# GLOBAL EXISTENCE/NONEXISTENCE FOR SIGN-CHANGING SOLUTIONS TO $u_t = \Delta u + |u|^p$ IN $R^d$

Ross G. Pinsky

Department of Mathematics
Technion-Israel Institute of Technology
Haifa, 32000 Israel
email:pinsky@math.technion.ac.il

Abstract. Consider the equation

$$(*)\qquad \begin{aligned} &u_t = \Delta u + |u|^p \text{ in } R^d \times (0,T);\\ &u(x,0) = \phi(x) \text{ in } R^d, \end{aligned}$$

up to a maximal time $T = T_\infty$, where $p > 1$. Let $p* = 1 + \frac{2}{d}$. It is a classical result that if $p \le p^*$, then there exist no nonnegative, global solutions to (*) for any choice of $\phi \gneq 0$; that is, necessarily $T_\infty < \infty$ and the solution blows up in some sense. On the other hand, if $p > p^*$, then there do exist nonnegative, global solutions for appropriate choices of $\phi$. A recent paper by Zhang [16] seems to be the first to consider the existence of global solutions for (*) with sign-changing initial data. He proved that if $p \le p^*$ and $\int_{R^d} \phi(x)dx > 0$, then the solution to (*) is not global. In this paper it is shown that this result continues to hold if $\int_{R^d} \phi(x)dx = 0$, and then it is shown that for each $p > 1$ and each $l \in (-\infty, 0)$, there exists a $\phi$ satisfying $\int_{R^d} \phi(x)dx = l$ and such that the corresponding solution to (*) is global, and there exists a $\phi$ satisfying $\int_{R^d} \phi(x)dx = l$ and such that the corresponding solution is non-global.

**1. Introduction.** The study of finite time blow-up and global existence for solutions of the semilinear parabolic equation

$$(1.1)\qquad \begin{aligned} &u_t = \Delta u + |u|^p,\ (x,t) \in R^d \times (0,T);\\ &u(x,0) = \phi(x),\ x \in R^d, \end{aligned}$$

up until a maximal time $T = T_\infty$, where $p > 1$ and $d \ge 1$ goes back to Fujita [4]. He proved that if $1 < p < 1 + \frac{2}{d}$, then there are no nonnegative, global solutions to (1.1) with nontrivial initial data; that is, necessarily, $T_\infty < \infty$ and the solution

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blows up in some sense. On the other hand, he showed that if $p > 1 + \frac{2}{d}$, then there exist nonnegative, global solutions with nontrivial initial data; that is, there are solutions with $T_\infty = \infty$. A number of authors extended this result to the critical case, $p = p^* \equiv 1 + \frac{2}{d}$, showing that there are no nonnegative global solutions [5,7,14]. While Fujita's work has continued to spawn much research on various related problems (see the survey papers [8,2]), a recent paper by Zhang [16] seems to be the first one to study the question of global existence for the above problem when the initial data changes sign. (The papers [9, 10] treat global existence for the equation $u_t = u_x x + |u|^{p-1}u$ in one dimension with sign-changing initial data, but this is a different problem.)

We now describe Zhang's result. Let $w$ denote the solution to the corresponding linear heat equation:

$$w_t = \Delta w, \ (x, t) \in R^d \times (0, T);$$

$$w(x, 0) = \phi(x), \ x \in R^d.$$

It seems clear that any solution to (1.1) will dominate $w$, however this has never been proven for sign-changing solutions. In Theorem 1 below, we show that under the growth condition (1.2), any solution to (1.1) must satisfy the integral equation (1.3), from which it follows in particular that any such solution dominates $w$. The representation (1.3) was proven for *all* nonnegative solutions without any growth condition [15]. Zhang proved the following result:

**Theorem (**Zhang, [16]). *Consider (1.1) with $\phi \in L^1(R^d)$. Define $p^* = 1 + \frac{2}{d}$.*

*i. If $p < p^*$ and $\int_{R^d} \phi(x)dx > 0$, then (1.1) has no global, classical solution which dominates the solution to the corresponding linear equation.*

*ii. If $p = p^*$, $\int_{R^d} \phi(x)dx > 0$ and $\phi^- \equiv -min(\phi(x), 0)$ is compactly supported, then (1.1) has no global, classical solution which dominates the solution to the corresponding linear equation.*

**Remark.** Zhang states his theorem without the domination condition, but in his proof he assumes that the solution to (1.1) is dominated by the solution to the corresponding linear heat equation.

If $\phi \leq 0$, then $v \equiv -u$ satisfies $v_t = \Delta v - |v|^p$ with $v(x,0) \geq 0$, and it is known that this equation has a unique solution and this solution is global, nonnegative and dominated by the corresponding solution to the linear heat equation. In light of this and Zhang's result, it remains to study what happens if $\phi$ changes sign and $\int_{R^d} \phi(x)dx \leq 0$.

