COMPARISON THEOREMS FOR THE SPECTRAL GAP OF DIFFUSIONS PROCESSES AND SCHRÖDINGER OPERATORS ON AN INTERVAL

Ross G. Pinsky

Department of Mathematics
Technion-Israel Institute of Technology
Haifa, 32000 Israel
email:pinsky@math.technion.ac.il

Abstract. We compare the spectral gaps and thus the exponential rates of convergence to equilibrium for ergodic one-dimensional diffusions on an interval. One of the results may be thought of as the diffusion analog of a recent result for the spectral gap of one-dimensional Schrödinger operators. We also discuss the similarities and differences between spectral gap results for diffusions and for Schrödinger operators.

1. Introduction and Statement of Results. The main point of this note is to elucidate the similarities and differences between comparison theorems for the spectral gap of ergodic one-dimensional diffusions $X(t)$ on an interval and of one-dimensional Schrödinger operators on an interval, and to prove a particular comparison result on the spectral gap for diffusions that can be considered as the analog of a recent comparison result on the spectral gap for Schrödinger operators. By ergodic, we mean that the distribution of the diffusion process converges to an invariant distribution as $t \to \infty$. We consider two types of ergodic diffusions $X(t)$ on the interval $[0, \gamma]$. Let

$$L = \frac{1}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}. \tag{1.1}$$

1991 *Mathematics Subject Classification.* 60J60, 35P15.
Key words and phrases. spectral gap, invariant measure, diffusion process, Schrödinger operator.
This research was supported by the Fund for the Promotion of Research at the Technion and by the V.P.R. Fund.

I. $X(t)$ is a reflected diffusion on $[0,\gamma]$ generated by L as in (1.1), where b is piecewise C^1 (but not necessarily continuous) on $[0,\gamma]$, with the Neumann (reflected) boundary condition $u'(0)=u'(\gamma)=0$.

II. $X(t)$ is a diffusion on $(0,\gamma)$ generated by L as in (1.1), where b is piecewise C^1 on $(0,\gamma)$ (but not necessarily continuous) and satisfies $\int_0 \exp(-\int_{\frac{\gamma}{2}}^x 2b(y)dy)dx = \int^\gamma \exp(-\int_{\frac{\gamma}{2}}^x 2b(y)dy)dx = \infty$. (This integral condition on b is necessary and sufficient for the process to never leave $(0,\gamma)$ [8, chapter 5]. In particular, it will be satisfied if $\lim_{x\to 0} xb(x) \geq \frac{1}{2}$ and $\lim_{x\to\gamma}(\gamma - x)b(x) \leq -\frac{1}{2}$.) No boundary condition is imposed because the endpoints are not regular boundary points; that is, the diffusion cannot reach them.

For each of these two types of diffusions, the invariant probability density is given by

$$\phi(x) = \frac{1}{c_b}\exp\Big(\int_{\frac{\gamma}{2}}^x 2b(y)dy\Big),$$

where $c_b = \int_0^\gamma \exp(\int_{\frac{\gamma}{2}}^x 2b(y)dy)dx$. Conversely,

$$b = \frac{\phi'}{2\phi}. \tag{1.2}$$

Note that ϕ is continuous and piecewise C^2 on $[0,\gamma]$.

Consider now a Schrödinger operator H of the form

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V$$

on $[0,\gamma]$, where V is piecewise continuous, with either the Neumann boundary condition ($u'(0)=u'(\gamma)=0$) or the Dirichlet boundary condition ($u(0)=u(\gamma)=0$). There is an intimate connection between the diffusion operators and the Schrödinger operators. Let λ_0^H denote the principal eigenvalue for such an H and let $q>0$ denote the corresponding eigenfunction. Then the h-transformed operator $(-H+\lambda_0^H)^q$ defined by $(-H+\lambda_0^H)^q u = \frac{1}{q}(-H+\lambda_0^H)(qu)$ is given by

$$(-H+\lambda_0^H)^q = \frac{1}{2}\frac{d^2}{dx^2} + \frac{\phi'}{2\phi}\frac{d}{dx},$$

where $\phi = q^2$. The right hand side above is just the operator L in (1.1). Conversely, starting from L as in (1.1) and using (1.2), define the h-transformed operator $L^{\frac{1}{\sqrt{\phi}}}$

by $L^{\frac{1}{\sqrt{\phi}}}u = \phi^{\frac{1}{2}}L(\phi^{-\frac{1}{2}}u)$. Then $-L^{\frac{1}{\sqrt{\phi}}} + \lambda_0^H = H$, with the potential $V = -\frac{q''}{2q} + \lambda_0^H$, where $q = \phi^{\frac{1}{2}}$. Let $\mathcal{V}_I$ ($\mathcal{V}_{II}$) denote the class of potentials for which the principal eigenfunction q of the Schrödinger equation with the Neumann (Dirichlet) boundary condition is such that $b = \frac{\phi'}{2\phi} = \frac{q'}{q}$ satisfies the conditions in the definition of type I (type II) diffusion. In particular, it is easy to see that $\mathcal{V}_I$ and $\mathcal{V}_{II}$ include all the smooth potentials on $[0, \gamma]$.

