SPECTRAL GAP AND RATE OF CONVERGENCE TO EQUILIBRIUM FOR A CLASS OF CONDITIONED BROWNIAN MOTIONS

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Abstract.

If a Brownian motion is *physically* constrained to the interval $[0,\gamma]$ by reflecting it at the endpoints, one obtains an ergodic process whose exponential rate of convergence to equilibrium is $\frac{\pi^2}{2\gamma^2}$. On the other hand, if Brownian motion is *conditioned* to remain in $(0,\gamma)$ up to time t, then in the limit as $t\to\infty$ one obtains an ergodic process whose exponential rate of convergence to equilibrium is $\frac{3\pi^2}{2\gamma^2}$. A recent paper [2] considered a different kind of physical constraint—when the Brownian motion reaches an endpoint, it is catapulted to the point $p\gamma$, where $p\in(0,\frac12]$, and then continues until it again hits an endpoint at which time it is catapulted again to $p\gamma$, etc. The resulting process—*Brownian motion physically returned to the point* $p\gamma$—is ergodic and the exponential rate of convergence to equilibrium is independent of p and equals $\frac{2\pi^2}{\gamma^2}$. In this paper we define a conditioning analog of the process physically returned to the point $p\gamma$ and study its rate of convergence to equilibrium.

1. Introduction and statement of results. Consider a free Brownian particle that one wants to constrain to the interval $[0,\gamma]$ in such a way that the resulting process is Markovian and ergodic. There are two classical ways to do this—one via a physical constraint and one constrained via conditioning. For the physical constraint, one introduces reflection at the two endpoints to obtain *reflecting Brownian motion.* The infinitesimal generator of this process is the Neumann Laplacian on $[0,\gamma]$, which we denote by $\mathcal{L}_N$. It is the closure of the operator $L=\frac12\frac{d^2}{dx^2}$ in $[0,\gamma]$ on the domain $\{u\in C^2([0,\gamma]):u'(0)=u'(\gamma)=0\}$. For the conditioning method,

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we need to introduce a bit of notation. Denote paths in $C([0,\infty),R)$ by ω and denote the value of a path at time t by $X(t)=X(t,\omega)$. Let P_x be Wiener measure on $C([0,\infty),R)$ supported on paths satisfying $X(0)=x$. Define the first exit time from $[0,\gamma]$ by $T_{0,\gamma}=\inf\{t\ge 0: X(t)\notin(0,\gamma)\}$. For $t>0$, one considers the conditioned process $P_x(X(\cdot)\in\cdot|T_{0,\gamma}>t)$. This process is a time inhomogeneous Markov process, and as $t\to\infty$, it converges weakly to a limiting diffusion process that we call *Brownian motion conditioned to remain in* $(0,\gamma)$, and which can be described as follows [5]. Let $\mathcal{L}_D$ denote the Dirichlet Laplacian on $[0,\gamma]$. It is the closure of the operator $L=\frac{1}{2}\frac{d^2}{dx^2}$ in $[0,\gamma]$ on the domain $\{u\in C^2([0,\gamma]): u(0)=u(\gamma)=0\}$. The principal eigenvalue for $-\mathcal{L}_D$ is $\frac{\pi^2}{2\gamma^2}$ and the corresponding eigenfunction is $\phi_0(x)=\sin\frac{\pi x}{\gamma}$. The limiting diffusion process obtained by the above described conditioning is generated by $(\mathcal{L}_D+\frac{\pi^2}{2\gamma^2})^{\phi_0}$, the h–transform of $\mathcal{L}_D+\frac{\pi^2}{2\gamma^2}$ via the h–function ϕ_0. (The h-transform $\mathcal{A}^\psi$ of an operator of the form $\mathcal{A}=\frac{1}{2}\frac{d^2}{dx^2}+b\frac{d}{dx}+V$ via the function ψ is defined by $\mathcal{A}^\psi f=\frac{1}{\psi}\mathcal{A}(\psi f)$ which, when written out, gives $\mathcal{A}^\psi=\frac{1}{2}\frac{d^2}{dx^2}+b\frac{d}{dx}+\frac{\psi'}{\psi}\frac{d}{dx}+\frac{\mathcal{A}\psi}{\psi}$.) When written out explicitly, one finds that a core for the generator of the process is $L=\frac{1}{2}\frac{d^2}{dx^2}+\frac{\phi_0'}{\phi_0}\frac{d}{dx}=\frac{1}{2}\frac{d^2}{dx^2}+\frac{\pi}{\gamma}\cot\frac{\pi x}{\gamma}\frac{d}{dx}$ on the domain $\{u\in C_b^2((0,\gamma))\}$.

Consider now the invariant probability density and the spectral gap for the operators $\mathcal{L}_N$ and $(\mathcal{L}_D+\frac{\pi^2}{2\gamma^2})^{\phi_0}$, which correspond respectively to the reflected Brownian motion on $[0,\gamma]$ and to the Brownian motion conditioned to remain in $(0,\gamma)$. The principal eigenvalue for both $-\mathcal{L}_N$ and $-(\mathcal{L}_D+\frac{\pi^2}{2\gamma^2})^{\phi_0}$ is of course 0, and the corresponding eigenfunction is 1. The invariant probability density solves the adjoint equation; that is, it is the eigenfunction corresponding to the eigenvalue 0 for the adjoint operator. Now $-\mathcal{L}_N$ is self-adjoint while the adjoint of $-(\mathcal{L}_D+\frac{\pi^2}{2\gamma^2})^{\phi_0}$ is seen to be $-(\mathcal{L}_D+\frac{\pi^2}{2\gamma^2})^{\frac{1}{\phi_0}}$. Denoting the invariant probability densities by $\mu^{(N)}$ and $\mu^{(D)}$, we find that

$$\mu^{(N)}(x)=\frac{1}{\gamma}$$

and

$$\mu^{(D)}(x)=\frac{2}{\gamma}\sin^2\frac{\pi x}{\gamma}.$$

The spectrum is invariant under h–transforms; thus the spectrum of $(\mathcal{L}_D+\frac{\pi^2}{2\gamma^2})^{\phi_0}$ coincides with that of $\mathcal{L}_D+\frac{\pi^2}{2\gamma^2}$. (Indeed, $(\mathcal{L}_D+\frac{\pi^2}{2\gamma^2})^{\phi_0}u=-\lambda u$ if and only if $(\mathcal{L}_D+\frac{\pi^2}{2\gamma^2})(\phi_0 u)=-\lambda\phi_0 u$, and $\phi_0 u$ satisfies the Dirichlet boundary condition if u is bounded.) The first positive eigenvalue for $-\mathcal{L}_N$ is given by $\lambda_1^{(N)}=\frac{\pi^2}{2\gamma^2}$, while for $-(\mathcal{L}_D+\frac{\pi^2}{2\gamma^2})$ it is given by $\lambda_1^{(D)}-\frac{\pi^2}{2\gamma^2}=\frac{2\pi^2}{\gamma^2}-\frac{\pi^2}{2\gamma^2}=\frac{3\pi^2}{2\gamma^2}$. Since the principal eigenvalue is 0, this first positive eigenvalue is the spectral gap. As is well-known, the spectral gap controls the rate of convergence to equilibrium. Letting $p^{(N)}(t,x,y)$ and $p^{(D)}(t,x,y)$ denote the transition probability densities for the two processes, one can show that