We will show that if $\int_{R^d} \phi(x)dx = 0$, then Zhang's result continues to hold. Then we will show that for each $l \in (-\infty, 0)$ and each $p > 1$, there exists a global solution with initial data $\phi$ satisfying $\int_{R^d} \phi(x)dx = l$, and there exists a non-global solution with initial data $\phi$ satisfying $\int_{R^d} \phi(x)dx = l$ . The reason one can obtain a global solution when $p \in (1, p^*]$ with sign-changing initial data $\phi$ satisfying $\int_{R^d} \phi(x)dx < 0$ is that for appropriate configurations of $\phi$, the diffusion from the region where $u$ is negative will prevail over the reaction term $|u|^p$ and drive the solution down below 0 everywhere. Once the solution is everywhere negative it remains so, as noted in the previous paragraph. On the other hand, when $p \in (1, p^*]$ and $\int_{R^d} \phi(x)dx \geq 0$, the diffusion cannot overcome the reaction term to drive the solution below 0, regardless of how $\phi$ is chosen.

In order to extend Zhang's result, we will will need a representation result and a comparison result, which we think are interesting in their own right and which we now state. Let $p(t,x,y) = (4\pi t)^{-\frac{d}{2}} \exp(-\frac{|y-x|^2}{4t})$.

**Theorem 1.** *Let $u \in C^{2,1}(R^d \times (0,T]) \cap C(R^d \times [0,T])$ satisfy (1.1) and assume that the initial data $\phi$ is Hölder continuous. Assume that there exists a $c > 0$ such that*

$$|u(x,t)| \leq c \exp(c|x|^2),\ (x,t) \in R^d \times [0,T]. \tag{1.2}$$

*Then $u(x,t)$ satisfies*

$$u(x,t) = \int_{R^d} p(t,x,y)\phi(y)dy + \int_0^t \int_{R^d} p(t-s,x,y)|u(y,s)|^p dyds,\ (x,t) \in R^d \times [0,T]. \tag{1.3}$$

**Remark.** As noted above, the representation (1.3) was proven in [15] for all non-negative solutions without any growth condition. We will prove Theorem 1 by amending that proof.

**Theorem 2.** *Let $\phi \in L^r(R^d) \cap L^\infty(R^d)$, for some $r \geq \max(p, \frac{d(p-1)}{2})$, and assume that $\phi$ is Hölder continuous. There exists a $T_\phi \in (0, \infty]$ and a unique solution $u_\phi$ to (1.3) on the time interval $[0, T_\phi)$. The solution $u_\phi(\cdot, t)$ is a continuous map from $[0, T_\phi)$ into $L^r(R^d) \cap L^\infty(R^d)$. If $T_\phi < \infty$, then $\lim_{t \to T_\phi} \max(||u_\phi^+(\cdot, t)||_r, ||u_\phi^+(\cdot, t)||_\infty) = \infty$ and $\sup_{0 \leq t < T_\phi} \max(||u_\phi^-(\cdot, t)||_r, ||u_\phi^-(\cdot, t)||_\infty) < \infty$. This solution is also a classical solution to (1.1).*

*Consider now the additional condition*

$$\lim_{|x| \to \infty} \phi(x) = 0. \tag{1.4}$$

*Then*

$$\lim_{|x| \to \infty} \sup_{0 \leq t \leq T} |u_\phi(x, t)| = 0, \quad \text{for } 0 < T < T_\phi, \tag{1.5}$$

*and solutions with different initial data can be compared as follows. If $\phi_1$ and $\phi_2$ satisfy (1.4) and $\phi_2 \geq \phi_1$, then $T_{\phi_2} \leq T_{\phi_1}$ and $u_{\phi_2} \geq u_{\phi_1}$ on $R^d \times [0, T_{\phi_2})$.*

**Remark.** Our proof of Theorem 2 relies on some fundamental results of Weissler [12,13,14]. The proof that the solution to (1.3) in Theorem 2 is in fact a classical solution to (1.1) relies on the method used in [15].

The next two theorems extend Zhang's result.