Under the above h-transforms, there is a one to one correspondence between Schrödinger operators with $\mathcal{V}_I$ potentials with the Neumann boundary condition and the class of diffusion operators of type I for which the drift $b = \frac{\phi'}{2\phi}$ satifies $\phi'(0) = \phi'(\gamma) = 0$. (The general type I diffusion with $b = \frac{\phi'}{2\phi}$ corresponds to a Schrödinger operator with the mixed boundary condition $\frac{u'}{u}(0) = \frac{\phi'}{2\phi}(0)$.) And under the h transform, there is a one to one correspondence between Schrödinger operators with $\mathcal{V}_{II}$ potentials with the Dirichlet boundary condition and the class of type II diffusion operators.

We now point out a few spectral theoretical facts. The operator L can be written in the form

$$Lu = \frac{1}{2}\phi^{-1}(\phi u')'.$$

Taking into account the Neumann boundary condition in the case of type I diffusion and the fact that ϕ vanishes at the endpoints in the case of type II diffusion, it follows that L is symmetric with respect to the density ϕ; that is, it is symmetric on $L^2([0, \gamma]; \phi)$. The operator $-L$ possesses a compact resolvent and thus there exists a sequence of eigenvalues $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le ...$ satisfying $\lim_{n\to\infty} \lambda_n = \infty$, and a corresponding complete orthonormal sequence of eigenfunctions $\{\psi_n\}_{n=0}^{\infty}$ in $L^2([0, \gamma]; \phi)$. The so-called *spectral gap*, the distance between λ_0 and λ_1, is then equal to λ_1. Since $\psi_0 = 1$, it follows from the mini-max principle that

$$\lambda_1 = \inf_{u: \int_0^\gamma u\phi dx = 0} \frac{\frac{1}{2}\int_0^\gamma (u')^2 \phi dx}{\int_0^\gamma u^2 \phi dx}. \tag{1.3}$$

The spectral gap is equal to the exponential rate of convergence of the process to equilibrium. Indeed, let $p(t, x, y) = P_x(X(t) \in dy)$ denote the transition probability density of the diffusion process. Then $p(t,x,y)$ can be written in the eigenfunction expansion

$$p(t,x,y) = \sum_{n=0}^{\infty} \exp(-\lambda_n t)\psi_n(x)\psi_n(y)\phi(y) = \phi(y) + \sum_{n=1}^{\infty} \exp(-\lambda_n t)\psi_n(x)\psi_n(y)\phi(y).$$

By the mini-max principle and comparison with the operator $\frac{1}{2}\frac{d^2}{dx^2}$, one can show that the eigenvalues grow sufficiently fast so that $\sum_{n=0}^{\infty} \exp(-\lambda_n t) < \infty$ for all $t > 0$ (that is, the semigroup is trace class). From these facts it follows immediately that

$$\lim_{t\to\infty} \frac{1}{t} \log ||p_\phi(t,\cdot,\cdot) - 1||_2 = -\lambda_1,$$

where $p_\phi(t,x,y) = \frac{p(t,x,y)}{\phi(y)}$ is the transition density with respect to the invariant measure and $||\cdot||_2$ refers to the norm on $L^2([0,\gamma]\times[0,\gamma];\phi\times\phi)$. In fact we have the stronger result,

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in(0,\gamma)} |p(t,x,y) - \phi(y)| = -\lambda_1.$$

For a proof of this, see [9] which treats a similar situation.

The h-transform is spectrally invariant; thus the eigenvalues of the Schrödinger operator H corresponding to L are given by $\{\lambda_0^H + \lambda_n\}_{n=0}^{\infty}$. In particular, the spectral gap—the difference between the first two eigenvalues—of the corresponding Schrödinger operator is equal to that of the diffusion operator.

We now cite several comparison theorems for the spectral gap of Schrödinger operators. The potential V is called a *single-well* potential if there exists an $x_0 \in [0,\gamma]$ such that V is nonincreasing on $[0,x_0]$ and nondecreasing on $[x_0,\gamma]$. If V satisfies the reverse inequalities, it is called a *single-barrier* potential. The point x_0 is called the *transition point.* The potential is called *symmetric* if it is symmetric with respect to $\frac{\gamma}{2}$. In the case that V is constant, the spectral gap can easily be computed; it equals $\frac{3\pi^2}{2\gamma^2}$ in the Dirichlet case and $\frac{\pi^2}{2\gamma^2}$ in the Neumann case.

Theorem L ([6]). *Let the potential V be convex. Then the spectral gap for the Schrödinger operator $H = -\frac{1}{2}\frac{d^2}{dx^2} + V$ is greater than or equal to $\frac{3\pi^2}{2\gamma^2}$ in the Dirichlet case and greater than or equal to $\frac{\pi^2}{2\gamma^2}$ in the Neumann case. In either case, equality holds only for constant potentials.*

In [1] it was shown for the Dirichlet case that if V is a symmetric, single-well (single-barrier) potential, then the spectral gap is greater than (less than) or equal to $\frac{3\pi^2}{2\gamma^2}$, with equality holding only for constant potentials. This was extended recently to single-well potentials with transition point $x_0 = \frac{\gamma}{2}$.