$$\lim_{t\to\infty}\frac{1}{t}\log\sup_{x,y\in(0,\gamma)}|p^{(N)}(t,x,y)-\frac{1}{\gamma}|=-\frac{1}{2}\frac{\pi^2}{\gamma^2},$$

and

$$\lim_{t\to\infty}\frac{1}{t}\log\sup_{x,y\in(0,\gamma)}|p^{(D)}(t,x,y)-\frac{2}{\gamma}\sin^2\frac{\pi y}{\gamma}|=-\frac{3}{2}\frac{\pi^2}{\gamma^2}.$$

(To prove this, one writes the transition probability density in an eigenfunction expansion as in the proof of (2.11) below. However, in the present case unlike in (2.11), all the eigenvalues and eigenfunctions can be calculated explicitly.) Thus, the exponential rate of convergence to equilibrium for Brownian motion conditioned to remain in $(0,\gamma)$ is three times the rate for reflected Brownian motion.

Now consider the following alternative physical conditioning. Fix $p\in(0,\frac{1}{2}]$. The Brownian particle starts from some point in $(0,\gamma)$ and runs freely until it hits 0 or γ. When it hits one of these endpoints, the particle is immediately catapulted to $p\gamma$ from where it continues its Brownian motion until it again hits an endpoint and is again catapulted to $p\gamma$, etc. We will call this process *Brownian motion physically returned to the point* $p\gamma$. (We consider $p\in(0,\frac{1}{2}]$ instead of in $(0,1)$ because the cases p and $1-p$ are equivalent by symmetry.) This process, which is of course discontinuous, was investigated in a recent paper [2]. Among other things, it was shown that the process is an ergodic Markov process with invariant measure

$$\mu^{phy,p}(x)=\begin{cases}\frac{2}{\rho\gamma^2}x, & \text{if } y\in[0,p\gamma];\\ \frac{2}{(1-p)\gamma^2}(\gamma-x), & \text{if } x\in[p\gamma,\gamma].\end{cases}$$

Furthermore, letting $p^{phy,p}(t,x,y)$ denote the corresponding transition probability density, it was shown that

$$\lim_{t\to\infty}\frac{1}{t}\log\sup_{x,y\in(0,\gamma)}|p^{(phy,p)}(t,x,y)-\mu^{(phy,p)}(y)|=-\frac{2\pi^2}{\gamma^2}.$$

(Actually, in [2] the authors have $\frac{\pi^2}{2\gamma^2}$ instead of $\frac{2\pi^2}{\gamma^2}$ on the right hand side above; they didn't realize that the two larger order terms that ostensibly contribute to the slower rate of decay in fact cancel one another.)

We were intrigued by the fact that the exponential rate of convergence to equilibrium for Brownian motion physically returned to $p\gamma$ is independent of p. Note that the exponential rate of convergence to equilibrium is four times as fast as that of reflected Brownian motion and a little faster than that of Brownian motion conditioned to remain in $(0,\gamma)$. In light of the above considerations, we were interested in constructing a Markov process which would constitute the conditioning analog of the Brownian motion physically returned to a point, and to investigate its spectral gap and rate of convergence to equilibrium. The process in question should be such that when it is in $(0,p\gamma)$ it looks like Brownian motion conditioned to hit $p\gamma$ before 0, and when it is in $(p\gamma,\gamma)$ it looks like Brownian motion conditioned to hit $p\gamma$ before γ. Let $T_y=\inf\{t\ge 0: X(t)=y\}$. Let $\psi_1(x)=P_x(T_{p\gamma}<T_0)=\frac{x}{p\gamma}$, for $x\in[0,p\gamma]$ and let $\psi_2(x)=P_x(T_{p\gamma}<T_\gamma)=\frac{\gamma-x}{(1-p)\gamma}$, for $x\in[p\gamma,\gamma]$. As is well-known [6, chapter 7], Brownian motion on $(0,p\gamma)$ conditioned to hit $p\gamma$ before hitting 0 corresponds to the h–transformed operator $(\frac{1}{2}\frac{d^2}{dx^2})^{\psi_1}=\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}$ and Brownian motion on $(p\gamma,\gamma)$ conditioned to hit $p\gamma$ before hitting γ corresponds to the h–transformed operator $(\frac{1}{2}\frac{d^2}{dx^2})^{\psi_2}=\frac{1}{2}\frac{d^2}{dx^2}-\frac{1}{\gamma-x}\frac{d}{dx}$. Thus the process we are looking for is the solution to the martingale problem for the operator

$$L_p=\begin{cases}\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}, & \text{for } x\in(0,p\gamma)\\ \frac{1}{2}\frac{d^2}{dx^2}-\frac{1}{\gamma-x}\frac{d}{dx}, & \text{for } x\in(p\gamma,\gamma).\end{cases}\tag{1.1}$$

Note that the drift has a discontinuity at $p\gamma$. However, since the drift is bounded in a neighborhood of $p\gamma$, this discontinuity causes no problem; the martingale problem is well-posed for bounded measurable drifts and continuous diffusion coefficients

[10]. (The drift *is* unbounded at the endpoints but this too is no problem; indeed, this is what prevents the process from leaving $(0,\gamma)$.) We will call this process *Brownian motion returned to the point* $p\gamma$ *by conditioning.* The process can also be obtained via a procedure similar to the one used to obtain the Brownian motion conditioned to remain in $(0,\gamma)$. For $\delta>0$, let $\tau_1^{[p\gamma,p\gamma+\delta]}=\inf\{t\ge 0: X(t)=p\gamma\}$, $\sigma_n^{[p\gamma,p\gamma+\delta]}=\inf\{t>\tau_n^{[p\gamma,p\gamma+\delta]}: X(t)=p\gamma+\delta\}$ and $\tau_{n+1}^{[p\gamma,p\gamma+\delta]}=\inf\{t>\sigma_n^{[p\gamma,p\gamma+\delta]}: X(t)=p\gamma\}$, for $n\ge 1$. Then $\sigma_n^{[p\gamma,p\gamma+\delta]}$ is the time of the n-th upcrossing of the interval $[p\gamma,p\gamma+\delta]$. Instead of conditioning on $T_{0,\gamma}>t$ and letting $t\to\infty$, as was done for Brownian motion conditioned to remain in $(0,\gamma)$, we condition on $\sigma_n^{[p\gamma,p\gamma+\delta]}<T_{0,\gamma}$ and let $n\to\infty$. The limiting process will not be Markovian, but if we then let $\delta\to 0$ we obtain the desired process. We have the following result. Let $C([0,\infty);(0,\gamma))$ denote the space of continuous functions $X(t)=X(t,\omega)$ on $[0,\infty)$ with values in $(0,\gamma)$ and let $\mathcal{F}_t,\ t\ge 0$ denote the standard filtration.