**Theorem 3.** *Consider (1.1) with Hölder continuous $\phi \in L^1(R^d)$. Define $p^* = 1 + \frac{2}{d}$.*
*i. Let $p < p^*$ and let $\phi \not\equiv 0$ satisfy $\int_{R^d} \phi(x)dx = 0$. Then (1.1) has no global classical solution satisfying (1.2). If in addition, $\phi \in L^\infty(R^d)$, then $\phi \in L^s(R^d)$, for all $s > 1$, and the solution $u_\phi$ from Theorem 2 is non-global.*
*ii. Let $p = p^*$ and let $\phi \not\equiv 0$ satisfy $\int_{R^d} \phi(x)dx = 0$. Assume in addition that $\phi^-(x) \leq c_1 \exp(-c_2|x|^2)$, for some $c_1, c_2 > 0$. Then (1.1) has no global classical solution satisfying (1.2). If in addition, $\phi \in L^\infty(R^d)$, then $\phi \in L^s(R^d)$, for all $s > 1$, and the solution $u_\phi$ from Theorem 2 is non-global.*

**Theorem 4.** *i. For each $p > 1$ and for each $l \in (-\infty, 0)$, there exists a classical global solution to (1.1) whose initial data $\phi$ changes sign and satisfies $\int_{R^d} \phi(x)dx = l$.*

*ii. For each $p > 1$ and for each $l \in (-\infty, 0)$, there exists a classical solution to (1.1) that is non-global and whose initial data $\phi$ is compactly supported and satisfies $\int_{R^d} \phi(x)dx = l$.*

**Remark.** From Zhang's theorem and Theorems 3 and 4, we conclude that the parameter $l \equiv \int_{R^d} \phi(x)dx$ governs the existence of global solutions when $l \geq 0$; namely, there are none. However, this one dimensional parameter, when negative, does not contain enough information to determine whether there will be global solutions.

The proof of Theorem 4 shows what type of initial data yield global solutions in (i) and what type of initial data yield non-global solutions in (ii). Let $B_R$ denote the open ball of radius $R$ centered at the origin. Fix $R_1, R_2, R_3$ satisfying $0 < R_1 < R_2 < R_3$. Let $\phi_1 \not\equiv 0$ be a continuous, nonnegative function supported in $B_{R_1}$ and let $\phi_2 \geq 0$ be a continuous function with support satisfying $B_{R_3} - B_{R_2} \subset \text{supp}(\phi_2) \subset R^d - B_{R_1}$. Let $\phi$ satisfy $\phi \leq \epsilon\phi_1 - \phi_2$. Then for sufficiently small $\epsilon > 0$, the solution with initial data $\phi$ will be global. (The construction in the proof of Theorem 4-i is a little different, but the reader can check that the proof does indeed show that the solution corresponding to $\phi$ as above will be global.) Note that the support of $\phi^-$ "surrounds" the support of $\phi^+$. The diffusion from the region of negative initial values surrounding the inner core of positive initial values sufficiently dominates the reaction term $|u|^p$ and drives the solution below 0 everywhere.

For initial data leading to a non-global solution, let $\phi_1, \phi_2 \gneq 0$ be continuous, compactly supported functions satisfying $\text{supp}(\phi_1) \cap \text{supp}(\phi_2) = \emptyset$ and $\int_{R^d} \phi_1(x)dx < \int_{R^d} \phi_2(x)dx$. Let $\phi$ satisfy $\phi \geq \lambda(\phi_1 - \phi_2)$. Then for sufficiently large $\lambda > 0$, the solution with initial data $\phi$ will be non-global.

We will prove Theorems 1 and 2 in section 2 and Theorems 3 and 4 in section 3.

## 2. Proofs of Theorems 1 and 2.

**Proof of Theorem 1.** As noted in the remark following the statement of the theorem, in the case that $\phi \geq 0$, it was proved in [15] that all nonnegative classical solutions to (1.1) satisfy (1.3). The method of proof was to show that the left hand side of (1.3) is larger or equal to the right hand side, and that the difference between the left hand and right hand side is a solution to the linear heat equation with zero initial data. Invoking the Widder uniqueness theorem for nonnegative solutions to the heat equation then gives equality in (1.3). In the case of general, not necessarily nonnegative initial data $\phi$, the method in [15] goes through to show that the difference between the left and right hand sides of (1.3) is a solution to the linear heat equation with zero initial data. The assumption (1.2) then guarantees that there exists a $C > 0$ such that the absolute value of this difference is bounded by $C\exp(C|x|^2)$ on the time interval $[0,T]$. As is well-known, such a solution must be identically zero [6]. □

For the proof of Theorem 2 we will need the following well-known comparison result.