Theorem H ([4]). *Let V be a single-well potential with transition point $\frac{\gamma}{2}$. Then the spectral gap for the Schrödinger operator $H = -\frac{1}{2}\frac{d^2}{dx^2} + V$ is greater than or equal to $\frac{3\pi^2}{2\gamma^2}$ in the Dirichlet case. Equality holds only for constant potentials.*

Remark. It was also shown in [4] that the reverse inequality does not always hold in the Dirichlet case for single-barrier potentials with transition point $x_0 = \frac{\gamma}{2}$.

The above results compare the spectral gap of Schrödinger operators with certain types of potentials to the corresponding spectral gaps for the Schrödinger operator with constant potential in the case of the Neumann or the Dirichlet boundary condition, or both. Now the Schrödinger operator with constant potential and the Neumann boundary condition corresponds to the reflected Brownian motion ($b \equiv 0$ and $\phi = \frac{1}{\gamma}$), while the Schrödinger operator with constant potential and the Dirichlet boundary condition corresponds to the so-called Brownian motion conditioned never to exit $(0, \gamma)$ ($b = \frac{\pi}{\gamma} \cot \frac{\pi}{\gamma} x$ and $\phi = \frac{2}{\gamma} \sin^2 \frac{\pi}{\gamma} x$) [9]. Thus, in the spirit of the above results for the spectral gap of Schrödinger operators, it will be natural to compare the spectral gap of diffusions with certain types of invariant densities (or equivalently, certain types of drifts) to those of the reflected Brownian motion and the Brownian motion conditioned never to exit $(0, \gamma)$.

In order to compare the spectral gap results for diffusions and for Schrödinger operators, we record explicitly how the drift and the invariant density of the diffusion process and the potential of the corresponding Schrödinger operator are related—in terms of the principal eigenfunction q and principal eigenvalue λ_0^H of the Schrödinger operator:

$$
\begin{aligned}
&\text{drift: } b = \frac{q'}{q} \\
(1.4) \qquad &\text{invariant density: } \phi = q^2 \\
&\text{potential: } V = -\frac{q''}{2q} + \lambda_0^H.
\end{aligned}
$$

In the sequel, depending on which is more convenient in the particular context, we will sometimes identify the diffusion by its drift b, sometimes by its invariant measure ϕ, and sometimes by both. We will use the notation $\lambda_1(b)$ to denote the dependence of λ_1 on the drift b and λ_1^ϕ to denote its dependence on the invariant measure ϕ.

There is an analog of Theorem L for diffusion operators which essentially goes back to Payne and Weinberger [7] (see also [11, Lemma 1]).

Proposition (Payne-Weinbeger).

$$\lambda_1^\phi \geq \frac{\pi^2}{2\gamma^2},$$

for all log-concave densities ϕ, with equality only in the case of reflected Brownian motion ($\phi = \frac{1}{\gamma}$).

Remark. In light of (1.4), it follows that Theorem L for Schrödinger operators requires the convexity of $-\frac{q''}{2q}$, while the Payne-Weinberger result for diffusions requires the log-concavity of q; these two conditions are of course not comparable in general.

We will obtain a diffusion analog of Theorem H. The natural analog of a single-well (single-barrier) potential in the diffusion case is a single-barrier (single-well) shaped invariant density ϕ. From (1.4) it is clear that a single-well (single-barrier) potential does not at all guarantee a single-barrier (single-well) invariant density and vice versa. It is also clear from (1.4) that even if the potential is single-well (single-barrier) and the invariant density is single-barrier (single-well), their transition points are not necessarily the same. Thus, the diffusion case and the Schrödinger case are not comparable in general.

Theorem 1. *Assume that b is anti-symmetric with respect to $\frac{\gamma}{2}$ (equivalently, ϕ is symmetric with respect to $\frac{\gamma}{2}$).*

i. Let $\hat{b} \not\equiv b$ satisfy $\hat{b} \geq b$ on $(0, \frac{\gamma}{2})$ and $\hat{b} \leq b$ on $(\frac{\gamma}{2}, \gamma)$. Then

$$\lambda_1(\hat{b}) > \lambda_1(b).$$

ii. Let $\hat{b} \not\equiv b$ satisfy $\hat{b} \leq b$ on $(0, \frac{\gamma}{2})$ and $\hat{b} \geq b$ on $(\frac{\gamma}{2}, \gamma)$. Then

$$\lambda_1(\hat{b}) < \lambda_1(b).$$

The following corollaries are the diffusion analogs of Theorem H.

Corollary 1.

i. If $b \geq 0$ on $(0, \frac{\gamma}{2})$ and $b \leq 0$ on $(\frac{\gamma}{2}, \gamma)$ (equivalently, ϕ is nondecreasing on $(0, \frac{\gamma}{2})$ and nonincreasing on $(\frac{\gamma}{2}, \gamma)$), then

$$\lambda_1(b) = \lambda_1^{\phi} \geq \frac{\pi^2}{2\gamma^2},$$

with equality only in the case of reflected Brownian motion ($b \equiv 0$, $\phi = \frac{1}{\gamma}$).

ii. If $b \leq 0$ on $(0, \frac{\gamma}{2})$ and $b \geq 0$ on $(\frac{\gamma}{2}, \gamma)$ (equivalently, ϕ is nonincreasing on $(0, \frac{\gamma}{2})$ and nondecreasing on $(\frac{\gamma}{2}, \gamma)$), then

$$\lambda_1(b) = \lambda_1^{\phi} \leq \frac{\pi^2}{2\gamma^2},$$

with equality only in the case of reflected Brownian motion ($b \equiv 0$, $\phi = \frac{1}{\gamma}$).

iii. Parts (i) and (ii) are not true in general if the point $\frac{\gamma}{2}$ is replaced by any other point in $(0, \gamma)$.