Proposition 1. *Let* $P_x^{cond,p}$, $x\in(0,\gamma)$, *denote the probability measures on* $C([0,\infty);(0,\gamma))$ *corresponding to the Brownian motion returned to the point* $p\gamma$ *by conditioning; that is,* $P_x^{cond,p}$ *is the solution to the martingale problem for the operator in (1.1). Then* $P_x^{cond,p}$ *can be obtained as the following weak limit:*

$$P_x^{cond,p}(\cdot)=\lim_{\delta\to 0}\lim_{n\to\infty}P(\cdot\ |\sigma_n^{[p\gamma,p\gamma+\delta]}<T_{0,\gamma}).$$

Remark. One can of course work just as well with upcrossings of $[p\gamma-\delta_1,p\gamma+\delta_2]$ with $\delta_1,\delta_2\ge 0$ and $\delta_1+\delta_2>0$, and then let $\delta_1,\delta_2\to 0$.

An alternative way to characterize the process under consideration is in terms of the Radon-Nikodym derivative of $P_x^{cond,p}$ with respect to Wiener measure P_x, which as will be seen below, can be written in terms of the local time at $p\gamma$. Let $l_t^x=l_t^x(\omega)$ denote the local time at x of a Brownian path $X(\cdot,\omega)$ up to time t. That is, $l_t^x(\omega)$ is the density of the occupation measure $L(\cdot,\omega)=\int_0^t 1_{(\cdot)}(X(s,\omega))ds$ with respect to Lebesgue measure.

Proposition 2.

$$\frac{dP_x^{cond,p}}{dP_x}\Big|_{\mathcal{F}_{t\wedge T_{0,\gamma}}}=\frac{\psi_p(X(t\wedge T_{0,\gamma}))}{\psi_p(x)}\exp\left(\frac{l^{p\gamma}_{t\wedge T_{0,\gamma}}}{2\gamma p(1-p)}\right),$$

where

$$\psi_p(x) = \begin{cases} \frac{x}{p\gamma}, & \textit{for } x \in [0, p\gamma], \\ \frac{\gamma - x}{(1-p)\gamma}, & \textit{for } x \in [p\gamma, \gamma]. \end{cases} \tag{1.2}$$

Remark. Note that the Radon-Nikodym derivative "rewards" paths that spend a lot of time at $p\gamma$ (through the term $\exp(\frac{l^{p\gamma}_{t\wedge T_{0,\gamma}}}{2\gamma p(1-p)})$), and "penalizes" paths that stray away from $p\gamma$ toward the endpoints (through the term $\psi_p(X(t \wedge T_{0,\gamma}))$).

We will prove the following theorem. Let $p^{cond,p}(t, x, y)$ denote the transition probability density for Brownian motion returned to the point $p\gamma$ by conditioning.

Theorem 1. *Brownian motion returned to the point $p\gamma$ by conditioning has invariant probability density given by*

$$\mu^{cond,p}(x) = \begin{cases} \frac{3x^2}{p^2\gamma^3}, & \textit{if} \ \ 0 < x \le p\gamma; \\ \frac{3(\gamma - x)^2}{(1-p)^2\gamma^3}, & \textit{if} \ \ p\gamma \le x < \gamma. \end{cases}$$

One has

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in(0,\gamma)} |p^{(cond,p)}(t, x, y) - \mu^{(cond,p)}(y)| = -\frac{Q(p)}{\gamma^2},$$

where $Q(p)$ is a continuous, strictly increasing function on $(0, \frac{1}{2}]$ satisfying

$$Q(0^+) = \frac{1}{2}\kappa_0^2\pi^2 = (1.022...)\pi^2,$$

where $\kappa_0 = 1.430...$ is the smallest positive root of $\tan \pi x = \pi x$, and

$$Q(\frac{1}{2}) = 2\pi^2.$$

In fact, for $p \in (0, \frac{1}{2})$, $Q(p) = \frac{q^2}{2}$, where q is the smallest positive root of the equation

$$\frac{q \sin q}{\sin pq \sin(1-p)q} = \frac{1}{p(1-p)}.$$

The theorem allows us to conclude that for Brownian motion returned to a point by conditioning, the exponential rate of convergence to equilibrium depends on the particular point, unlike in the case of Brownian motion physically returned to a point. For $p = \frac{1}{2}$, the exponential rate of convergence to equilibrium is equal to that of Brownian motion physically returned to a point, but for all other p the rate is slower. Even the slowest rate is more than twice as fast as the rate for reflected Brownian motion. For p sufficiently close to 0 (sufficiently close to $\frac{1}{2}$), the rate is less than (greater than) that of Brownian motion conditioned to remain in $(0, \gamma)$.

Let

$$A_p = -\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x) \tag{1.3}$$

denote the one-dimensional Schrodinger operator on $[0, \gamma]$ with δ-potential $V_p(x) = -\frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x)$ and with the Dirichlet boundary condition. More precisely, A_p is defined via the quadratic form $q(\phi) = \frac{1}{2}\int_0^\gamma (\phi'(x))^2 dx - \frac{1}{2\gamma p(1-p)}\phi^2(p\gamma)$ operating on functions $\phi \in H_0^1((0, \gamma))$. See [7, Theorem X.17 (p. 167) and example 3 (page 168)]. The following result is an immediate corollary of the proof of Theorem 1.

Corollary 1. *Let $\lambda_0(p)$ and $\lambda_1(p)$ denote the first two eigenvalues of the Schrodinger operator A_p in (1.3) with the Dirichlet boundary condition. Then $\lambda_0(p) = 0$ and $\lambda_1(p) = \frac{Q(p)}{\gamma^2}$, where Q is as in Theorem 1.*

Remark. Consider a one-dimensional Schrodinger operator of the form $A = -\frac{1}{2}\frac{d^2}{dx^2} + V$ on $[0, \gamma]$ with the Dirichlet boundary condition, and denote its first two eigenvalues by λ_0 and λ_1. One calls V a *single-well potential* if there exists a $c \in [0, \gamma]$ such that V is nonincreasing on $[0, c]$ and nondecreasing on $[c, \gamma]$. The point c is called the *transition point*. It is known that if V is a single-well potential with transition point $\frac{\gamma}{2}$ or if V is a convex, single-well potential, then the spectral gap $\lambda_1 - \lambda_0$ satisfies the following inequality ([1,3,4]):

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{2\gamma^2}, \tag{1.4}$$

and equality holds if and only if V is constant. Note that the potential $V_p(x) = -\frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x)$ is a nonconvex, single-well potential with transition point $p\gamma$. By (1.4) it follows that $\lambda_1(\frac{1}{2}) - \lambda_0(\frac{1}{2}) \geq \frac{3\pi^2}{2\gamma^2}$, and in fact by Corollary 1 we have

$\lambda_1(\frac{1}{2})-\lambda_0(\frac{1}{2})=\frac{2\pi^2}{\gamma^2}$. On the other hand, the inequality $\lambda_1(p)-\lambda_0(p)\ge\frac{3\pi^2}{2\gamma^2}$ does not hold for p sufficiently close to 0.