**Proposition 1.** *For some $T \in (0,\infty)$, let $u, v \in C^{2,1}(R^d \times (0,T]) \cap C(R^d \times [0,T])$ be bounded on $R^d \times [0,T]$ and satisfy $\Delta u + |u|^p - u_t \leq \Delta v + |v|^p - v_t$, for $(x,t) \in R^d \times (0,T]$. If $u(x,0) \geq v(x,0)$, for $x \in R^d$, and $\liminf_{|x|\to\infty} \inf_{t\in[0,T]}(u(x,t) - v(x,t)) \geq 0$, then*

$$u \geq v \text{ in } R^d \times [0,T].$$

**Proof.** Let $Z = u - v$. Then $\Delta Z - Z_t + (\frac{|u(x,t)|^p - |v(x,t)|^p}{u(x,t)-v(x,t)} I_{\{u(x,t)\neq v(x,t)\}})Z \leq 0$. Since $f(x) = |x|^p$ is a locally Lipschitz function and since $u$ and $v$ are bounded, we have $\Delta Z - Z_t + V(x,t)Z \leq 0$, where $V$ is bounded. Also, $Z(x,0) \geq 0$ and $\liminf_{|x|\to\infty} \inf_{t\in[0,T]} Z(x,t) \geq 0$. It then follows from the standard linear parabolic maximum principle that $Z \geq 0$; that is, $u \geq v$. □

**Proof of Theorem 2.** In the first paragraph of the statement of the theorem, if one deletes the claim that $u$ is a classical solution, and if one deletes $L^\infty$ and replaces the claim that $\lim_{t\to T_\phi} \max(||u^+_\phi(\cdot,t)||_r, ||u^+_\phi(\cdot,t)||_\infty) = \infty$ and $\sup_{0\leq t<T_\phi} \max(||u^-_\phi(\cdot,t)||_r, ||u^-_\phi(\cdot,t)||_\infty) < \infty$ with $\lim_{t\to T_\phi} ||u_\phi(\cdot,t)||_r = \infty$, then the claims in this paragraph follow from the results of Weissler [12,13] along with his comments in the last paragraph on page 31 in [14]. By [11, chapter 6, Theorem 1.4], the same result holds if one works with the $L^\infty$- norm. Thus, combining the two results allows one to work with $L^r(R^d) \cap L^\infty(R^d)$. The method in [15] can now be used to show that $u_\phi$ is in fact a classical solution to (1.1). (See the proofs of Propositions 3.1 and 3.2 in [15] and note that parts (i)-(iv) of Theorem 1.1 in [15] hold for $u_\phi$ since $u_\phi \in L^r$ for some $r \geq p$.)

We now turn to (1.5) and the comparison claim in the second paragraph of the statement of the theorem, after which we return to deal with the switch from norm blow up of $u_\phi$ to norm blow up of $u_\phi^+$ and norm boundedness of $u_\phi^-$. In fact, the comparison claim follows immediately from Proposition 1 and (1.5). To prove (1.5), we utilize the representation (1.3). Fix $T \in (0, T_\phi)$. Making a change of variables, we have

$$\int_{R^d} p(t,x,y)\phi(y)dy = \int_{R^d} p(1,0,z)\phi(x + t^{\frac{1}{2}}z)dz.$$

Using this along with the dominated convergence theorem and the assumption that $\lim_{|x|\to\infty} \phi(x) = 0$ shows that

$$\lim_{|x|\to\infty} \sup_{0\leq t\leq T} \int_{R^d} p(t,x,y)\phi(y)dy = 0. \tag{2.1}$$

Fix $\epsilon \in (0,T)$. Let $M = \sup_{0\leq t\leq T} ||u_\phi(\cdot,t)||_\infty$. Then

$$\int_0^t \int_{R^d} p(t-s,x,y)|u_\phi(y,s)|^p dyds \leq M^p\epsilon, \text{ for } t \in (0,\epsilon] \tag{2.2}$$

and

$$\begin{aligned} &\int_0^t \int_{R^d} p(t-s,x,y)|u_\phi(y,s)|^p dyds \\ &\leq M^p\epsilon + \int_0^{t-\epsilon} \int_{R^d} p(t-s,x,y)|u_\phi(y,s)|^p dyds, \text{ for } t \in (\epsilon,T]. \end{aligned} \tag{2.3}$$

Assume now that $r > p$. The case $r = p$ is treated similarly but without the implementation of Hölder's inequality. Let $l > 1$ satisfy $pl = r$ and let $q$ satisfy

