 \leq k \leq n.
$$

**Remark 1.** Let $s(n,k) = (-1)^{n-k}|s(n,k)|$ denote the corresponding signed Stirling number of the first kind, and let $s = \{s(n,k)\}$. Corollary 1 complements the following well-known result [8].

$sS = Ss - Id$; that is

$$
\sum_{j=1}^{\infty} s(n,j)S(j,k) = \sum_{j=1}^{\infty} S(n,j)s(j,k) = \begin{cases} 
1, \text{ if } n = k; \\
0, \text{ if } n \neq k.
\end{cases}
$$

**Remark 2.** We were unable to find the identity (1.2) in the published journal literature, however we did find it implicitly in an unpublished arXiv article by Donald Knuth [5] and in the Mathematics Stack Exchange online [7]. Knuth’s method is via so-called convolution matrices. The proof in [7] is via repeated use of generating functions. In contrast, our proof of the identity is completely combinatorial, and presents a natural combinatorial structure that the formula is enumerating.

**Remark 3.** A complicated proof of Theorem 1 using certain two-parameter generating functions appears in [4].
Remark 4. In light of Corollary 1, it is natural to consider the quantity \( \sum_{j=1}^{\infty} S(n,j)|s(j,k)| \). We can give a combinatorial description of this quantity, however we don’t know how to evaluate it. Let \( \mathcal{LL}(n) = \bigcup_{k=1}^{n} \mathcal{LL}(n,k) \) denote the total collection of lists of lists of the numbers in \([n]\). Let \( \nu \in \mathcal{LL}(n) \); so \( \nu \in \mathcal{LL}(n,k) \), for some \( k \). Label each of the \( k \) lists by the smallest element in the list. Call any one of the \( k \) lists a right-to-left minimum if its label is smaller than the labels on all of the lists that appear to its right. Denote the total number of right-to-left minima by \( \hat{M}_{R/L}(\nu) \). For example, consider \( \nu \in \mathcal{LL}(n,k) \) given by \( 3,2,7/6,4/9,1/5,8 \). Then the list \( 3,2,7 \) gets the label \( 2 \), the list \( 6,4 \) gets the label \( 4 \), the list \( 9,1 \) gets the label \( 1 \), and the list \( 5,8 \) gets the label \( 5 \). Thus, the lists \( 9,1 \) and \( 5,8 \) are right-to-left minima and \( \hat{M}_{R/L}(\nu) = 2 \). We have

\[
\sum_{j=1}^{\infty} S(n,j)|s(j,k)| = |\{ \nu \in \mathcal{LL}(n) : \hat{M}_{R/L}(\nu) = k \}|.
\]

The explanation for this is as follows. For each \( j \in \{k, \cdots, n\} \), there are \( S(n,j) \) different ways to arrange the numbers in \([n]\) into \( j \) nonempty subsets. Given such a partition, we label the \( j \) subsets as above, according to their smallest element. These \( j \) labels can be arranged into \( j! \) different permutations, of which \( |s(j,k)| \) of them will have \( k \) right-to-left minima.

2. Proof of Theorem 1

Note that from the definitions, it follows that

\[
|S_{n}(n,k)| = S(n,k).
\]