The two propositions and the theorem will be proved in the next section.

2. Proof of Results.

Proof of Proposition 1. For ease of notation we will write τ_n^δ for $\tau_n^{[p\gamma,p\gamma+\delta]}$ and σ_n^δ for $\sigma_n^{[\rho\gamma,\rho\gamma+\delta]}$. For $n\ge1$, let $\mathcal{F}_{\tau_n^\delta}^{\sigma_n^\delta}=\sigma(X(t),\tau_n^\delta<t\le\sigma_n^\delta)$ and $\mathcal{F}_{\sigma_n^\delta}^{\tau_{n+1}^\delta}=\sigma(X((t),\sigma_n^\delta<t\le\tau_{n+1}^\delta)$. Also, define $\mathcal{F}_{\sigma_0^\delta}^{\tau_1^\delta}=\sigma(X(t),0\le t\le\tau_1^\delta)$. Let $P_x^{(n,\delta)}(X(\cdot)\in\cdot)\equiv P(X(\cdot)\in\cdot|\sigma_n^\delta<T_{0,\gamma})$. It follows from the definitions that for $m\le n$, the process corresponding to $P_x^{(n,\delta)}$, restricted to $\mathcal{F}_{\tau_m^\delta}^{\sigma_m^\delta}$, is Brownian motion conditioned to hit $p\gamma+\delta$ before hitting 0. This conditioned Brownian motion on $(0,p\gamma+\delta)$ is the diffusion corresponding to the h-transformed operator $(\frac{1}{2}\frac{d^2}{dx^2})^{\psi_{1\delta}}=\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}$ on $(0,p\gamma+\delta)$, where $\psi_{1,\delta}(x)=P_x(T_{p\gamma+\delta}<T_0)=\frac{x}{p\gamma+\delta}$. Note that this operator has the same form as has the restriction of L_p (in (1.1)) to the interval $(0,p\gamma)$. Thus, for $f\in C_0^\infty(R)$ and $m\le n$,

$$(2.1)\quad f(X(\tau_m^\delta\vee(t\wedge\sigma_m^\delta)))-\int_{\tau_m^\delta}^{\tau_m^\delta\vee(t\wedge\sigma_m^\delta)}L_pf(X(s))ds \text{ is a } P_x^{(n,\delta)}-\text{ martingale.}$$

Similarly, for $m<n$, it follows that on $\mathcal{F}_{\sigma_m^\delta}^{\tau_{m+1}^\delta}$, the process corresponding to $P_x^{(n,\delta)}$ is Brownian motion conditioned to hit $p\gamma$ before hitting γ. This conditioned Brownian motion on $(p\gamma,\gamma)$ is the diffusion corresponding to the h-transformed operator $(\frac{1}{2}\frac{d^2}{dx^2})^{\psi_2}=\frac{1}{2}\frac{d^2}{dx^2}-\frac{1}{\gamma-x}\frac{d}{dx}$ on $(p\gamma,\gamma)$, where $\psi_2(x)=P_x(T_{p\gamma}<T_\gamma)=\frac{\gamma-x}{(1-p)\gamma}$. This operator is the restriction of L_p (in (1.1)) to the interval $(p\gamma,\gamma)$. Thus, for $f\in C_0^\infty(R)$ and $m<n$,

$$(2.2)\quad f(X(\sigma_m^\delta\vee(t\wedge\tau_{m+1}^\delta)))-\int_{\sigma_m^\delta}^{\sigma_m^\delta\vee(t\wedge\tau_{m+1}^\delta)}L_pf(X(s))ds \text{ is a } P_x^{(n,\delta)}-\text{ martingale.}$$

Since $P_x^{(n,\delta)}|_{\mathcal{F}_{\sigma_m^\delta}}=P_x^{(m,\delta)}|_{\mathcal{F}_{\sigma_m^\delta}}$, for $m<n$, and since $\lim_{n\to\infty}P_x^{(n,\delta)}(\sigma_n^\delta\le t)=0$, for all $t\ge0$, it follows that there exists a measure $P_x^{(\delta)}$ on $\sigma(\cup_{n=1}^\infty\mathcal{F}_{\sigma_n^\delta})$ such that $P_x^{(\delta)}|_{\mathcal{F}_{\sigma_n^\delta}}=P_x^{(n,\delta)}|_{\mathcal{F}_{\sigma_n^\delta}}$ [10, Theorem 1.3.5]. Since $\lim_{n\to\infty}\sigma_n^\delta=\infty$ a.s. $[P_x^{(\delta)}]$, it follows that $\sigma(\cup_{n=1}^\infty\mathcal{F}_n)=\sigma(\cup_{n=1}^\infty\mathcal{F}_{\sigma_n^\delta})$; thus, $P_x^{(\delta)}$ is defined on $\mathcal{F}_\infty$.

From (2.1) and (2.2), it follows that for $f \in C_0^\infty(R)$,

(2.3) $f(X(t)) - \int_0^t L_p f(X(s)) 1_{[p\gamma, p\gamma+\delta]^c}(X(s)) ds - B_f^{(\delta)}(t)$ is a $P_x^{(\delta)}-$ martingale,

where $B_f^{(\delta)}(t)$ satisfies

(2.4)
$$|B_f^{(\delta)}(t)| \le M \int_0^t I_{[p\gamma, p\gamma+\delta]}(X(s)) ds,$$
$$\text{with } M = \sup_{x \in [p\gamma, p\gamma+\delta]} \max\{\frac{1}{x}|f'(x)|, \frac{1}{\gamma - x}|f'(x)|, \frac{1}{2}|f''(x)|\}.$$