$\frac{1}{q} + \frac{1}{l} = 1$. For some constant $C_\epsilon > 0$, we have

$$
\begin{aligned}
&\int_0^{t-\epsilon} \int_{R^d} p(t-s,x,y)|u_\phi(y,s)|^p dyds \\
&\le \int_0^{t-\epsilon} ds (\int_{R^d} p^{\frac{q}{2}}(t-s,x,y)dy)^{\frac{1}{q}} (\int_{R^d} p^{\frac{l}{2}}(t-s,x,y)|u_\phi(y,s)|^r dy)^{\frac{1}{l}} \\
&\le C_\epsilon \int_0^{t-\epsilon} ds (\int_{R^d} p^{\frac{l}{2}}(t-s,x,y)|u_\phi(y,s)|^r dy)^{\frac{1}{l}}, \\
&\le C_\epsilon (t-\epsilon)^{1-\frac{1}{l}} (\int_0^{t-\epsilon} \int_{R^d} p^{\frac{l}{2}}(t-s,x,y)|u_\phi(y,s)|^r dyds)^{\frac{1}{l}}, \text{ for } t \in (\epsilon, T],
\end{aligned}
\tag{2.4}
$$

where the final step uses Jensen's inequality. Also,

$$
\begin{aligned}
&\int_0^{t-\epsilon} \int_{R^d} p^{\frac{l}{2}}(t-s,x,y)|u_\phi(y,s)|^r dyds \\
&\le \int_0^{T-\epsilon} \int_{R^d} (4\pi\epsilon)^{-\frac{dl}{4}} \exp(-\frac{l|y-x|^2}{8T})|u_\phi(y,s)|^r dyds, \text{ for } t \in (\epsilon, T].
\end{aligned}
\tag{2.5}
$$

Now $u_\phi \in L^r(R^d \times [0, T-\epsilon])$ since $u_\phi(\cdot, t)$ is continuous from $[0, T_\phi)$ to $L^r(R^d)$. Therefore, by the dominated convergence theorem, the right hand side of (2.5) converges to 0 as $|x| \to \infty$. It now follows from (2.2)-(2.5) and the fact that $\epsilon > 0$ is arbitrary that

$$
\lim_{|x|\to\infty} \sup_{0 \le t \le T} \int_0^t \int_{R^d} p(t-s,x,y)|u_\phi(y,s)|^p dyds = 0. \tag{2.6}
$$

Now (1.5) follows from (2.1), (2.6) and (1.3).

With the comparison result proved, we can now prove that if $T_\phi < \infty$, then the norm of $u_\phi^+$ blows up and the norm of $u_\phi^-$ does not. By the comparison result, we have $u_{-\phi^-} \le u_\phi$. But $v \equiv -u_{-\phi^-}$ satisfies $v_t = \Delta v - |v|^p$ with nonnegative initial data. It is known that there exists a unique solution to this equation and that solution is global, nonnegative and dominated by the corresponding solution to the linear heat equation [3]. We conclude then that $u_\phi^-(x,t) \le \int_{R^d} p(t,x,y)\phi^-(y)dy$. Thus, by Jensen's inequality, $||u_\phi^-(\cdot,t)||_s \le ||\phi^-||_s$, for all $s \in [1,\infty]$ and all $t \in [0, T_\infty)$. This shows that neither the $L^r$-norm nor the $L^\infty$-norm of $u_\phi^-$ blows up. On the other hand, since either $||u_\phi(\cdot,t)||_r$ or $||u_\phi(\cdot,t)||_\infty$ does blow up when $t \to T_\phi$, it follows that either $||u_\phi^+(\cdot,t)||_r$ or $||u_\phi^+(\cdot,t)||_\infty$ blows up as $t \to T_\infty$. □

**3. Proofs of Theorems 3 and 4.**

**Proof of Theorem 3.** Let $\phi$ be Hölder continuous. In addition, assume one of the following two situations:

*1*: $\phi \in L^1(R^d)$ satisfies $\int_{R^d} \phi(x)dx = 0$ and $u_\phi$ is a global solution to (1.1) satisfying (1.2)

or

*2*: $\phi \in L^1(R^d) \cap L^\infty(R^d)$, in which case $\phi \in L^s(R^d)$ for all $s > 1$, and the solution $u_\phi$ in Theorem 2 is global.

Then by Theorem 1 in the former case and by Theorem 2 in the latter case, $u_\phi$ satisfies (1.3). Integrating (1.3) shows that

$$\int_{R^d} u_\phi(x,1)dx = \int_0^1 \int_{R^d} |u_\phi(x,s)|^p dxds > 0. \tag{3.1}$$

Let $v(x,t) = u_\phi(x,t+1)$, for $t \geq 0$. Then $v$ solves (1.1) with $\phi(x)$ replaced by $u_\phi(x,1)$. In the case of situation (1) above, $v$ satisfies (1.2) since $u$ does. In the case of situation (2), $v(x,0) = u_\phi(x,1) \in L^r(R^d) \cap L^\infty(R^d)$. Thus, it follows from Theorem 1 for situation (1) and from the uniqueness part of Theorem 2 for situation (2) that

$$v(x,t) = \int_{R^d} p(t,x,y)u_\phi(y,1)dy + \int_0^t \int_{R^d} p(t-s,x,y)|v(y,s)|^p dyds. \tag{3.2}$$