Using this along with the fact that \( |s(n,n)| = 1 \), the case \( j = n \) in (1.1) follows immediately. Assume now that \( k \leq j \leq n-1 \). An arbitrary element \( \nu \) of \( S_{n}(j,k) \) can be built as follows. Choose \( n-j \) integers \( 2 \leq a_{1} < \cdots < a_{n-j} \leq n \) to be the integers that are not right-to-left minima. Now choose an element of \( \mathcal{SL}_{j}(j,k) \). By (2.1), there are \( S(j,k) \) possible elements to choose from. Relabel the integers \( 1,2,\cdots,j \) appearing in this element of \( \mathcal{SL}_{j}(j,k) \) by the integers in \([n] - \{a_{i}\}_{i=1}^{n-j}\) , where 1 is replaced by the smallest integer in \([n] - \{a_{i}\}_{i=1}^{n-j}\) , 2 is replaced by the second smallest integer in \([n] - \{a_{i}\}_{i=1}^{n-j}\) , etc. Call this object \( \tilde{\nu} \). Finally, enter the \( n-j \) integers \( \{a_{i}\}_{i=1}^{n-j} \) into \( \tilde{\nu} \), one
at a time, according to the order \(a_1, a_2, \cdots, a_{n-j}\). The resulting element \(\nu\) belongs to \(SL(n,k)\). In order that \(\nu\) indeed belong to \(SL_j(n,k)\), we must enter these \(n-j\) integers in such a way that none of them will be right-to-left minima. Therefore, \(a_1\) needs to be entered immediately to the left of one of the integers \(1, \cdots, a_1 - 1\) (and in the same subset as that integer). Once \(a_1\) has been entered, then \(a_2\) must be entered immediately to the left of one of the integers \(1, \cdots, a_2 - 1\) (and in the same subset as that integer), etc. Thus, there are \(\prod_{i=1}^{n-j} (a_i - 1)\) different ways to enter the integers \(\{a_i\}_{i=1}^{n-j}\). As an example, let \(n = 9, k = 3, j = 6, a_1 = 3, a_2 = 6, a_3 = 7,\) and let the other integers be arranged as 149/28/5. Then \(a_1 = 3\) can be entered in \(a_1 - 1 = 2\) possible positions: to the left of 1 or to the left of 2. Let’s say we choose the latter possibility; then we have 149/328/5. Now \(a_2 = 6\) can be entered in \(a_2 - 1 = 5\) possible positions: to the left of 1,2,3,4 or 5. Let’s say we choose the first of these possibilities; then we have 6149/328/5. Finally, \(a_3 = 7\) can be entered in \(a_3 - 1 = 6\) possible positions: to the left of 1,2,3,4,5 or 6.

From the above argument, we conclude that

\[
|SL_j(n,k)| = S(j,k) \sum_{\{a_i\}_{i=1}^{n-j} \subset [n]} \prod_{i=1}^{n-j} (a_i - 1).
\]

Define

\[
A_{n,j} = \sum_{\{a_i\}_{i=1}^{n-j} \subset [n]} n-j \prod_{i=1}^{n-j} (a_i - 1), j = 1, \cdots, n
\]

(by convention, \(A_{n,n} = 1\)). In light of (2.2), to complete the proof, we need to show that

\[
(2.3) \quad A_{n,j} = |s(n,j)|.
\]

Using the convention that \(|s(n,0)| = 0\), recall \([8]\) that the unsigned Stirling numbers of the first kind are uniquely determined by the conditions

\[
(2.4) \quad |s(n,n)| = 1, n \geq 1;
|s(n + 1, j)| = n|s(n,j)| + |s(n, j - 1)|, n \geq j \geq 1.
\]

We now show that the \(\{A_{n,j}\}\) also satisfy the recursive equation in (2.4) and the boundary condition there, thereby proving (2.3).
Since $A_{n,n} = 1$, the boundary condition in (2.4) is satisfied. For the recursive equation in (2.4), we break up the sum defining $A_{n+1,j}$ into two parts, one involving those $\{a_i\}_{i=1}^{n+1-j}$ for which $a_{n+1-j} \neq n + 1$, and one involving those $\{a_i\}_{i=1}^{n+1-j}$ for which $a_{n+1-j} = n + 1$:

$$A_{n+1,j} = \sum_{\{a_i\}_{i=1}^{n+1-j} \subset [n+1]} \prod_{i=1}^{n+1-j} (a_i - 1) = \sum_{\{a_i\}_{i=1}^{n+1-j} \subset [n]} \prod_{i=1}^{n+1-j} (a_i - 1) +$$

$$\sum_{\{a_i\}_{i=1}^{n+1-j} \subset [n]} \left( \prod_{i=1}^{n-j} (a_i - 1) \right) \cdot n = A_{n,j-1} + nA_{n,j}.$$

Thus $\{A_{n,j}\}$ satisfies the recursive equation in (2.4). \hfill \Box

References


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