From (2.3), it is clear that one can find a constant A_f such that $g(X(t)) - A_f t$ is a $P_x^{(\delta)}$-submartingale for all small $\delta > 0$ and all translates g of f (that is, $g(\cdot) = f(x_0 + \cdot)$ for some $x_0 \in R$). It then follows from [10, Theorem 1.4.6] that the measures $P_x^{(\delta)}$ are tight. From (2.4), it follows that $\lim_{\delta \to 0} P_x^{(\delta)}(|B_f^{(\delta)}(t)| > \epsilon) = 0$, for all $\epsilon > 0$ and $t > 0$. Thus, letting $\delta \to 0$ and using (2.3), we conclude that $f(X(t)) - \int_0^t L_p f(X(s)) ds$ is a Q-martingale for every accumulation point Q of $\{P_x^{(\delta)}\}$. This completes the proof since there exists a unique solution to the martingale problem for the operator L_p in (1.1). □

Proof of Proposition 2. The drift in (1.1) is given by $\frac{\psi_p'}{\psi_p}$, where ψ_p is as in (1.2). Thus, from Girsanov's theorem [6,9,10] it follows that

(2.5)
$$\frac{dP_x^{cond,p}}{dP_x}|_{\mathcal{F}_{t \wedge T_{0,\gamma}}} = \exp(\int_0^{t \wedge T_{0,\gamma}} \frac{\psi_p'}{\psi_p}(X(s)) dX(s) - \frac{1}{2} \int_0^{t \wedge T_{0,\gamma}} (\frac{\psi_p'}{\psi_p})^2 (X(s)) ds).$$

We have $\psi_p'((p\gamma)^+) - \psi_p'((p\gamma)^-) = -\frac{1}{\gamma p(1-p)}$. Furthermore, $\psi_p''(x) = 0$ for $x \neq p\gamma$. Thus, in the distributional sense, $\psi_p''(x) = -\frac{1}{\gamma p(1-p)} \delta_{p\gamma}(x)$, where $\delta_y(\cdot)$ is the Dirac delta-function at y. In particular then, $\frac{\psi_p''(x)}{2\psi_p(x)} = -\frac{1}{2\gamma p(1-p)} \delta_{p\gamma}(x)$. Thus, the generalized Ito formula for functions whose second derivatives are measures [9, IV-45] gives

(2.6)
$$\int_0^{t \wedge T_{0,\gamma}} \frac{\psi_p'}{\psi_p}(X(s)) dX(s) = \log \frac{\psi_p(X(t \wedge T_{0,\gamma}))}{\psi_p(x)} + \frac{1}{2} \int_0^{t \wedge T_{0,\gamma}} (\frac{\psi_p'}{\psi_p})^2 (X(s)) ds$$
$$+ \frac{1}{2\gamma p(1-p)} l_{t \wedge T_{0,\gamma}}^{p\gamma}.$$

Substituting (2.6) into (2.5), we obtain

$$\frac{dP_x^{cond,p}}{dP_x}|_{\mathcal{F}_{t \wedge T_{0,\gamma}}} = \frac{\psi_p(X(t \wedge T_{0,\gamma}))}{\psi_p(x)} \exp(\frac{l_{t \wedge T_{0,\gamma}}^{p\gamma}}{2\gamma p(1-p)}).$$

□

Proof of Theorem 1. The Brownian motion returned to the point $p\gamma$ by conditioning corresponds to the operator L_p in (1.1). Recall from the proof of Proposition 2 that $\frac{\psi_p''(x)}{2\psi_p(x)} = -\frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x)$, where ψ_p is as in (1.2). Consequently we can write the operator L_p as an h-transform in the following way:

$$L_p = (\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x))^{\psi_p}. \tag{2.7}$$

We need to impose the Dirichlet boundary condition on the operator $\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x)$ since ψ_p vanishes at 0 and γ. Note then that

$$L_p = (-A_p)^{\psi_p}, \tag{2.7$'$}$$

where A_p is the operator defined in the paragraph containing (1.3). It is easy to check that the adjoint of an operator of the form $\mathcal{A}^h$ is $\tilde{\mathcal{A}}^{\frac{1}{h}}$. Since $\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x)$ is self adjoint, we have $\tilde{L}_p = (\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x))^{\frac{1}{\psi_p}}$. The invariant probability density $\mu^{(cond,p)}$ must solve $\tilde{L}_p\mu^{(cond,p)} = 0$. Thus, $\mu^{(cond,p)}$ will be of the form $v\psi_p$, where v solves $(\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x))v = 0$. That is, v is the eigenfunction corresponding to the principal eigenvalue 0 for the operator $\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x)$. It follows that v must be linear on $[0, p\gamma]$, linear on $[p\gamma, \gamma]$, vanish at 0 and at γ, and have a jump discontinuity in its first derivative at $p\gamma$ of magnitude $-\frac{1}{\gamma p(1-p)}v(p\gamma)$. This gives $v(x) = a\psi_p(x)$, for some $a > 0$. Choosing a so that $v\psi$ is a probability density, we find that $\mu^{(cond,p)} = v\psi_p$ is given by the expression in the statement of the theorem.

We now turn to the spectrum of $-L_p$. We will show first that $-L_p$ has a compact resolvent; that is, there exists a sequence $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le ...$ of eigenvalues satisfying $\lim_{n\to\infty}\lambda_n = \infty$ and a corresponding complete sequence of eigenfunctions. By (2.7) and the spectral invariance under h-transforms, it suffices to show this for the operator $-(\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x))$ with the Dirichlet boundary condition. The quadratic form corresponding to this operator is given by $Q(u) = \int_0^\gamma \frac{(u'(x))^2}{2}dx - \frac{1}{2\gamma p(1-p)}u^2(p\gamma)$. Letting

$$\hat{\lambda}_n = \sup_{v_1,\dots,v_{n-1}} \inf_{u\in[v_1,\dots,v_{n-1}]^\perp, \int_0^\gamma u^2dx=1} Q(u),$$

where the infimum and the supremum are over functions u which vanish at 0 and γ, it follows from the mini-max principle [8] that if $\lim_{n\to\infty}\hat{\lambda}_n=\infty$, then the operator has a compact resolvent and $\lambda_n=\hat{\lambda}_n$. Using the boundary condition on u and the normalization, along with the inequality $|ab|\le\epsilon\frac{1}{2}a^2+\frac{1}{\epsilon}\frac{1}{2}b^2$, for $a,b\in R$ and $\epsilon>0$, we have

$$u^2(p\gamma)=\int_0^{p\gamma}(2uu')(x)dx\le\epsilon\int_0^{\gamma}\frac{(u'(x))^2}{2}dx+\frac{2}{\epsilon}.$$

Thus,

$$(1-\epsilon)\int_0^{\gamma}\frac{(u'(x))^2}{2}dx-\frac{1}{\epsilon\gamma p(1-p)}\le Q(u)\le\int_0^{\gamma}\frac{(u'(x))^2}{2}dx. \tag{2.8}$$

However, $\int_0^{\gamma}\frac{(u'(x))^2}{2}dx$ is the quadratic form for $\frac{1}{2}\frac{d^2}{dx^2}$, and as is well known, the eigenvalues for this operator are $\{\frac{\pi^2n^2}{2}\}_{n=1}^{\infty}$. Using this with (2.8) and the mini-max principle, it follows that $\lim_{n\to\infty}\frac{\hat{\lambda}_n}{n^2}=\frac{\pi^2}{2}$, completing the proof.