From (1.3), $u_\phi^-(x,1) \leq \int_{R^d} p(t,x,y)\phi^-(y)dy$. Since $\phi^- \in L^1(R^d)$, it follows that $u_\phi^-(\cdot,1) \in L^1(R^d)$. It also follows that if $\phi^-(x) \leq c_1 \exp(-c_2|x|^2)$, then $u_\phi^-(x,1) \leq c_3 \exp(-c_4|x|^2)$ for some $c_3, c_4 > 0$. It is possible that $u_\phi(\cdot,1) \notin L^1(R^d)$; that is, the right hand side of (3.1) may equal $\infty$. However, we note that the proof of Zhang's theorem goes through with no change even if $\int_{R^d} \phi(x)dx = \infty$. From these facts along with (3.1) and (3.2), it follows from Zhang's theorem that $v$, and thus also $u_\phi$, is not a global solution. This contradicts the assumption that $u_\phi$ is a global solution. □

**Proof of Theorem 4.** *i.* We will use the notation $S_t f(x) = \int_{R^d} p(t,x,y)f(y)dy = \int_{R^d} (4\pi t)^{-\frac{d}{2}} \exp(-\frac{|y-x|^2}{4t})f(y)dy$, for any $f$. Let $B_R$ denote the open ball of radius $R$ centered at the origin. Fix $R_1, R_2, R_3$ satisfying $0 < R_1 < R_2 < R_3$. Let $\phi_1 \not\equiv 0$ be a continuous function satisfying $0 \leq \phi_1 \leq 1$ and supported in $B_{R_1}$. Let $\phi_2$ be a continuous, compactly supported function satisfying $0 \leq \phi_2 \leq 1$, $\inf_{x \in B_{R_3} - B_{R_2}} \phi_2(x) > 0$, and $\mathrm{supp}(\phi_2) \subset R^d - B_{R_1}$. Then $S_1\phi_1(x) \leq C_1 \exp(-\frac{|x|^2}{4}) \int_{B_{R_1}} \exp(\frac{1}{2}(x,y))dy$ and $S_1\phi_2(x) \geq C_2 \exp(-\frac{|x|^2}{4}) \int_{B_{R_3}-B_{R_2}} \exp(\frac{1}{2}(x,y))dy$, for constants $C_1, C_2 > 0$. From this it follows that $\lim_{|x|\to\infty} \frac{S_1\phi_1(x)}{S_1\phi_2(x)} = 0$. Thus, we may choose $\epsilon > 0$ so small that

$$2\epsilon S_1\phi_1(x) - S_1\phi_2(x) \leq 0, \text{ for all } x \in R^d. \tag{3.3}$$

Let $v$ solve the inhomogeneous linear equation

$$\begin{aligned} &v_t = \Delta v + \lambda(\epsilon S_t\phi_1 + S_t\phi_2); \\ &v(x,0) = \lambda(\epsilon\phi_1 - 2\phi_2). \end{aligned} \tag{3.4}$$

Then we have

$$\begin{aligned} v(x,t) &= \lambda(\epsilon S_t\phi_1(x) - 2S_t\phi_2(x) + \int_0^t S_{t-s}(\epsilon S_s\phi_1 + S_s\phi_2)(x)ds) \\ &= \lambda(\epsilon S_t\phi_1(x) - 2S_t\phi_2(x) + \epsilon t S_t\phi_1(x) + tS_t\phi_2(x)) \\ &= \lambda(\epsilon(1+t)S_t\phi_1(x) + (t-2)S_t\phi_2(x)). \end{aligned} \tag{3.5}$$

By (3.3) and (3.5), we have

$$v(x,1) = \lambda(2\epsilon S_1\phi_1(x) - S_1\phi_2(x)) \leq 0, \text{ for all } x \in R^d. \tag{3.6}$$

Also, from (3.5) we have

$$\begin{aligned} |v(x,t)|^p &\leq 2^p\lambda^p((2\epsilon S_t\phi_1(x))^p + (2S_t\phi_2(x))^p) \\ &= (4\lambda\epsilon)^p(S_t\phi_1(x))^p + (4\lambda)^p(S_t\phi_2(x))^p, \text{ for } x \in R^d \text{ and } t \in [0,1]. \end{aligned} \tag{3.7}$$

Since $0 \leq \phi_1, \phi_2 \leq 1$, it follows that $0 < S_t\phi_1, S_t\phi_2 \leq 1$; therefore $(S_t\phi_i)^p \leq S_t\phi_i$. Consequently, in light of (3.7) we can choose $\lambda > 0$ so small that

$$|v(x,t)|^p \leq \lambda(\epsilon S_t\phi_1(x) + S_t\phi_2(x)), \text{ for } x \in R^d \text{ and } t \in [0,1]. \tag{3.8}$$