Corollary 2. *Let*

$$b_0(x) = \frac{\pi}{\gamma} \cot \frac{\pi}{\gamma} x.$$

i. If $b \geq b_0(x)$ on $(0, \frac{\gamma}{2})$ and $b \leq b_0(x)$ on $(\frac{\gamma}{2}, \gamma)$, then

$$\lambda_1(b) \geq \frac{3\pi^2}{2\gamma^2},$$

with equality only in the case of Brownian motion conditioned never to exit $(0, \gamma)$ ($b = b_0(x)$).

ii. If $b \le b_0(x)$ on $(0, \frac{\gamma}{2})$ and $b \ge b_0(x)$ on $(\frac{\gamma}{2}, \gamma)$, then

$$\lambda_1(b) \le \frac{3\pi^2}{2\gamma^2},$$

with equality only in the case of Brownian motion conditioned never to exit $(0, \gamma)$ $(b = b_0(x))$.

Remark 1. Corollary 2 and parts (i) and (ii) of Corollary 1 follow immediately as particular cases of Theorem 1.

Remark 2. Corollaries 1 and 2 give lower bounds on the spectral gap for single-barrier shaped invariant densities and upper bounds for single-well shaped invariant densities. However, as we have seen from Theorem H and the remark following it, for Schrödinger operators with the Dirichlet boundary condition, the corresponding lower bound holds for single-well potentials, but the corresponding upper bound for single-barrier potentials does not always hold. A corresponding result for the Schrödinger operator with the Neumann boundary condition and single-well or single-barrier potentials does not seem to be known.

We now investigate how one may obtain arbitrarily large and arbitrarily small spectral gaps; that is, we consider sequences $b_n = \frac{\phi_n'}{2\phi_n}$ for which $\lim_{n\to\infty} \lambda_1(b_n) = \lim_{n\to\infty} \lambda_1^{\phi_n}$ equals ∞ or 0. We begin by noting the following fact:

$$\lambda_1^{\phi} \le \frac{1}{2Var(\phi)}. \tag{1.5}$$

To prove (1.5), let $\mu = \int_0^\gamma x\phi dx$ be the expected value under ϕ, and let $u(x) = x - \mu$. Then $\int_0^\gamma u\phi dx = 0$; thus, by (1.3),

$$\lambda_1^{\phi} \le \frac{\frac{1}{2}\int_0^\gamma (u')^2\phi dx}{\int_0^\gamma u^2\phi dx} = \frac{1}{2Var(\phi)}.$$

From (1.5) we find that a necessary condition for an arbitrarily large spectral gap is that $\lim_{n\to\infty} Var(\phi_n) = 0$, from which it follows that all accumulation points of the measures $\{\phi_n dx\}$ are degenerate δ-measures. However, in fact there exists a necessary condition that is much stronger than (1.5). For $0 \le c < d \le \gamma$, let $\phi^{[c,d]}$

denote the density obtained by restricting ϕ to $[c,d]$ and renormalizing. Thus,

$$\phi^{[c,d]}(x) = \begin{cases} \frac{\phi(x)}{\int_c^d \phi(y)dy}, & \text{if } x \in [c,d]; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.

$$\lambda_1^{\phi} \le \frac{1}{2\sup_{0\le c<d\le\gamma} Var(\phi^{[c,d]})}.$$

Thus, a necessary condition for

$$\lim_{n\to\infty} \lambda_1^{\phi_n} = \infty$$

is that

$$\lim_{n\to\infty} \sup_{0\le c<d\le\gamma} Var(\phi_n^{[c,d]}) = 0. \tag{1.6}$$

Remark. We suspect that condition (1.6) is sufficient (or close to it), but have no proof.

The following result gives an explicit construction of sequences that result in arbitrarily large and arbitrarily small spectral gaps.

Proposition 1.

i. Let $x_1 \in (0,\gamma)$ and let b_n satisfy $b_n(x) \ge n$ for $x \in [0,x_1)$ and $b_n(x) \le -n$ for $x \in (x_1,\gamma]$. Then

$$\lim_{n\to\infty} \lambda_1(b_n) = \lim_{n\to\infty} \lambda_1^{\phi_n} = \infty.$$

ii. Let $0 \le c_1 < c_2 < d_1 < d_2 \le \gamma$ and let b_n satisfy $b_n(x) \le -n$, for $x \in [c_1,c_2]$ and $b_n(x) \ge n$, for $x \in [d_1,d_2]$. Then

$$\lim_{n\to\infty} \lambda_1(b_n) = \lim_{n\to\infty} \lambda_1^{\phi_n} = 0.$$

Remark 1. Note that it is "harder" to obtain an arbitrarily large spectral gap than to obtain an arbitrarily small one. Indeed, for an arbitrarily large one, it follows from Theorem 2 that the drifts $\{b_n\}$ must be tightly controlled uniformly on every subinterval of $[0,\gamma]$. However, by Proposition 1, it is possible to have an arbitrarily small spectral gap while controlling $\{b_n\}$ on two fixed subintervals of any given length.