We now show that

$$\lim_{t\to\infty}\frac{1}{t}\log\sup_{x,y\in(0,\gamma)}|p^{(cond,p)}(t,x,y)-\mu^{(cond,p)}(y)|=-\lambda_1. \tag{2.9}$$

We can write L_p from (1.1) (or (2.7)) in divergence form as

$$L_p=\frac{1}{2}\frac{1}{\mu^{(cond,p)}}\frac{d}{dx}(\mu^{(cond,p)}\frac{d}{dx}). \tag{2.10}$$

(Recall that up to a multiplicative constant, $\mu^{(cond,p)}$ is equal to ψ_p^2.) It thus follows that L_p is self adjoint on $L^2_{\mu^{(cond,p)}}((0,\gamma))$. Since $\mu^{(cond,p)}$ vanishes at the endpoints, we impose no boundary conditions. Let $\{\phi_n\}_{n=0}^{\infty}$ be an orthonormal sequence of eigenfunctions corresponding to the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. In particular, $\phi_0=1$. Now, $\tilde{L}_p$, the adjoint (with respect to Lebesgue measure) of L_p, is easily seen to be self-adjoint with respect to the density $\frac{1}{\mu^{(cond,p)}}$. Thus, it possesses a sequence of eigenfunctions $\{\tilde{\psi}_n\}_{n=0}^{\infty}$, orthonormal in $L^2_{\frac{1}{\mu^{(cond,p)}}}((0,\gamma))$, corresponding to the same eigenvalues. One can check that $\tilde{\phi}_n=\phi_n\mu^{(cond,p)}$. In particular, $\tilde{\phi}_0=\mu^{(cond,p)}$. Since $p^{(cond,p)}(t,x,y)$ solves $u_t=L_pu$ as a function of t and x, solves $u_t=\tilde{L}_pu$ as a function of t and y, and satisfies $p^{(cond,p)}(0,x,y)=\delta_x(y)$, it follows that the equality

(2.11)

$$p^{(cond,p)}(t,x,y) = \sum_{n=0}^{\infty} \exp(-\lambda_n t)\phi_n(x)\tilde{\phi}_n(y) = \sum_{n=0}^{\infty} \exp(-\lambda_n t)\phi_n(x)\phi_n(y)\mu^{(cond,p)}(y)$$
$$= \mu^{(cond,p)}(y) + \sum_{n=1}^{\infty} \exp(-\lambda_n t)\phi_n(x)\phi_n(y)\mu^{(cond,p)}(y)$$

holds in L^2. If we show that the series $\sum_{n=1}^{\infty} \exp(-\lambda_n t)\phi_n(x)\phi_n(y)\mu^{(cond,p)}(y)$ converges uniformly over $x, y \in [0,\gamma] \times [0,\gamma]$, for all $t > 0$, then (2.11) will hold pointwise and also, clearly, (2.9) will hold.

In order to show this uniform convergence, we need to estimate ϕ_n. Before entering into calculations, we investigate the smoothness of ϕ_n at the singularity $p\gamma$. By (2.7) and (2.7′), L_p is the h transform of $-A_p$ with h-function ψ_p. Thus, $\phi_n = \frac{f_n}{\psi_p}$, where f_n is the eigenfunction corresponding to the eigenvalue λ_n for the operator $-A_p$. Since $\frac{1}{2}f_n'' + \frac{1}{2\gamma p(1-p)}\delta_{p\gamma} f_n = -\lambda_n f_n$, it follows that $f_n'(p\gamma^+) - f_n'(p\gamma^-) = -\frac{1}{\gamma p(1-p)} f_n(p\gamma)$. From (1.2), $\psi_p'(p\gamma^+) - \psi_p'(p\gamma^-) = -\frac{1}{\gamma p(1-p)}\psi_p(p\gamma)$. From these facts, it follows that ϕ_n' is continuous at $p\gamma$—the singularity at $p\gamma$ first enters into ϕ_n at the level of the second derivative.

We now estimate ϕ_n. We begin by showing that

$$\lim_{x\to 0} \mu^{(cond,p)}(x)\phi_n'(x) = \lim_{x\to\gamma} \mu^{(cond,p)}(x)\phi_n'(x) = 0. \tag{2.12}$$

In fact, (2.12) follows from a direct calculation using the facts that $\phi_n = \frac{f_n}{\psi_p}$, $\mu^{(cond,p)} = a\psi_p^2$, for some $a > 0$, and $f_n(0) = f_n(\gamma) = \psi_p(0) = \psi_p(\gamma)$.

Recall that by the normalization, $\int_0^\gamma \phi_n^2(x)\mu^{(cond,p)}(x)dx = 1$. Consider the equation $L_p\phi_n = -\lambda_n\phi_n$, where L_p is written as in (2.10). Since $(\mu^{(cond,p)}\phi_n')' = -2\lambda_n\phi_n\mu^{(cond,p)}$, it follows from (2.12) and the fact that ϕ_n' is continuous that

$$\phi_n'(x) = -\frac{2\lambda_n}{\mu^{(cond,p)}(x)} \int_0^x \phi_n(y)\mu^{(cond,p)}(y)dy. \tag{2.13}$$

Using Cauchy-Schwarz and the fact that $\mu^{(cond,p)} = a\psi_p^2$, for some $a > 0$, we obtain from (2.13)

$$
\begin{aligned}
|\phi_n'(x)| &\le \frac{2\lambda_n}{\mu^{(cond,p)}(x)}\int_0^x |\phi_n(y)|\mu^{(cond,p)}(y)dy \\
&\le \frac{2\lambda_n}{\mu^{(cond,p)}(x)}(\int_0^x \mu^{(cond,p)}(y)dy)^{\frac{1}{2}}(\int_0^x \phi_n^2(y)\mu^{(cond,p)}(y)dy)^{\frac{1}{2}} \\
&\le \frac{2\lambda_n}{\mu^{(cond,p)}(x)}(\int_0^x \mu^{(cond,p)}(y)dy)^{\frac{1}{2}} = \frac{2\lambda_n}{a^{\frac{1}{2}}\psi_p^2(x)}(\int_0^x \psi_p^2(y)dy)^{\frac{1}{2}}.
\end{aligned} \tag{2.14}
$$

Since $\psi_p(0)=0$ and $\psi_p'(0)\neq 0$, it follows that

$$
\frac{1}{\psi_p^2(x)}(\int_0^x \psi_p^2(y)dy)^{\frac{1}{2}} \le Cx^{-\frac{1}{2}}, \quad \text{for } x\in[0,\frac{\gamma}{2}], \quad \text{and some } C>0. \tag{2.15}
$$