From (3.4) and (3.8) we obtain

$$\Delta v + |v|^p - v_t \leq 0 \text{ in } R^d \times [0,1]. \tag{3.9}$$

Let $\phi \equiv \lambda(\epsilon\phi_1 - 2\phi_2)$ and let $u_\phi$ denote the solution to (1.1) from Theorem 2 with maximal time $T_\phi$. Since $\phi_i$ is compactly supported, $\lim_{|x|\to\infty} \sup_{0 \leq t \leq 1} S_t\phi_i(x) = 0$;

thus, it follows from (3.5) that $\lim_{|x|\to\infty}\sup_{0\le t\le 1}|v(x,t)|=0$. From (1.4) and (1.5), it follows that $\lim_{|x|\to\infty}\sup_{0\le t\le T_\phi-\delta}u_\phi(x,t)=0$, for any $\delta>0$. Using this along with (3.9), (1.1) and Proposition 1, we conclude that

$$u_\phi \le v \text{ in } R^d\times[0,1\wedge T_\phi). \tag{3.10}$$

By Theorem 2, if $T_\phi<\infty$, then $\lim_{t\to T_\phi}||u_\phi^+(\cdot,t)||_s=\infty$, for some $s\in[p,\infty]$. Therefore, it follows from (3.10) that $T_\phi>1$. Using this with (3.6) and (3.10), we have

$$u_\phi(x,1)\le 0, \text{ for } x\in R^d. \tag{3.11}$$

Now let $v(x,t)=u_\phi(x,t+1)$. By (3.11), $v(x,0)=u_\phi(x,1)\le 0$. By Theorem 2, $v(\cdot,0)\in L^r(R^d)\cap L^\infty(R^d)$, where $r$ is as in Theorem 2. By the uniqueness in Theorem 2, $v$ is the solution from Theorem 2 with initial data $u_\phi(\cdot,1)\le 0$. Now $V=-v$ satisfies $V_t=\Delta V-|V|^p$ and $V(x,0)\ge 0$. As noted earlier, there exists a unique, nonnegative, global solution to this equation. We conclude then that $u_\phi$ is a global solution.

We have now proved that a global solution exists with initial data $\lambda(\epsilon\phi_1-2\phi_2)$, for sufficiently small $\epsilon,\lambda>0$. By Theorem 3, if follows that $\int_{R^d}\lambda(\epsilon\phi_1(x)-2\phi_2(x))dx<0$. The absolute value of this integral can be made arbitrarily small by decreasing $\lambda$. Thus, we have shown that there exists a global solution with continuous, compactly supported, sign-changing initial data $\phi$ with $\int_{R^d}\phi(x)dx<0$ and $|\int_{R^d}\phi(x)dx|$ arbitrarily small.

Now let $l\in(-\infty,0)$. By the above paragraph there exists a continuous, compactly supported $\phi_3$ with $\int_{R^d}\phi_3(x)dx>l$ and such that the solution $u_{\phi_3}$ in Theorem 2 corresponding to initial data $\phi_3$ is global; that is, $T_{\phi_3}=\infty$. Choose a continuous, compactly supported $\phi_4$ such that $\phi_4\le\phi_3$ and $\int_{R^d}\phi_4(x)dx=l$. Let $u_{\phi_4}$ be the corresponding solution from Theorem 2 and let $T_{\phi_4}$ denote its maximal time. From the comparison result in Theorem 2 it follows that $T_{\phi_4}\ge T_{\phi_3}=\infty$; thus, $u_{\phi_4}$ is a global solution. □

*ii.* Let $\phi_1$ be a continuous, compactly supported, sign-changing function satisfying $\int_{R^d} \phi_1(x)dx < 0$. Let $\lambda > 0$ and let $w$ be the solution of the linear heat equation corresponding to (1.1) with initial data $\lambda\phi_1$; that is,

$$w(x,t) = \lambda \int_{R^d} p(t,x,y)\phi_1(y)dy. \tag{3.12}$$

Let $v$ be the solution to the inhomogeneous linear equation

$$\begin{aligned} &v_t = \Delta v + (w^+)^p \text{ in } R^d \times (0,\infty); \\ &v(x,0) = \lambda\phi_1(x) \text{ in } R^d. \end{aligned} \tag{3.13}$$

We have

$$v(x,t) = \lambda S_t\phi_1(x) + \int_0^t S_{t-s}(w^+)^p(x,s)ds. \tag{3.14}$$

Let $u_{\lambda\phi_1}$ be the solution described in Theorem 2 for the initial data $\lambda\phi_1$, and let $T_{\lambda\phi_1}$ denote its maximal time. Since $u_{\lambda\phi_1}$ satisfies (1.3), it follows from (3.12) that

$$u_{\lambda\phi_1} \geq w \ \text{ up until time } T_{\lambda\phi_1}. \tag{3.15}$$