Remark 2. In part (ii), generically, ϕ_n will be shaped like a well. As $n \to \infty$, the height above the x-axis at the bottom of the well approaches 0.

Remark 3. At the expense of a lot of additional calculations, one can use continuous drifts in part (i).

For general estimates on the spectral gap of multidimensional diffusions, see for example [2] and [3]. For general estimates on the spectral gap of multidimensional Schrödinger operators, see for example [10], [12], [5] and [11].

We prove Theorem 1 and part (iii) of Corollary 1 in section 2. The proofs of Theorem 2 and Proposition 1 are given in section 3.

2. Proofs of Theorem 1 and Part (iii) of Corollary 1. There exists a natural monotonicity property for principal eigenvalues which is lacking for second eigenvalues. We exploit a simple technique to convert the study of second eigenvalues to the study of certain principal eigenvalues. We will define two principal eigenvalues. The precise definition will depend on whether we are considering type I or type II diffusions, but the same notation will be used in both cases. Consider first type I. For $x \in (0,\gamma)$, let $\lambda_0^{ND}(b;0,x)$ denote the principal eigenvalue for L on $[0,x]$ with the Neumann boundary condition at 0 ($u'(0)=0$) and the Dirichlet boundary condition at x ($u(x)=0$), and let $\lambda_0^{DN}(b;x,\gamma)$ denote the principal eigenvalue for L on $[x,\gamma]$ with the Dirichlet boundary condition at x ($u(x)=0$) and the Neumann boundary condition at γ ($u'(\gamma)=0$). Now consider type II. For $x \in (0,\gamma)$, let $\lambda_0^{ND}(b;0,x)$ denote the principal eigenvalue for L on $(0,x]$ with no boundary condition at 0 and the Dirichlet boundary condition at x ($u(x)=0$), and let $\lambda_0^{DN}(b;x,\gamma)$ denote the principal eigenvalue for L on $[x,\gamma)$ with the Dirichlet boundary condition at x ($u(x)=0$) and no boundary condition at γ.

We will need the following monotonicity result.

Proposition 2. *As a function of* b, $\lambda_0^{ND}(b;0,x)$ *is strictly increasing and* $\lambda_0^{DN}(b;x,\gamma)$

is strictly decreasing, while as a function of x, $\lambda_0^{ND}(b; 0, x)$ *is strictly decreasing and* $\lambda_0^{DN}(b; x, \gamma)$ *is strictly increasing.*

Remark. Using probabilistic considerations, it is easy to show non-strict monotonicity. To obtain the strict monotonicity, we resort to an analytic proof, which we postpone until the end of the section.

The following proposition allows us to convert the study of the second eigenvalue into the study of certain principal eigenvalues.

Proposition 3. *There exists a unique point* $x_0 = x_0(b) \in (0, \gamma)$ *such that*

$$\lambda_1(b) = \lambda_0^{ND}(b; 0, x_0(b)) = \lambda_0^{DN}(b; x_0(b), \gamma).$$

Proof. The infimum in (1.3) is of course attained at the eigenfunction u_1 which satisfies $\frac{1}{2}\phi^{-1}(\phi u_1')' = -\lambda_1 u_1$, along with the boundary condition $u_1'(0) = u_1'(\gamma) = 0$ in the case of type I diffusion. For either type of diffusion, it follows from a uniqueness theorem for ordinary differential equations that $u_1(0) \neq 0$. In the course of the proof of Proposition 2 below, we will prove that u_1' is negative on $[0, \gamma]$. Thus, since $\int_0^\gamma u_1 \phi dx = 0$, it follows that there exists a unique point $x_0 = x_0(b) \in (0, \gamma)$ at which u_1 vanishes. (At least in the case of type I diffusion, this actually follows from classical oscillation theory for Sturm-Liouville operators.) Noting that $\frac{1}{2}\phi^{-1}(\phi u_1')' = -\lambda_1 u_1$ on $(0, x_0]$ and on $[x_0, \gamma)$, that u_1 maintains its sign on each of these intervals, that $u_1(x_0) = 0$ and, in the case of type I diffusion, that $u_1'(0) = u_1'(\gamma) = 0$, it follows that u_1 on $(0, x_0(b)]$ must be the eigenfunction corresponding to the principal eigenvalue $\lambda_0^{ND}(b; 0, x_0(b))$ and that u_1 on $[x_0(b), \gamma)$ must be the eigenfunction corresponding to the principal eigenvalue $\lambda_0^{DN}(b; x_0(b), \gamma)$. We conclude that $\lambda_1 = \lambda_1(b)$ satisfies

$$\lambda_1(b) = \lambda_0^{ND}(b; 0, x_0(b)) = \lambda_0^{DN}(b; x_0(b), \gamma).$$

Finally, by the strict monotonicity in Proposition 2, $\lambda_0^{ND}(b; 0, x) \neq \lambda_0^{DN}(b; x, \gamma)$, if $x \neq x_0(b)$. $\square$

Propositions 2 and 3 yield the following corollaries.