From (2.14) and (2.15) along with the corresponding calculation starting from γ instead of 0, it follows that

$$
|\phi_n'(x)| \le \begin{cases} \frac{C\lambda_n}{x^{\frac{1}{2}}}, \text{ if } x\in[0,\frac{1}{2}\gamma] \\ \frac{C\lambda_n}{(\gamma-x)^{\frac{1}{2}}}, \text{ if } x\in[\frac{1}{2}\gamma,\gamma]. \end{cases} \tag{2.16}
$$

Since $\int_0^\gamma \phi_n^2(x)\mu^{(cond,p)}(x)dx=1$, there exists an x_n such that $|\phi_n(x_n)|=1$. From this and (2.16) if follows that $|\phi_n(x)|\le 1+C_1\lambda_n$, for some $C_1>0$. Using this along with the fact proved above that $\lim_{n\to\infty}\frac{\lambda_n}{n^2}=\frac{\pi^2}{2}$, it follows that the series $\sum_{n=1}^\infty \exp(-\lambda_n t)\phi_n(x)\phi_n(y)\mu^{(cond,p)}(y)$ converges uniformly.

In light of (2.9), to complete the proof of the theorem it remains to show that $\lambda_1=\frac{Q(p)}{\gamma^2}$, where Q is as in the statement of the theorem. As we've noted above, λ_1 is the first positive eigenvalue for the operator $\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x)$ with the Dirichlet boundary condition. It follows that λ_1 is the smallest positive number for which there exists a function $\hat{\phi}_1$ satisfying

$$
\begin{aligned}
&\frac{1}{2}\hat{\phi}_1''(x)+\frac{1}{2\gamma p(1-p)}\delta_{p\gamma}(x)\hat{\phi}_1(x)=-\lambda_1\hat{\phi}_1(x), \text{ for } x\in(0,\gamma); \\
&\hat{\phi}_1(0)=\hat{\phi}_1(\gamma)=0.
\end{aligned}
$$

Since $\frac{1}{2}\hat{\phi}_1''=-\lambda_1\hat{\phi}_1$ on $(0,p\gamma)$ and $\hat{\phi}_1(0)=0$, we have $\hat{\phi}_1(x)=\sin\frac{qx}{\gamma}$, $x\in[0,p\gamma]$, for some $q>0$, in which case

$$
\lambda_1=\frac{q^2}{2\gamma^2}. \tag{2.17}
$$

Then since $\frac{1}{2}\hat{\phi}_1'' = -\lambda_1\hat{\phi}_1$ on $(p\gamma, \gamma)$, we have $\hat{\phi}_1(x) = d_1 \cos\frac{qx}{\gamma} + d_2 \sin\frac{qx}{\gamma}$, for $x \in (p\gamma, \gamma)$. The condition $\hat{\phi}_1(\gamma) = 0$ gives

$$d_1 \cos q + d_2 \sin q = 0, \tag{2.18}$$

while the continuity requirement at $x = p\gamma$ gives

$$\sin qp = d_1 \cos qp + d_2 \sin qp. \tag{2.19}$$

Finally, in order to take care of the Dirac δ-function, we need

$$\hat{\phi}_1'(p\gamma^+) - \hat{\phi}_1'(p\gamma^-) = -\frac{1}{\gamma p(1-p)}\hat{\phi}_1(p\gamma),$$

which gives

$$-qd_1 \sin qp + qd_2 \cos qp - q\cos qp = -\frac{1}{p(1-p)} \sin qp. \tag{2.20}$$

The determinant of the matrix corresponding to the 2×2 equation arising from (2.18) and (2.19) for d_1, d_2 is equal to $\sin qp \cos q - \cos qp \sin q = -\sin(1-p)q$, while the inhomogeneous term is the vector $(0, \sin qp)$. Thus, there can only be a solution if both $\sin qp$ and $\sin(1-p)q$ vanish or if both do not vanish. We now restrict q to the interval $(0, 2\pi)$, in which case $\sin qp \neq 0$. Therefore, we must have $\sin(1-p)q \neq 0$ for there to be a solution. Solving (2.18) and (2.19), we obtain

$$\begin{aligned} d_1 &= \frac{\sin q \sin pq}{\sin(1-p)q}; \\ d_2 &= \frac{-\cos q \sin pq}{\sin(1-p)q}. \end{aligned} \tag{2.21}$$

Substituting (2.21) in (2.20) and using the identity $\sin q \sin pq + \cos q \cos pq = \cos(1-p)q$, we have

$$q\frac{\sin pq}{\sin(1-p)q}\cos(1-p)q + q\cos pq = \frac{1}{p(1-p)}\sin pq. \tag{2.22}$$

Dividing by $\sin pq$, then obtaining a common denominator and using the identity $\sin pq \cos(1-p)q + \cos pq \sin(1-p)q = \sin pq$, we can rewrite (2.22) as

$$\frac{q \sin q}{\sin pq \sin(1-p)q} = \frac{1}{p(1-p)}.$$

Defining

$$f(x) = \frac{\sin x}{x},$$

we arrive at the equation

$$\frac{f(q)}{f((1-p)q)f(pq)} = 1. \tag{2.23}$$

We first show that the smallest positive root $q = q(p)$ of (2.23) does not fall in the interval $(0, \pi)$. For fixed $q \in (0, \pi)$, let $h(p) = f((1-p)q)f(pq)$. Then

(2.24)

$$h'(p) = qf((1-p)q)f'(pq) - qf(pq)f'((1-p)q) = qh(p)\left(\frac{f'}{f}(pq) - \frac{f'}{f}((1-p)q)\right).$$

We have

$$\frac{f'}{f}(x) = \cot x - \frac{1}{x}. \tag{2.25}$$

Since $f(x) > 0$ and $\cot x - \frac{1}{x} < 0$ for $x \in (0, \pi)$, we conclude from (2.24) and (2.25) that $f((1-p)q)f(pq)$ is decreasing as a function of $p \in (0, \frac{1}{2}]$. Using this along with the fact that $\lim_{p \to 0} \frac{f(q)}{f((1-p)q)f(pq)} = 1$ shows that $\frac{f(q)}{f((1-p)q)f(pq)} < 1$ for $p \in (0, \frac{1}{2}]$ and $q \in (0, \pi)$. We conclude that the smallest positive root $q = q(p)$ of (2.23) does not fall in $(0, \pi)$.