Let $Z = u_{\lambda\phi_1} - v$. Then by (3.13) and (3.15), $\Delta Z - Z_t = -|u_{\lambda\phi_1}|^p + (w^+)^p \leq 0$ up until time $T_{\lambda\phi_1}$. Since $\lim_{|x|\to\infty} \sup_{0\leq t\leq T} |v(x,t)| = 0$, for any $T > 0$, and since by (1.5), $\lim_{|x|\to\infty} \sup_{0\leq t\leq T} |u_{\lambda\phi_1}(x,t)| = 0$, for any $T \in (0, T_{\lambda\phi_1})$, we have $\lim_{|x|\to\infty} \sup_{0\leq t\leq T} |Z(x,t)| = 0$, for any $T \in (0, T_{\lambda\phi_1})$. It then follows from the linear maximum principle that $Z \geq 0$ up until time $T_{\lambda\phi_1}$; that is,

$$u_{\lambda\phi_1} \geq v \text{ up until time } T_{\lambda\phi_1}. \tag{3.16}$$

From (3.14), we have

(3.17)
$$\begin{aligned} &\int_{R^d} v(x,1)dx = \lambda \int_{R^d} \phi_1(x)dx + \int_0^1 \int_{R^d} (w^+(x,s))^p dxds \\ &= \lambda \int_{R^d} \phi_1(x)dx + \lambda^p \int_0^1 \int_{R^d} (0 \vee \int_{R^d} (4\pi s)^{-\frac{d}{2}} \exp(-\frac{|y-x|^2}{4s})\phi_1(y)dy)^p dxds. \end{aligned}$$

Since $\phi_1$ changes sign, $0 \vee \int_{R^d} (4\pi s)^{-\frac{d}{2}} \exp(-\frac{|y-x|^2}{4s})\phi_1(y)dy \gneq 0$ on $R^d \times [0,1]$. Therefore, it follows from (3.17) that if $\lambda > 0$ is chosen sufficiently large, then

$\int_{R^d} v(x,1)dx > 0$. Using this with (3.16) shows that for sufficiently large $\lambda$, either $T_{\lambda\phi_1} \le 1$ or $u_{\lambda\phi_1}(\cdot,1) \in L^r(R^d) \cap L^\infty(R^d)$ and

$$\int_{R^d} u_{\lambda\phi_1}(x,1)dx > 0. \tag{3.18}$$

In the latter case, it follows from the uniqueness in Theorem 2 that $U(x,t) \equiv u_{\lambda\phi_1}(x,t+1)$ has maximal time $T_{\lambda\phi_1} - 1$ and satisfies

$$\begin{aligned} &U(x,t) = \int_{R^d} p(t,x,y)u_{\lambda\phi_1}(y,1)dy + \int_0^t \int_{R^d} p(t-s,x,y)|U(y,s)|^p dyds, \\ &\text{for } x \in R^d \text{ and } t \in [0, T_{\lambda\phi_1} - 1). \end{aligned} \tag{3.19}$$

From (3.18) and (3.19) and Zhang's theorem, it follows that $U$ is not global; hence $T_{\lambda\phi_1} < \infty$.

We have now proved that a non-global solution exists with initial data $\lambda\phi_1$ for sufficiently large $\lambda > 0$, where $\phi_1$ is continuous and compactly supported and satisfies $\int_{R^d} \phi_1(x)dx < 0$. This shows that there exist non-global solutions with continuous, compactly supported initial data $\phi$ satisfying $\int_{R^d} \phi(x)dx < 0$ and $|\int_{R^d} \phi(x)dx|$ arbitrarily large.

Now let $l \in (-\infty, 0)$. By the above paragraph, there exists a continuous, compactly supported $\phi_2$ such that $\int_{R^d} \phi_2(x)dx < l$ and such that the solution $u_{\phi_2}$ from Theorem 2 is not global; that is $T_{\phi_2} < \infty$. Choose a continuous, compactly supported $\phi_3$ such that $\phi_3 \ge \phi_2$ and $\int_{R^d} \phi_3(x)dx = l$, and let $u_{\phi_3}$ be the corresponding solution from Theorem 2 with maximal time $T_{\phi_3}$. By the comparison result in Theorem 2 we have $T_{\phi_3} \le T_{\phi_2} < \infty$; thus $u_{\phi_3}$ is not global. □

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Ross Pinsky
Department of Mathematics
Technion-Israel Institute of Technology
Haifa, 32000 Israel
e-mail:pinsky@math.technion.ac.il