Corollary 3. *For any* $x \in (0, \gamma)$,

$$\min(\lambda_0^{ND}(b; 0, x), \lambda_0^{DN}(b; x, \gamma)) \leq \lambda_1(b) \leq \max(\lambda_0^{ND}(b; 0, x), \lambda_0^{DN}(b; x, \gamma)),$$

with strict inequality holding if $\lambda_0^{ND}(b; 0, x) \neq \lambda_0^{DN}(b; x, \gamma)$.

Proof. By Proposition 2, $\lambda_0^{ND}(b; 0, x)$ is strictly decreasing in x and $\lambda_0^{DN}(b; x, \gamma))$ is strictly increasing in x. The corollary follows from this along with Proposition 3. □

Corollary 4. *Let* $x_0(b)$ *be as in Proposition 3.*
i. If $\hat{b} \neq b$ *satisfies* $\hat{b} \geq b$ *on* $(0, x_0(b))$ *and* $\hat{b} \leq b$ *on* $(x_0(b), \gamma)$*, then* $\lambda_1(\hat{b}) > \lambda_1(b)$.
ii. If $\hat{b} \neq b$ *satisfies* $\hat{b} \leq b$ *on* $(0, x_0(b))$ *and* $\hat{b} \geq b$ *on* $(x_0(b), \gamma)$*, then* $\lambda_1(\hat{b}) < \lambda_1(b)$.

Proof. The proof is immediate from Corollary 3 and Proposition 2. □

Proof of Theorem 1. Theorem 1 now follows directly from Corollary 4 and the fact that, by symmetry considerations, $x_0(b) = \frac{\gamma}{2}$. □

Proof of part (iii) of Corollary 1. Let $x_1 < \frac{\gamma}{2}$. We will give an example where $b \geq 0$ on $[0, x_1]$ and $b \leq 0$ on $[x_1, \gamma]$, yet $\lambda_1(b) < \frac{\pi^2}{2\gamma^2}$. The other cases are treated similarly. Choose x_2, x_3, x_4 satisfying $x_1 < x_2 < x_3 < x_4 < \frac{\gamma}{2}$ and choose $\delta > 0$ sufficiently small so that $\lambda_0^{DN}(-\delta; x_4, \gamma) < \frac{\pi^2}{2\gamma^2}$. This is possible because by Proposition 2, $\lambda_0^{DN}(0; x_4, \gamma) < \lambda_0^{DN}(0; \frac{\gamma}{2}, \gamma)$ and a direct calculation gives $\lambda_0^{DN}(0; \frac{\gamma}{2}, \gamma) = \frac{\pi^2}{2\gamma^2}$. Let b_0 satisfy $b_0 \geq 0$ on $[0, x_1]$ and $b_0(x_1) = 0$. For $N > 0$, let b_N satisfy $b_N = b_0$ on $[0, x_1]$, $b_N \leq 0$ on $[x_1, \gamma]$, $b_N = -\delta$ on $[x_4, \gamma]$ and $b_N \leq -N$ on $[x_2, x_3]$. A comparison result similar to Proposition 2 shows that $\lambda_0^{ND}(b_N; 0, x_4) \leq \lambda_0^{ND}(b_N; x_2, x_3)$. Thus, $\lambda_0^{ND}(b_N; 0, x_4) \leq \lambda_0^{ND}(b_N; x_2, x_3) = \lambda_0^{ND}(-N; x_2, x_3)$, and a direct calculation, which we leave to the reader, shows that $\lim_{N\to\infty} \lambda_0^{ND}(-N; x_2, x_3) = 0$. Therefore, $\lambda_0^{ND}(b_N; 0, x_4) < \frac{\pi^2}{2\gamma^2}$, for sufficiently large N. By construction $\lambda_0^{DN}(b_N; x_4, \gamma) = \lambda_0^{DN}(-\delta; x_4, \gamma) < \frac{\pi^2}{2\gamma^2}$. Thus, by Corollary 1, $\lambda_1(b_N) < \frac{\pi^2}{2\gamma^2}$, for sufficiently large N. □

We now return to prove Proposition 2.

Proof of Proposition 2. We prove the proposition for $\lambda_0^{ND}(b;0,x)$. We first show that $\lambda_0^{ND}(b;0,x)$ is strictly decreasing in x. Fix x_1 and x_2 with $0 < x_1 < x_2 < \gamma$ and let $u_i > 0$, $i = 1,2$, denote the eigenfunction corresponding to the principal eigenvalue $\lambda^{ND}(b;0,x_i)$. Multiply the equation $\frac{1}{2}(\phi u_1')' = -\lambda^{ND}(b;0,x_1)\phi u_1$ by u_2 and integrate both sides from 0 to x_1. Integrating by parts twice on the left hand side and using the fact that $(u_1' u_2 \phi)(0) = (u_1 u_2' \phi)(0) = 0$ for both type I and type II diffusion, and that $u_1(x_1) = 0$, we obtain

$$(\lambda_0^{ND}(b;0,x_2) - \lambda_0^{ND}(b;0,x_1)) \int_0^{x_1} u_1(x)u_2(x)\phi(x)dx = (u_1' u_2 \phi)(x_1). \tag{2.1}$$

Strict monotonicity follows from (2.1) and the fact that, by the Hopf maximum principal, $u_1'(x_1) < 0$.