Consider now $q \in [\pi, 2\pi)$. Note that $0 < f(x) < 1$ for $x \in (0, \pi)$ and $f(x) < 0$ for $x \in (\pi, 2\pi)$. We have $pq < \frac{1}{2}q < \pi$; thus $0 < f(pq) < 1$. Therefore, in order for the left hand side of (2.23) to be positive, we need $q > \frac{\pi}{1-p}$. Since our calculations have been under the assumption that $q < 2\pi$, it follows in particular that if $p = \frac{1}{2}$, then there is no solution $q \in [\pi, 2\pi)$. On the other hand the reader can easily check that (2.18)-(2.20) hold with $p = \frac{1}{2}, d_1 = 0, d_2 = 1$ and $q = 2$. Thus, it follows that $q(\frac{1}{2}) = 2\pi$. Therefore, from (2.17) we obtain in the notation of the theorem $Q(\frac{1}{2}) = 2\pi^2$.

From now on, we assume that $p \in (0, \frac{1}{2})$. Recalling from the statement of the theorem that κ_0 is the smallest positive root of $\tan \pi x = \pi x$, it follows from (2.25) that f is decreasing on $[0, \kappa_0\pi]$ and increasing on $[\kappa_0\pi, 2\pi]$. Using this along with the fact that $0 < f(pq) < 1$, we conclude that (2.23) cannot occur if $q > \frac{\pi}{1-p}$ but $q \leq \kappa_0\pi$. Thus a necessary condition for (2.23) to hold is that

$$q > \max(\frac{\pi}{1-p}, \kappa_0\pi). \tag{2.26}$$

For fixed $p \in (0, \frac{1}{2})$ the left hand side of (2.23) equals $+\infty$ at $q = (\frac{\pi}{1-p})^+$ and equals 0 at $q = (2\pi)^-$. From this and (2.26) we conclude that the smallest positive root $q = q(p)$ of (2.23) satisfies

$$\max(\frac{\pi}{1-p}, \kappa_0\pi) < q(p) < 2\pi. \tag{2.27}$$

Using (2.27) along with (2.17) allows us to conclude in the notation of the theorem that

$$\max(\frac{1}{2(1-p)^2}, \frac{\kappa_0^2}{2})\pi^2 < Q(p) < 2\pi^2.$$

We will now show that if we set $(1-p)q = \kappa_0\pi$, then

$$\frac{f(q)}{f((1-p)q)f(pq)} = \frac{f(\kappa_0\pi + \frac{p\kappa_0\pi}{1-p})}{f(\kappa_0\pi)f(\frac{p\kappa_0\pi}{1-p})} < 1 \ \text{ for sufficiently small } p > 0. \tag{2.28}$$

As we noted above, for fixed $p \in (0, \frac{1}{2})$, $\frac{f(q)}{f((1-p)q)f(pq)}$ equals $+\infty$ at $q = (\frac{\pi}{1-p})^+$; thus, we conclude from (2.28), (2.27) and (2.23) that $\kappa_0\pi < q(p) < \frac{\kappa_0\pi}{1-p}$ for small $p > 0$. Letting $p \to 0$ then shows that $q(0^+) = \kappa_0\pi$. In the notation of the theorem, this gives $Q(0^+) = \frac{\kappa_0^2\pi^2}{2}$.

To prove (2.28), we expand $f(\kappa_0\pi + \frac{p\kappa_0\pi}{1-p})$ about $\kappa_0\pi$ and $f(\frac{p\kappa_0\pi}{1-p})$ about 0. One checks that $f'(0) = 0$ and $f''(0) = -\frac{1}{3}$. Thus, $\frac{f''}{f}(0) = -\frac{1}{3}$. Since f reaches a minimum at $\kappa_0\pi$, we have $f'(\kappa_0\pi) = 0$. Using the fact that $\tan\kappa_0\pi = \kappa_0\pi$, one can check that $\frac{f''}{f}(\kappa_0\pi) = -1$. Thus, we have

$$\begin{aligned}&\frac{f(\kappa_0\pi + \frac{p\kappa_0\pi}{1-p})}{f(\kappa_0\pi)f(\frac{p\kappa_0\pi}{1-p})} = \left(1 - \frac{1}{2}(\frac{p\kappa_0\pi}{1-p})^2 + o(p^2)\right)\left(\frac{1}{1 - \frac{1}{6}(\frac{p\kappa_0\pi}{1-p})^2 + o(p^2)}\right),\\ &= 1 - \frac{1}{3}(\frac{p\kappa_0\pi}{1-p})^2 + o(p^2), \text{ as } p \to 0,\end{aligned}$$

which proves (2.28).

It remains to prove that $Q(p)$ is monotone. Since $Q(p) = \frac{1}{2}q^2(p)$, it suffices to show that $q(p)$ is monotone. By the implicit function theorem, $q(p)$ is differentiable for $p \in (0, \frac{1}{2})$. We will show that $q'(p) > 0$. Differentiating the equation $\frac{f(q)}{f((1-p)q)f(pq)} = 1$ with respect to p and doing some algebra, we obtain

$$\frac{q'(p)}{q(p)} = \frac{\frac{f'}{f}(pq) - \frac{f'}{f}((1-p)q)}{\frac{f'}{f}(q) - p\frac{f'}{f}(pq) - (1-p)\frac{f'}{f}((1-p)q)}. \tag{2.29}$$

We will show that both the numerator and the denominator on the right hand side of (2.29) are negative.

Using (2.25) and a trigonometric identity, we have

$$\begin{aligned} \frac{f'}{f}(pq) + \frac{f'}{f}((1-p)q) &= \cot pq - \frac{1}{pq} + \cot(1-p)q - \frac{1}{(1-p)q} \\ &= -\frac{1}{qp(1-p)} + \frac{\sin q}{\sin pq \sin(1-p)q}. \end{aligned} \tag{2.30}$$

From the equality $\frac{f(q)}{f((1-p)q)f(pq)} = 1$ and the definition of f, we have

$$\sin pq \sin(1-p)q = p(1-p)q \sin q.$$

Substituting this into (2.30) gives

$$\frac{f'}{f}(pq) + \frac{f'}{f}((1-p)q) = 0. \tag{2.31}$$

Since $\frac{f'}{f}(x) < 0$, for $x \in (0, \pi)$, and since $pq \in (0, \pi)$, it follows from (2.31) that the numerator on the right hand side of (2.29) is negative. Using (2.31) again, we have

$$\frac{f'}{f}(q) - p\frac{f'}{f}(pq) - (1-p)\frac{f'}{f}((1-p)q) = \frac{f'}{f}(q) + (1-2p)\frac{f'}{f}(pq). \tag{2.32}$$

By (2.27), $q = q(p) \in (\kappa_0\pi, 2\pi)$. Since $\frac{f'}{f}(x) < 0$ for $x \in (\kappa_0\pi, 2\pi)$ and since $\frac{f'}{f}(pq) < 0$, it follows from (2.32) that the denominator on the right hand side of (2.29) is also negative. This completes the proof of the monotonicity. $\square$

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