We now show that $\lambda_0^{ND}(b;0,x)$ is strictly increasing in b. Fix $x \in (0,\gamma]$ and let b_1, b_2 be drifts satisfying $b_1 \le b_2$ on $[0,x]$ and $b_1 \not\equiv b_2$. Define $b(t,y) = (1-t)b_1(y) + tb_2(y)$ and $\phi(t,y) = \exp(\int_{\frac{\gamma}{2}}^{y} 2b(t,z)dz)$, and use the notation $\lambda(t)$ for $\lambda_0^{ND}(b(t,\cdot);0,x)$ and $u(t,y)$ for the corresponding principal eigenfunction, normalized by $\int_0^x u^2(t,y)\phi(t,y)dy = 1$. To complete the proof, we will show that $\lambda(t)$ is strictly increasing. We will use a prime to denote differentiation in y and the subscript t to denote differentiation in t. Differentiating

$$\frac{1}{2}u''(t,y) + b(t,y)u'(t,y) = -\lambda(t)u(t,y)$$

in t gives

$$\frac{1}{2}u_t'' + bu_t' + \lambda u_t = -\lambda_t u - b_t u'. \tag{2.2}$$

Since $u(t,x) = 0$, we also have $u_t(t,x) = 0$. Similarly, in the case of type I diffusion, we have $u_t'(t,0) = 0$. Rewrite the left hand side of (2.2) as $\frac{1}{2}\frac{1}{\phi}(\phi u_t')'$, multiply the equation by ϕu and integrate both sides. Integrating by parts twice shows that the left hand side vanishes; thus

$$\lambda_t(t) = -\int_0^x u(t,y)u'(t,y)\phi(t,y)b_t(t,y)dy. \tag{2.3}$$

Now $b_t = b_2 - b_1 \gneq 0$ by assumption, and we will show presently that $u' < 0$ on $[0, \gamma]$. Thus, we conclude from (2.3) that $\lambda(t)$ is strictly increasing.

We turn to the proof that $u' < 0$. Using the fact that $u_1'(0) = 0$ in the case of type I diffusion and the fact that $\phi(0) = 0$ in the case of type II diffusion, we have for either type

$$u_1'(x) = -\frac{2\lambda_1}{\phi} \int_0^x \phi u_1(y) dy. \tag{2.4}$$

From (2.4) and the fact that $\int_0^\gamma u_1 \phi dx = 0$, it follows that there exists an $x_1 \in (0, \gamma]$ such that $u_1'(x_1) = 0$ and u_1' is negative on $[0, x_1)$. To complete the proof that $u_1' < 0$ on $(0, \gamma)$ we will show that $x_1 = \gamma$. Integrating $\frac{1}{2}\phi^{-1}(\phi u_1')' = -\lambda_1 u_1$ from 0 to x_1 and using the fact that $u_1'(x_1) = 0$ and that either $u_1'(0) = 0$ or $\phi(0) = 0$, we obtain

$$\lambda_1 = \frac{\frac{1}{2}\int_0^{x_1} (u_1')^2 \phi dx}{\int_0^{x_1} u_1^2 \phi dx}. \tag{2.5}$$

Let

$$v(x) = \begin{cases} u_1(x), & \text{if } x \in [0, x_1]; \\ u_1(x_1), & \text{if } x \in [x_1, \gamma]. \end{cases}$$

Define $a = \int_0^\gamma v\phi dx$. Letting $w = v - a$, we have $\int_0^\gamma w\phi dx = 0$ and

$$\lambda_1 \le \frac{\frac{1}{2}\int_0^\gamma (w')^2 dx}{\int_0^\gamma w^2 \phi dx} = \frac{\frac{1}{2}\int_0^{x_1} (u_1')^2 dx}{\int_0^{x_1} u_1^2 \phi dx + (u_1(x_1) - a)^2 \int_{x_1}^\gamma \phi dx}, \tag{2.6}$$

where the inequality follows from (1.3). Now

$$u_1(x_1) - a = u_1(x_1) - \int_0^{x_1} u_1 \phi dx - u_1(x_1) \int_{x_1}^\gamma \phi dx = \int_0^{x_1} (u_1(x_1) - u_1(x))\phi(x) dx < 0,$$

where the inequality follows from the fact that u_1 is strictly decreasing on $[0, x_1]$. In particular, $(u_1(x_1) - a)^2 \neq 0$. Thus, we conclude that $x_1 = \gamma$ since, by (2.5), the right hand side of (2.6) would be less than λ_1 if $x_1 < \gamma$. □

3. Proofs of Theorem 2 and Proposition 1.

Proof of Theorem 2. Let $\delta > 0$ and choose an interval $[c_\delta, d_\delta]$ satisfying $0 < c_\delta < d_\delta < \gamma$ and such that $\sup_{0 \le c < d \le \gamma} Var(\phi^{[c,d]}) \le Var(\phi^{[c_\delta, d_\delta]}) + \delta$. Note that $\frac{(\phi^{[c_\delta,d_\delta]})'}{\phi^{[c_\delta,d_\delta]}} = \frac{\phi'}{\phi} = 2b$ on $[c_\delta, d_\delta]$. Let $\lambda_1^{(\delta)}$ denote the first nonzero eigenvalue for L

on $[c_\delta, d_\delta]$ with the Neumann boundary condition at the endpoints. Then by (1.5) along with the definition of $[c_\delta, d_\delta]$, we have

$$\lambda_1^{(\delta)} \le \frac{1}{2Var(\phi^{[c_\delta,d_\delta]})} \le \frac{1}{2(\sup_{0\le c<d\le\gamma} Var(\phi^{[c,d]}) - \delta)}. \tag{2.7}$$

By Proposition 3, there exists an $x_0^{(\delta)} \in (c_\delta, d_\delta)$ such that

$$\lambda_1^{(\delta)} = \lambda_0^{ND}(b; c_\delta, x_0^{(\delta)}) = \lambda_0^{DN}(b; x_0^{(\delta)}, d_\delta). \tag{2.8}$$

By the comparison result similar to Proposition 2 that was noted in the proof of Corollary 1-iii, we have

$$\lambda_0^{ND}(b; 0, x_0^{(\delta)}) \le \lambda_0^{ND}(b; c_\delta, x_0^{(\delta)}) \text{ and } \lambda_0^{DN}(b; x_0^{(\delta)}, \gamma) \le \lambda_0^{DN}(b; x_0^{(\delta)}, d_\delta). \tag{2.9}$$

From (2.7)-(2.9) it follows that

$$\max(\lambda_0^{ND}(b; 0, x_0^{(\delta)}),\ \lambda_0^{DN}(b; x_0^{(\delta)}, \gamma)) \le \frac{1}{2(\sup_{0\le c<d\le\gamma} Var(\phi^{[c,d]}) - \delta)}. \tag{2.10}$$

The theorem now follows from (2.10), Corollary 3 and the fact that $\delta > 0$ is arbitrary. □

Proof of Proposition 1. *i.* By Corollary 3, it is enough to show that $\lim_{n\to\infty} \lambda_0^{ND}(n; 0, x_1) = \infty$ and $\lim_{n\to\infty} \lambda_0^{DN}(-n; x_1, \gamma) = \infty$. These are direct calculations that we leave to the reader.

ii. By the comparison result similar to Proposition 2 that was used in the proofs of Corollary 1-iii and Theorem 2, we have $\lambda_0^{ND}(b_n; 0, c_2) \le \lambda_0^{ND}(b_n; c_1, c_2) = \lambda_0^{ND}(-n; c_1, c_2)$. A direct calculation which we leave to the reader shows that $\lim_{n\to\infty} \lambda_0^{ND}(-n; c_1, c_2) = 0$. Therefore, we have $\lim_{n\to\infty} \lambda_0^{ND}(b_n; 0, c_2) = 0$. A similar argument shows that $\lim_{n\to\infty} \lambda_0^{ND}(b_n; c_2, \gamma) = 0$. We then conclude from Corollary 3 that $\lim_{n\to\infty} \lambda_1(b_n) = 0$. □

References

1. Ashbaugh, M. and Benguria, R., *Optimal lower bound for the gap between the first two eigenvalues of one-dimensional Schrödinger operators with symmetric single-well potentials*, Proc. Amer. Math. Soc. **105** (1989), 419-424.

2. Chen, M. F. and Wang, F. Y., *Estimation of the first eigenvalue of second order elliptic operators*, J. Funct. Anal. **131** (1995), 345–363.
3. Chen, M. F. and Wang, F. Y., *Estimation of spectral gap for elliptic operators*, Trans. Amer. Math. Soc. **349** (1997), 1239–1267.
4. Horváth, M., *On the first two eigenvalues of Sturm-Liouville operators*, Proc. Amer. Math. Soc. **131** (2003), 1215-1224.
5. Kirsch, W. and Simon, B., *Comparison theorems for the gap of Schrödinger operators*, J. Funct. Anal. **75** (1987), 396–410.
6. Lavine, R., *The eigenvalue gap for one-dimensional convex potentials*, Proc. Amer. Math. Soc. **121** (1994), 815-821.
7. Payne, L. E. and Weinberger, H. F., *An optimal Poincare' inequality for convex domains*, Arch. Rational Mech. Anal. **5** (1960), 286–292.
8. Pinsky, R.G., *Positive Harmonic Functions and Diffusion*, Cambridge University Press, 1995.
9. Pinsky, R., *Spectral gap and rate of convergence to equilibrium for a class of conditioned Brownian motions, submitted.*
10. Singer, I. M., Wong, B., Yau, S.-T. and Yau, S. S.-T., *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4) **12** (1985), 319–333.
11. Smits, R. G., *Spectral gaps and rates to equilibrium for diffusions in convex domains*, Michigan Math. J. **43** (1996), 141–157.
12. Yu, Q. H. and Zhong, J. Q., *Lower bounds of the gap between the first and second eigenvalues of the Schrödinger operator*, Trans. Amer. Math. Soc. **294** (1986), 341–349.